From Little Strings to M5-branes via a Quasi-Topological Sigma Model on Loop Group

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String and M-Theory: The New Geometry of the 21st Century

Outline of Talk

- Introduction and Motivation
- Summary of Results
- Main Body of the Talk
- Conclusion and Future Directions

Introduction and Motivation

In this talk, we will discuss a **quasi-topological twist** of a 2d $\mathcal{N} = (2, 2)$ nonlinear sigma model (NLSM) on $\mathbb{C}P^1$ with target space the based loop group $\Omega SU(k)$.

The motivations for doing so are to:

- Describe the ground and half-excited states of the 6d A_{k-1} $\mathcal{N} = (2, 0)$ **little string theory**.
- Obtain a physical **derivation and generalization** of a mathematical relation by Braverman-Finkelberg which defines a **geometric Langlands correspondence for surfaces**.
- Elucidate the **1/2 and 1/4 BPS sectors** of the **M5-brane worldvolume theory.**

Introduction and Motivation

This talk is based on

- M.-C. Tan et al., *Little Strings, Quasi-Topological Sigma Model on Loop Group, and Toroidal Lie Algebras*, Nucl.Phys. B928, 469-498 (2018)

Built on earlier insights in

- M.-C. Tan, *Two-Dimensional Twisted Sigma Models And The Theory of Chiral Differential Operators*, Adv.Theor.Math.Phys. 10, 759-851 (2006).
- M.-C. Tan, *Five-Branes in M-Theory and a Two-Dimensional Geometric Langlands Duality*, Adv.Theor.Math.Phys. 14, 179-224 (2010).
- R. Dijkgraaf, *The Mathematics of Fivebranes*, Documenta Mathematica, 133-142 (1998).

Summary of Results

1. In the quasi-topological sigma model with target $\Omega SU(k)$, there is a scalar supercharge $\overline{Q}_+$ which generated supersymmetry survives on a $\mathbb{C}P^1$ worldsheet, whereby in the $\overline{Q}_+$-cohomology, we have the following currents that generate the following toroidal algebra $\mathfrak{su}(k)_{\mathrm{tor}}$:

$$\boxed{[J^{an_1}_{m_1}, J^{bn_2}_{m_2}] = if^{ab}_c J^{c\{n_1+n_2\}}_{m_1+m_2} + c_1 n_1 \delta^{ab}\delta^{\{n_1+n_2\}0}\delta_{\{m_1+m_2\}0} + c_2 m_1 \delta^{ab}\delta^{\{n_1+n_2\}0}\delta_{\{m_1+m_2\}0}}$$

2. In the topological subsector of the sigma model, we have instead the following affine algebra $\mathfrak{su}(k)_{\mathrm{aff}}$:

$$\boxed{[J^{an_1}_0, J^{bn_2}_0] = if^{ab}_c J^{c\{n_1+n_2\}}_0 + c_1 n_1 \delta^{ab}\delta^{\{n_1+n_2\}0}}$$

Summary of Results

3. Via a theorem by Atiyah in [1], the left-excited states (in the DLCQ) of the 6d A_{k-1} $\mathcal{N}=(2,0)$ little string theory (LST) on $\mathbb{R}^{5,1}$ can be related to the $\overline{Q}_+$-cohomology of the quasi-topological sigma model. In turn, we find that

$$\boxed{\text{left-excited spectrum of 6d } A_{k-1}\ (2,0)\ \text{LST} = \text{modules of } \mathfrak{su}(k)_{\text{tor}}}$$

4. Likewise, the ground states (in the DLCQ) of the 6d A_{k-1} $\mathcal{N}=(2,0)$ LST on $\mathbb{R}^{5,1}$ can be related to the topological subsector of the sigma model. In turn, we find that

$$\boxed{\text{ground spectrum of 6d } A_{k-1}\ (2,0)\ \text{LST} = \text{modules of } \mathfrak{su}(k)_{\text{aff}}}$$

5. This means (via the ground states) that we have

$$\boxed{\mathrm{IH}^*(\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)) = \widehat{su}(k)^N_{c_1}}$$

i.e., the intersection cohomology of the moduli space $\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)$ of $SU(k)$-instantons forms a finite submodule over $\mathfrak{su}(k)_{\mathrm{aff}}$. This is just the Braverman-Finkelberg relation in [2]

