Higher AGT Correspondences, $\mathcal{W}$-algebras, and Higher Quantum Geometric Langlands Duality from M-Theory

Meng-Chwan Tan

National University of Singapore

String and M-Theory: The New Geometry of the 21st Century

Presentation Outline

- Summary of Talk
- Lightning Review: A 4d AGT Correspondence for Compact Lie Groups
- A 5d/6d AGT Correspondence for Compact Lie Groups
- $\mathcal{W}$-algebras and Higher Quantum Geometric Langlands Duality
- Supersymmetric Gauge Theory, $\mathcal{W}$-algebras and a Quantum Geometric Langlands Correspondence
- Higher Geometric Langlands Correspondences from M-Theory
- Conclusion

Summary of Talk

This talk is based on:

M.-C. Tan, "Higher AGT Correspondences, $\mathcal{W}$-algebras, and Higher Quantum Geometric Langlands Duality from M-Theory", Adv. Theor. Math. Phys. 22: 429-507, 2018, [arXiv:1607.08330]

which is a culmination of the insights and work over the years in:

M.-C. Tan, "M-Theoretic Derivations of 4d-2d Dualities: From a Geometric Langlands Duality for Surfaces, to the AGT Correspondence, to Integrable Systems", JHEP **07** (2013) 171, [arXiv:1301.1977].

M.-C. Tan, "An M-Theoretic Derivation of a 5d and 6d AGT Correspondence, and Relativistic and Elliptized Integrable Systems", JHEP **12** (2013) 31, [arXiv:1309.4775].

M.-C. Tan, "Quasi-Topological Gauged Sigma Models, The Geometric Langlands Program, And Knots", Adv. Theor. Math. Phys. **19**, 277-450, 2015, [arXiv:hep-th/1111.0691].

Summary of Talk

In this talk, we will, via a pair of dual M-theory compactifications with M5-branes wrapping hyperkahler four manifolds which spacetime BPS spectra are therefore expected to be equivalent, present an M-theoretic derivation of a 5d and 6d AGT correspondence for arbitrary compact Lie groups, namely

$$\boxed{Z^{\text{pure, 5d}}_{\text{inst}, G}(\epsilon_1, \epsilon_2, \vec{a}, \beta, \Lambda) = \langle G_G | G_G \rangle} \tag{1}$$

where the *coherent state*

$$\boxed{|G_G\rangle \in \widehat{\mathcal{W}^q}({}^L\mathfrak{g}_{\text{aff}})} \tag{2}$$

and

$$\boxed{Z^{\text{lin, 6d}}_{\text{inst}, SU(N)}(q_1, \epsilon_1, \epsilon_2, \vec{m}, \beta, R_6) = \langle\, \tilde{\Phi}^{\mathbf{w}}_{\mathbf{v}}(z_1)\, \tilde{\Phi}^{\mathbf{v}}_{\mathbf{u}}(z_2)\, \rangle_{\mathbf{T}^2}} \tag{3}$$

where the vertex operators

$$\boxed{\tilde{\Phi}^{\mathbf{c}}_{\mathbf{d}} : \widehat{\mathcal{W}^{q,v}}({}^L\mathfrak{su}(N)_{\text{aff}}) \to \widehat{\mathcal{W}^{q,v}}({}^L\mathfrak{su}(N)_{\text{aff}})} \tag{4}$$

Summary of Talk

from which we can obtain identities of various $\mathcal{W}$-algebras which underlie a quantum geometric Langlands duality and its higher analogs,