6. This also means (via the left-excited states) that we have

$$\boxed{H^*_{\check{\mathrm{C}}\mathrm{ech}}(\widehat{\Omega}^{ch}_{\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)}) = \widehat{\widehat{su}}(k)^N_{c_1,c_2}}$$

i.e., the $\check{\mathrm{C}}\mathrm{ech}$-cohomology of the sheaf $\widehat{\Omega}^{ch}_{\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)}$ of Chiral de Rham Complex on $\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)$ forms a submodule over $\mathfrak{su}(k)_{\mathrm{tor}}$. This is a novel, physically-derived generalization of the Braverman-Finkelberg relation.

7. Using the relevant SUSY algebras, one can show the correspondence between the ground states of the little string and the 1/2 BPS sector of the M5-brane worldvolume theory, from which we can compute the 1/2 BPS sector partition function to be

$$\boxed{Z_{1/2} = \sum_{\widehat{\lambda}'} \chi^{\widehat{\lambda}'}_{\widehat{su}(k)_{c_1}}(p)}$$

It is a cousin of a modular form which transforms as a representation of $SL(2,\mathbb{Z})$.

There is an instrinsic $SL(2,\mathbb{Z})$ symmetry in the M5-brane worldvolume theory on $\mathbb{R}^{5,1}$!

Emerges as gauge-theoretic S-duality of 4d $\mathcal{N}=4$ SYM after compactifying on T^2.

8. Likewise, one can show the correspondence between the left-excited states of the little string and the 1/4 BPS sector of the M5-brane worldvolume theory, from which we can compute the 1/4 BPS sector partition function to be

$$\boxed{Z_{1/4} = q^{\frac{1}{24}} \sum_{\widehat{\lambda}} \chi^{\widehat{\lambda}}_{\widehat{su}(k)_{c_1}}(p) \frac{1}{\eta(\tau)}}$$

It is a cousin of an automorphic form which transforms as a representation of $SO(2,2;\mathbb{Z})$.

There is an intrinsic $SO(2,2;\mathbb{Z})$ symmetry of the M5-brane worldvolume theory on $\mathbb{R}^{5,1}$!

Emerges as string-theoretic T-duality of little strings after compactifying on T^2.

LET'S EXPLAIN HOW WE GOT THESE RESULTS

Based Loop Group ΩG

A **loop group** LG is the group consisting of maps from the unit circle S^1 to a (Lie) group G:

$$f : S^1 \to G. \qquad (1)$$

Parametrize S^1 by $e^{i\theta}$. If we impose the based point condition

$$f(\theta = 0) \to I, \qquad (2)$$

we get the ***based*** loop group ΩG. One can show that

$$\Omega G = LG/G, \qquad (3)$$

i.e., it is a G-equivariant subset of LG endowed with an LG-action.

ΩG also admits a closed nondegenerate symplectic two-form ω. The complex and symplectic structures of ΩG are compatible, and conspire to make it an infinite-dimensional Kähler manifold.

Based Loop Group ΩG

Let ξ and η be elements of $\Omega\mathfrak{g}$, the based loop algebra. Then, expanding them in the $L\mathfrak{g}$ basis gives

$$\begin{aligned} \xi(\theta) &= \xi_n e^{in\theta} = \xi_{an} T^a e^{in\theta}, \\ \eta(\theta) &= \eta_n e^{in\theta} = \eta_{an} T^a e^{in\theta}, \end{aligned} \tag{4}$$

where $n \in \mathbb{Z}$ and $a = 1, \ldots, \dim\, \mathfrak{g}$. The based point condition (2), which can be written as $e^{i\xi(\theta=0)} = 1$, then translates to $\sum_n \xi_{an} T^a = 0$.

The metric of ΩG is

$$g_{am,bn} = |n| \delta_{n+m,0} \operatorname{Tr}(T_a T_b). \tag{5}$$

If we denote $T^a e^{im\theta} \equiv T^{am}$, we have

$$[T^{am}, T^{bn}] = i f_c^{ab} T^{c(m+n)}. \tag{6}$$

The $\mathcal{N}=(2,2)$ Sigma Model on $\mathbb{C}P^1$ with Target $\Omega SU(k)$

The action of the $\mathcal{N}=(2,2)$ supersymmetric sigma model on $\mathbb{C}P^1$ with $\Omega SU(k)$ target space is

$$S=\int d^2z\Big(g_{am,b\overline{n}}(\frac{1}{2}\partial_z\phi^{am}\partial_{\overline{z}}\overline{\phi}^{b\overline{n}}+\frac{1}{2}\partial_{\overline{z}}\phi^{am}\partial_z\overline{\phi}^{b\overline{n}}+\overline{\psi}_-^{b\overline{n}}D_{\overline{z}}\psi_-^{am}+\psi_+^{am}D_z\overline{\psi_+}^{b\overline{n}})$$
$$-R_{am,c\overline{p},bn,d\overline{q}}\psi_+^{am}\psi_-^{bn}\overline{\psi}_-^{c\overline{p}}\overline{\psi}_+^{d\overline{q}}\Big), \tag{7}$$

where

$$\begin{aligned}\phi^{a(-n)}&=\overline{\phi}^{an},\\ \psi_{\mp}^{a(-n)}&=\overline{\psi}_{\pm}^{an}.\end{aligned} \tag{8}$$

and

$$\begin{aligned}D_{\overline{z}}\psi_-^{am}&=\partial_{\overline{z}}\psi_-^{am}+\Gamma^{am}_{bn,cp}\partial_{\overline{z}}\phi^{bn}\psi_-^{cp},\\ D_z\overline{\psi}_+^{a\overline{m}}&=\partial_z\overline{\psi}_+^{a\overline{m}}+\Gamma^{a\overline{m}}_{b\overline{n},c\overline{p}}\partial_z\overline{\phi}^{b\overline{n}}\overline{\psi}_+^{c\overline{p}}.\end{aligned} \tag{9}$$

Quasi-Topological A-Model on $\mathbb{C}P^1$ with Target $\Omega SU(k)$

We may twist the $\mathcal{N} = (2,2)$ sigma model, i.e., shift the spin of the fields by their $U(1)_{\mathrm{R}}$-charges. Let us consider the A-twist. The fermionic fields then become the following scalars/one-forms

$$\begin{aligned}
\psi_+^{am} &\to \rho_{\bar{z}}^{am}, \\
\overline{\psi}_+^{am} &\to \overline{\chi}^{am}, \\
\psi_-^{am} &\to \chi^{am}, \\
\overline{\psi}_-^{am} &\to \overline{\rho}_z^{am},
\end{aligned} \tag{10}$$

and we can write

$$\begin{aligned}
S = \int d^2z \Big(& g_{am,bn} (\partial_{\bar{z}} \phi^{am} \partial_z \overline{\phi}^{bn} + \overline{\rho}_z^{bn} D_{\bar{z}} \chi^{am} + \rho_{\bar{z}}^{am} D_z \overline{\chi}^{bn}) \\
& - R_{cp,bn,dq,am} \overline{\rho}_z^{cp} \chi^{bn} \overline{\chi}^{dq} \rho_{\bar{z}}^{am} + \int \Phi^* \omega \\
& = S_{pert.} + \int \Phi^* \omega,
\end{aligned} \tag{11}$$

where the map $\Phi : \mathbb{C}P^1 \to \Omega SU(k)$ is of integer degree N.

Like the fermion fields, there are two (nilpotent) scalar supercharges $\overline{Q}_+$ and Q_-, which SUSYs are therefore preserved on a worldsheet of any genus. In particular, $\overline{Q}_+$ generates the transformations

$$
\begin{aligned}
\delta\phi^{am} &= 0, \\
\delta\overline{\phi}^{am} &= \bar{\epsilon}_-\overline{\chi}^{am}, \\
\delta\rho^{am}_{\bar{z}} &= -\bar{\epsilon}_-\partial_{\bar{z}}\phi^{am}, \\
\delta\overline{\rho}^{am}_{z} &= -\bar{\epsilon}_-\overline{\Gamma}^{am}_{bn,cp}\overline{\chi}^{bn}\overline{\rho}^{cp}_{z}, \\
\delta\chi^{am} &= 0, \\
\delta\overline{\chi}^{am} &= 0,
\end{aligned}
\tag{12}
$$

where $\bar{\epsilon}_-$ is a scalar grassmanian parameter.