$$
\begin{array}{ccc}
\boxed{\mathcal{W}_{\mathrm{aff},k}(\mathrm{g}) = \mathcal{W}_{\mathrm{aff},{}^Lk}({}^L\mathrm{g})} & \underset{\epsilon_2 \nrightarrow 0}{\overset{\epsilon_2 \to 0}{\rightleftarrows}} & \boxed{Z(U(\hat{\mathrm{g}})_{\mathrm{crit}}) = \mathcal{W}_{\mathrm{cl}}({}^L\mathrm{g})} \\
\beta \to 0 \uparrow\downarrow \beta \nrightarrow 0 & & \beta \to 0 \uparrow\downarrow \beta \nrightarrow 0 \\
\boxed{\mathcal{W}^{q,t}_{\mathrm{aff},k}(\mathfrak{g}) = \mathcal{W}^{t,q}_{\mathrm{aff},{}^Lk}({}^L\mathfrak{g})} & \underset{\epsilon_2 \nrightarrow 0}{\overset{\epsilon_2 \to 0}{\rightleftarrows}} & \boxed{Z(U_q(\hat{\mathfrak{g}})_{\mathrm{crit}}) = \mathcal{W}^{q}_{\mathrm{cl}}({}^L\mathfrak{g})} \\
R_6 \to 0 \uparrow\downarrow R_6 \nrightarrow 0 & & R_6 \to 0 \uparrow\downarrow R_6 \nrightarrow 0 \\
\boxed{\mathcal{W}^{q,t,v}_{\mathrm{aff},k}(\mathfrak{g}) = \mathcal{W}^{t,q,v}_{\mathrm{aff},{}^Lk}({}^L\mathfrak{g})} & \underset{\epsilon_2 \nrightarrow 0}{\overset{\epsilon_2 \to 0}{\rightleftarrows}} & \boxed{Z(U_{q,v}(\hat{\mathfrak{g}})_{\mathrm{crit}}) = \mathcal{W}^{q,v}_{\mathrm{cl}}({}^L\mathfrak{g})}
\end{array}
\tag{5}
$$

(g is arbitrary while $\mathfrak{g}$ is simply-laced)

Summary of Talk

whence we will be able to

(i) elucidate the sought-after connection between the 4d gauge-theoretic realization of the geometric Langlands correspondence by Kapustin-Witten [2, 3] and its algebraic 2d CFT formulation by Beilinson-Drinfeld [4],

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g))_{\mathcal{B}_{\text{`t-Hooft}}} & \overset{\text{String Duality}}{\longleftrightarrow} & \mathrm{M}^{\mathrm{flat}}_{{}^L G_{\mathbb{C}}}(\Sigma_g)_{\widehat{\mathcal{W}}({}^L\mathfrak{g})_{\text{“Wilson”}}} \\
S\text{-duality} \ \updownarrow \ \text{KW realization} & & \text{BD formulation} \ \updownarrow \ \text{FF-duality} \\
\mathrm{D}^{\mathrm{flat}}_{{}^L G_{\mathbb{C}}}(\Sigma_g)_{{}^L\mathcal{B}_{\mathrm{Wilson}}} & \overset{\text{String Duality}}{\longleftrightarrow} & \mathrm{M}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g))_{\widehat{\mathcal{W}}(\mathfrak{g})_{\text{“'t-Hooft”}}}
\end{array} \qquad (6)$$

(ii) explain what the higher 5d and 6d analogs of the geometric Langlands correspondence for simply-laced Lie (Kac-Moody) groups G ($\widehat{G}$), ought to involve, namely for 5d and G,

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\mathrm{S}^1}_{\mathrm{H.S.}}(G,\Sigma_g))\text{-module}} \Longleftrightarrow \boxed{\text{circle-valued flat } {}^LG\text{-bundle on } \Sigma_g} \tag{7}$$

for 5d and $\widehat{G}$

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\hat{\mathrm{S}}^1\times\mathrm{S}^1}_{\mathrm{H.S.}}(G,\Sigma))\text{-module}} \Longleftrightarrow \boxed{\text{circle-valued flat } \widehat{{}^LG}\text{-bundle on } \Sigma} \tag{8}$$

for 6d and G

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\mathrm{S}^1}_{\mathrm{H.S.}}(G,\Sigma_{1,2}))\text{-mod}} \Longleftrightarrow \boxed{\text{elliptic-valued flat } {}^LG\text{-bundle on } \Sigma_{1,2}} \tag{9}$$

for 6d and $\widehat{G}$

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\hat{\mathrm{S}}^1\times\mathrm{S}^1}_{\mathrm{H.S.}}(G,\Sigma_1))\text{-mod}} \Longleftrightarrow \boxed{\text{elliptic-valued flat } \widehat{{}^LG}\text{-bundle on } \Sigma_1} \tag{10}$$