Quasi-Topological A-Model on $\mathbb{C}P^1$ with Target $\Omega SU(k)$

- The action (11) can be cast into the form

$$S = \int d^2z \, \{\overline{Q}_+, W'(t)\} + \cdots + tN \tag{13}$$

  where $W'(t)$ is a metric-dependent combination of fields with metric scale t, and the ellipsis indicates additional terms which are metric-independent but depend on the complex structure of the target space.

- Although the stress tensor T_{zz} (i.e. $\delta S/\delta g_{zz}$) is $\overline{Q}_+$-closed, it is generically **not** $\overline{Q}_+$-exact; only $T_{\overline{z}\overline{z}}$ is $\overline{Q}_+$-exact. So, the correlation function of $\overline{Q}_+$-closed (but not exact) observables $\widetilde{\mathcal{O}}$ is not completely independent of arbitrary deformations the worldsheet metric g. This is the **quasi-topological** A-model.

- Path integral localizes to $\overline{Q}_+$-fixed points, and from (12), these are **holomorphic maps** from $\mathbb{C}P^1$ to $\Omega SU(k)$.

- The $\overline{Q}_+$-cohomology of the model has **ground and left-excited states**, and the relevant operator observables $\widetilde{\mathcal{O}}$ of **holomorphic dimension zero and positive** are Čech cohomology classes of the sheaf $\widehat{\Omega}^{ch}$ of chiral de Rham complex on $\mathcal{M}(\mathbb{C}P^1 \xrightarrow[hol.]{N} \Omega SU(k))$ [3].
- A correlation function of observables $\widetilde{\mathcal{O}}$ has the form

$$\langle \prod_\gamma \widetilde{\mathcal{O}}_\gamma \rangle = \sum_N e^{-tN} \Big(\int_{F_N} \mathcal{D}\phi \mathcal{D}\overline{\phi} \mathcal{D}\rho_{\overline{z}} \mathcal{D}\overline{\rho}_z \mathcal{D}\chi \mathcal{D}\overline{\chi} \; e^{-\int d^2z(\{\overline{Q}_+, W'(t)\}+\dots)} \prod_\gamma \widetilde{\mathcal{O}}_\gamma \Big). \tag{14}$$

Notice that

$$\frac{d}{dt}\Big(\int_{F_N} \mathcal{D}\phi \dots \mathcal{D}\overline{\chi} \; e^{-\int d^2z(\{\overline{Q}_+, W'(t)\}+\dots)} \prod_\gamma \widetilde{\mathcal{O}}_\gamma \Big) = \langle \{\overline{Q}_+, \dots\} \rangle_{pert.} = 0 \tag{15}$$

so we can compute the path integral over F_N in (14), henceforth denoted as $\langle \prod_\gamma \widetilde{\mathcal{O}}_\gamma \rangle_{pert.}$, at any convenient value of t, whilst keeping the original value of t in the constant factor e^{-tN} (due to worldsheet instantons).

Appearance of Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- **Isometries** of the target space inherited as **worldsheet symmetries** of the sigma model.

- Since $\Omega G \cong LG/G$, our sigma model ought to have an $LSU(k)$ symmetry on the worldsheet.

- Indeed, the corresponding **Noether currents**, the J's, which charges generate a symmetry of the sigma model, can be shown to obey a current algebra associated with $LSU(k)$.

- As the J's generate a symmetry, they act to leave the $\overline{Q}_+$-cohomology of operator observables invariant. Thus, they ought to be $\overline{Q}_+$-closed (but not exact), and are therefore also **in the $\overline{\mathbf{Q}}_+$-cohomology**, as one can verify.

Appearance of Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- We can conveniently compute the correlation functions of the J's and T_{zz} via a large t limit, as explained in (14)-(15), and as OPEs, they are (in worldsheet instanton sector N)

$$J_z^{an_1}(z)J_z^{bn_2}(w) \sim \frac{if_c^{ab}J_z^{c\{n_1+n_2\}}(w)}{z-w}, \tag{16}$$

and

$$T_{zz}(z)J_z^{ak}(w) \sim \frac{J_z^{ak}(w)}{(z-w)^2} + \frac{\partial J_z^{ak}(w)}{(z-w)}. \tag{17}$$

Appearance of Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- Laurent expanding, these correspond to the **double loop algebra** $LL\mathfrak{su}(k)$

$$[J^{an_1}_{m_1}, J^{bn_2}_{m_2}] = if^{ab}_c J^{c\{n_1+n_2\}}_{m_1+m_2}, \tag{18}$$

and

$$[L_n, J^{ak}_m] = -mJ^{ak}_{n+m}. \tag{19}$$

- In the **holomorphic dimension zero sector**, the corresponding operator $L_0 = \oint dzzT_{zz}$ must act trivially, i.e., be $\overline{Q}_+$-exact, and from (19), we see that $m = 0$, whence $LL\mathfrak{su}(k)$ reduces to the **loop algebra** $L\mathfrak{su}(k)$:

$$[J^{an_1}_0, J^{bn_2}_0] = if^{ab}_c J^{c\{n_1+n_2\}}_0. \tag{20}$$

This is also the **topological sector**, since T_{zz} is also $\overline{Q}_+$-exact.

Appearance of Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- Our aforementioned J's were derived from a classical Lagrangian density, and there would be quantum corrections.

- This means that the aforementioned algebras ought to modified as well. Specifically, they will acquire central extensions.

- This leads us to a **toroidal lie algebra** $\mathfrak{su}(k)_{\text{tor}}$:

$$\boxed{[J^{an_1}_{m_1}, J^{bn_2}_{m_2}] = if^{ab}_c J^{c\{n_1+n_2\}}_{m_1+m_2} + c_1 n_1 \delta^{ab}\delta^{\{n_1+n_2\}0}\delta_{\{m_1+m_2\}0} + c_2 m_1 \delta^{ab}\delta^{\{n_1+n_2\}0}\delta_{\{m_1+m_2\}0}} \tag{21}$$

and **affine Lie algebra** $\mathfrak{su}(k)_{\text{aff}}$:

$$\boxed{[J^{an_1}_{0}, J^{bn_2}_{0}] = if^{ab}_c J^{c\{n_1+n_2\}}_{0} + c_1 n_1 \delta^{ab}\delta^{\{n_1+n_2\}0}} \tag{22}$$

in the $\overline{Q}_+$-cohomology (for some $c_{1,2}$).

Modules over the Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- Now, acting on a ground state $|0\rangle$ (which is $\overline{Q}_+$-closed) with the generators of $\mathfrak{su}(k)_{\mathrm{tor}}$, we have the states

$$J^{a\{-n_1\}}_{-m_1} J^{b\{-n_2\}}_{-m_2} J^{c\{-n_3\}}_{-m_3} \ldots |0\rangle, \tag{23}$$

  where $m_j, n_i \geq 0$.

- They span a module $\widehat{\widehat{su}}(k)^N_{c_1,c_2}$ over the toroidal Lie algebra $\mathfrak{su}(k)_{\mathrm{tor}}$ of levels c_1 and c_2.

- These states have nonzero holomorphic dimension (according to (19)), and can be shown to be elements of the $\overline{Q}_+$-cohomology.

- Thus, via the state-operator correspondence, we have

$$H^*_{\check{\mathrm{C}}\mathrm{ech}}\Big(\widehat{\Omega}^{ch}_{\mathcal{M}(\mathbb{C}P^1 \xrightarrow[hol.]{N} \Omega SU(k))}\Big) = \widehat{\widehat{su}}(k)^N_{c_1,c_2}. \tag{24}$$

Modules over the Toroidal and Affine $SU(k)$ Algebra in the $\overline{Q}_+$-Cohomology

- In the topological sector where $m_i = 0$, the states are

$$J_0^{a\{-n_1\}} J_0^{b\{-n_2\}} J_0^{c\{-n_3\}} \dots |0\rangle, \tag{25}$$

  where $n_i \geq 0$.

- They span a module $\widehat{su}(k)^N_{c_1}$ over the affine Lie algebra $\mathfrak{su}(k)_{\text{aff}}$ of level c_1.

- These states have zero holomorphic dimension (according to (19)), and persist as elements of the $\overline{Q}_+$-cohomology.

- Thus, via the state-operator correspondence, and the fact that the zero holomorphic dimension chiral de Rham complex is just the de Rham complex [3], we have

$$H^*_{L^2}(\mathcal{M}(\mathbb{C}P^1 \xrightarrow[hol.]{N} \Omega SU(k))) = \widehat{su}(k)^N_{c_1}. \tag{26}$$

The 6d A_{k-1} $\mathcal{N} = (2,0)$ Little String Theory

- Little string theories (LST) exist in 6d spacetimes, and reduce to interacting local QFTs when string length $l_s \to 0$.

- The 6d A_{k-1} $\mathcal{N} = (2,0)$ LST, in particular, reduces to the 6d A_{k-1} $\mathcal{N} = (2,0)$ superconformal field theory - has **no known classical action**. Rich theory, so corresponding LST must be at least just as rich.

- It is also the worldvolume theory of a stack of NS5-branes in type IIA string theory, whereby the fundamental strings which reside within the branes with coupling $g_s \to 0$ (whence bulk d.o.f., including gravity, decouple) and $l_s \not\to 0$, are the little strings.

The Ground and Left-Excited Spectrum of the 6d A_{k-1} $(2,0)$ LST

- The discrete lightcone quantization (DLCQ) of the LST on $\mathbb{R}^{5,1}$ describes it as a 2d $\mathcal{N} = (4,4)$ sigma model on $S^1 \times R$ with target $\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)$, the moduli space of $SU(k)$ N-instantons on $\mathbb{R}^4$. Here, $k =$ no. of branes, $N =$ units of discrete momentum along the S^1 [4].

- The ground states of the LST are given by sigma model states annihilated by all the supercharges, i.e., they correspond to harmonic forms and thus L^2-cohomology classes of $\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)$.

- The left-excited states of the LST are given by sigma model states annihilated by the four chiral supercharges, i.e., they correspond to Čech cohomology classes of the sheaf $\widehat{\Omega}^{ch}_{\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)}$.

The Ground and Left-Excited Spectrum of the 6d A_{k-1} $(2,0)$ LST

- According to Atiyah [1], we have the identification

$$\mathcal{M}_G^N(\mathbb{R}^4) \cong \mathcal{M}(\mathbb{C}P^1 \xrightarrow[hol.]{N} \Omega G). \tag{27}$$

- In turn, this means we can identify

$$H^*_{L^2}(\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)) \cong H^*_{L^2}(\mathcal{M}(\mathbb{C}P^1 \xrightarrow[hol.]{N} \Omega SU(k))) \tag{28}$$

and

$$H^*_{\text{Čech}}(\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)) \cong H^*_{\text{Čech}}\left(\widehat{\Omega}^{ch}_{\mathcal{M}(\mathbb{C}P^1 \xrightarrow[hol.]{N} \Omega SU(k))}\right) \tag{29}$$

The Ground and Left-Excited Spectrum of the 6d A_{k-1} $(2,0)$ LST

- Thus, from (28) and (26), we find that

$$\boxed{\text{ground spectrum of 6d } A_{k-1} \ (2,0) \text{ LST} = \text{modules of } \mathfrak{su}(k)_{\text{aff}}} \tag{30}$$

- Similarly, from (29) and (24), we find that

$$\boxed{\text{left-excited spectrum of 6d } A_{k-1} \ (2,0) \text{ LST} = \text{modules of } \mathfrak{su}(k)_{\text{tor}}} \tag{31}$$

Deriving the Braverman-Finkelberg Relation and its Generalization

- From (26) and (28), we also find (c.f. [5]) that

$$\boxed{\mathrm{IH}^*(\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)) = \widehat{su}(k)^N_{c_1},} \tag{32}$$

This is the **Braverman-Finkelberg relation** in [2].

- From (24) and (29), we also find that

$$\boxed{H^*_{\text{Čech}}(\widehat{\Omega}^{ch}_{\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)}) = \widehat{\widehat{su}}(k)^N_{c_1,c_2}.} \tag{33}$$

This is a novel **generalization of the Braverman-Finkelberg relation**.

The M5-brane Worldvolume Theory

- M-theory does not have a perturbative coupling constant, since there is no dilaton-field in 11d supergravity.

- The dual relationship between M2-branes and M5-branes must therefore be different from more standard weak-strong coupling dualities between strings and fivebranes in $d = 10$.

- It is thus reasonable to view M-theory as self-dual in the sense that the dynamics of the M2-branes ought to describe the dynamics of the M5-branes and vice-versa.

- The aforementioned point, and the fact that type IIA fundamental strings (which originate from M2-branes that wrap the 11th circle) and NS5-branes (which originate from M5-branes) are dual in $d=10$, suggest that the M5-brane worldvolume theory ought to be described by M-strings arising from M2-branes ending on it and wrapping the 11th circle.

The M5-brane Worldvolume Theory

- The setup of the k NS5-branes with type IIA fundamental strings bound to it as little strings, has an M-theoretic interpretation.

- They can be regarded as k M5-branes with M2-branes ending on them in one spatial direction (as M-strings) and wrapping the 11th circle of radius R in the other spatial direction, observed at low energy scales $<< R^{-1}$.

- As such, the low energy DLCQ of this **worldvolume theory of $\mathbf{k}$ M5-branes** can also be understood via the LST described as a $\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)$ sigma model [6].

1/2 BPS sector of M5-brane Worldvolume Theory - I

- The 6d $\mathcal{N} = (2,0)$ supersymmetry algebra of the M5-brane worldvolume theory in DLCQ is

$$\begin{aligned} \{\mathcal{Q}^{a\alpha}, \mathcal{Q}^{b\beta}\} &= \epsilon^{\alpha\beta}(\mathcal{H}\mathbf{1}^{ab} + \mathcal{P}_L^m \Gamma_m^{ab}), \\ \{\mathcal{Q}^{a\dot{\alpha}}, \mathcal{Q}^{b\dot{\beta}}\} &= \epsilon^{\dot{\alpha}\dot{\beta}}(\mathcal{H}\mathbf{1}^{ab} + \mathcal{P}_R^m \Gamma_m^{ab}), \end{aligned} \tag{34}$$

  where $a = 1, \ldots, 4$ and $\alpha(\dot{\alpha}) = 1, 2$ are Lorentz and chiral (anti-chiral) R-symmetry indices, respectively.

- The 1/2 BPS sector of the worlvolume theory consists of states which obey the four chiral relations

$$\varepsilon_{A\alpha}\mathcal{Q}^{A\alpha}|BPS\rangle = 0, \qquad A = 3, 4, \tag{35}$$

  and anti-chiral relations

$$\varepsilon_{A\dot{\alpha}}\mathcal{Q}^{A\dot{\alpha}}|BPS\rangle = 0, \qquad A = 3, 4. \tag{36}$$

- Since the M-strings that live on the M5-brane worldvolume are 1/2 BPS objects, they only preserve eight of the sixteen supersymmetries of the worldvolume theory on their worldsheet.

- The eight preserved supercharges on each M- and therefore little string worldsheet obey the supersymmetry algebra

$$\begin{aligned}\{\mathscr{Q}^{\dot{a}\alpha}, \mathscr{Q}^{\dot{b}\beta}\} &= 2\epsilon^{\dot{a}\dot{b}}\epsilon^{\alpha\beta}\mathscr{L}_0, \\ \{\overline{\mathscr{Q}}^{\dot{a}\dot{\alpha}}, \overline{\mathscr{Q}}^{\dot{b}\dot{\beta}}\} &= 2\epsilon^{\dot{a}\dot{b}}\epsilon^{\dot{\alpha}\dot{\beta}}\overline{\mathscr{L}}_0,\end{aligned} \tag{37}$$

  where $\dot{a}, \dot{b} = 1, 2$ are Lorentz indices, and $\mathscr{L}_0$ and $\overline{\mathscr{L}}_0$ are the left and right-moving parts of the Hamiltonian on the worldsheet.

1/2 BPS sector of M5-brane Worldvolume Theory - III

- Observe that the eight preserved worldsheet supercharges are evenly divided into four chiral ($\mathscr{Q}^{\dot{a}\alpha}$) and four anti-chiral ($\overline{\mathscr{Q}}^{\dot{a}\dot{\alpha}}$) under the R-symmetry of the worldvolume theory. In fact, they correspond to the worldvolume supercharges in (35) and (36), respectively, since $\dot{a}, \dot{b} = 1, 2$ is actually $A, B = 3, 4$.

- Hence, the 1/2 BPS sector, spanned by states obeying the four chiral and four anti-chiral relations (35) and (36), is also spanned by states annihilated by the four left-moving and four right-moving supercharges on the worldsheet.

- From the supersymmetry algebra (37), we find that these states are necessarily ground states annihilated by the Hamiltonian $\mathscr{H} = \mathscr{L}_0 + \overline{\mathscr{L}}_0$.

- Thus, the low energy **1/2 BPS sector** of the M5-brane theory is captured by the **ground states** of the LST.

- Therefore, according to (30), the partition function of the 1/2 BPS sector ought to be given by summing representations of $\mathfrak{su}(k)_{\mathrm{aff}}$. In particular, it is computed to be

$$\boxed{Z_{1/2} = \sum_{\widehat{\lambda}'} \chi^{\widehat{\lambda}'}_{\widehat{su}(k)_{c_1}}(p)} \tag{38}$$

  where χ is a character of the module in complex parameter p, and $\widehat{\lambda}'$ is a dominant highest weight.

- This is a cousin of a modular form which transforms as a representation of $SL(2,\mathbb{Z})$.

- There is an instrinsic $SL(2,\mathbb{Z})$ symmetry in the M5-brane worldvolume theory on $\mathbb{R}^{5,1}$!

- Emerges as gauge-theoretic S-duality of 4d $\mathcal{N}=4$ SYM after compactifying on T^2.

1/4 BPS sector of M5-brane Worldvolume Theory - I

- The 1/4 BPS sector of the worlvolume theory consists of states which obey just the four anti-chiral relations in (36)

$$\varepsilon_{A\dot{\alpha}}\mathcal{Q}^{A\dot{\alpha}}|BPS\rangle = 0, \qquad A = 3, 4. \tag{39}$$

- Hence, the 1/4 BPS sector is also spanned by states annihilated by just the four right-moving supercharges on the worldsheet.

- From the supersymmetry algebra (37), we find that these states have eigenvalues $\overline{\mathscr{L}}_0 = 0$ but $\mathscr{L}_0 \neq 0$.

- Thus, the low energy **1/4 BPS sector** of the M5-brane theory is captured by the **left-excited states** of the LST.

- Therefore, according to (31), the partition function of the 1/4 BPS sector ought to be given by summing representations of $\mathfrak{su}(k)_{\text{tor}}$. In particular, it is computed to be

$$\boxed{Z_{1/4} = q^{\frac{1}{24}} \sum_{\widehat{\lambda}} \chi^{\widehat{\lambda}}_{\widehat{su}(k)_{c_1}}(p) \frac{1}{\eta(r)}} \tag{40}$$

  where q is a complex parameter and η is the Dedekind eta function.

- This is a cousin of an automorphic form which transforms as a representation of $SO(2,2;\mathbb{Z})$.

- There is an intrinsic $SO(2,2;\mathbb{Z})$ symmetry of the M5-brane worldvolume theory on $\mathbb{R}^{5,1}$!

- Emerges as string-theoretic T-duality of little strings after compactifying on T^2.

Conclusion

- We have explained how a quasi-topological $\Omega SU(k)$ sigma model can be used to help us (i) understand the 6d A_{k-1} $(2,0)$ LST; (ii) derive and generalize the Braverman-Finkelberg relation; (iii) understand the M5-brane worldvolume theory.

- Notably, we find that the chiral spectrum of the little string is furnished by representations of a toroidal algebra, and the BPS spectra of the M5-brane worldvolume theory are closely related to modular and automorphic forms.

- Consistent with these aforementioned physical results is a geometric Langlands correspondence for surfaces – the Braverman-Finkelberg relation – and its generalization, which we also physically derived.

- We see a nice interconnection between string theory, M-theory, geometric representation theory and number theory.

Future Directions

- To ascertain the **full chiral plus anti-chiral spectrum** of the the 6d A_{k-1} $(2,0)$ LST. We expect it to be furnished by representations of a holomorphic plus antiholomorphic (positive-moded) toroidal algebra.

- Gauge the $\Omega SU(k)$ sigma model to obtain a derivation and **generalization of the AGT correspondence**, which we expect will relate the equivariant Cech-cohomology of the sheaf of chiral de Rham complex on $\mathcal{M}^N_{SU(k)}(\mathbb{R}^4)$ to toroidal W-algebras.

- Go **beyond the BPS sector** of the M5-brane worldvolume theory as captured by the full spectrum of the LST. We expect the corresponding worldvolume partition function to consist of the $1/4$ BPS partition function with an extra Dedekind eta function in $\bar{r}$.

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