(iii) demonstrate Nekrasov-Pestun-Shatashvili's recent result in [5], which relates the moduli space of 5d/6d $\mathcal{N}=1$ G ($\widehat{G}$)-quiver $SU(K_i)$ gauge theories to the representation theory of quantum/elliptic affine (toroidal) G-algebras, namely for 5d and G-quiver,

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},y_1,y_2}_{\mathrm{S^1\text{-}mono},\mathbf{k}} \Longleftrightarrow \chi_q(V_i) = T_i(z), \quad V_i \in \mathrm{Rep}\,[U_q^{\mathrm{aff}}(\mathfrak{g})_{\{\mathrm{C_x}\}_{z_1,z_2}}]} \tag{11}$$

for 5d and $\widehat{G}$-quiver

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},k}_{\hat{\mathrm{S}}^1\times\mathrm{S^1\text{-}inst}} \Longleftrightarrow \chi_q(\hat{V}_i) = \hat{T}_i(z), \quad \hat{V}_i \in \mathrm{Rep}\,[U_q^{\mathrm{aff}}(\hat{\mathfrak{g}})_{\mathrm{C_x}}]} \tag{12}$$

for 6d and G-quiver

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},y_1,y_2}_{\mathrm{S^1\text{-}mono},\mathbf{k}} \Longleftrightarrow \chi_{q,v}(V_i) = T_i(z), \quad V_i \in \mathrm{Rep}\,[U_{q,v}^{\mathrm{ell}}(\mathfrak{g})_{\{\mathrm{C_x}\}_{z_1,z_2}}]} \tag{13}$$

for 6d and $\widehat{G}$-quiver

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},k}_{\hat{\mathrm{S}}^1\times\mathrm{S^1\text{-}inst}} \Longleftrightarrow \chi_{q,v}(\hat{V}_i) = \hat{T}_i(z), \quad \hat{V}_i \in \mathrm{Rep}\,[U_{q,v}^{\mathrm{ell}}(\hat{\mathfrak{g}})_{\mathrm{C_x}}]} \tag{14}$$

Punchline?

M-THEORY KNOWS EVERYTHING, AND MORE!

Let's now explain how we can derive these results

Lightning Review: An M-Theoretic Derivation of the 4d Pure AGT Correspondence for Compact Groups

Via a chain of string dualities in a background of fluxbranes as introduced in [6, 7], we have the dual M-theory compactifications

$$\underbrace{\mathbb{R}^4|_{\epsilon_1,\epsilon_2} \times \Sigma_{n,t}}_{N \text{ M5-branes}} \times \mathbb{R}^5|_{\epsilon_3;\, x_{6,7}} \quad \Longleftrightarrow \quad \mathbb{R}^5|_{\epsilon_3;\, x_{4,5}} \times \underbrace{\mathcal{C} \times TN_N^{R\to 0}|_{\epsilon_3;\, x_{6,7}}}_{1 \text{ M5-branes}}, \qquad (15)$$

where $n = 1$ or 2 for $G = SU(N)$ or $SO(N+1)$ (N even), and

$$\underbrace{\mathbb{R}^4|_{\epsilon_1,\epsilon_2} \times \Sigma_{n,t}}_{N \text{ M5-branes + OM5-plane}} \times \mathbb{R}^5|_{\epsilon_3;\, x_{6,7}} \Longleftrightarrow \mathbb{R}^5|_{\epsilon_3;\, x_{4,5}} \times \underbrace{\mathcal{C} \times SN_N^{R\to 0}|_{\epsilon_3;\, x_{6,7}}}_{1 \text{ M5-branes}}, \qquad (16)$$

where $n = 1$, 2 or 3 for $G = SO(2N)$, $USp(2N-2)$ or G_2 (with $N = 4$).

Here, $\epsilon_3 = \epsilon_1 + \epsilon_2$, the surface $\mathcal{C}$ has the same topology as $\Sigma_{n,t} = \mathbf{S}^1_n \times \mathbb{I}_t$, and we have an M9-brane at each tip of $\mathbb{I}_t$. The radius of $\mathbf{S}^1_n$ is given by β, which is much larger than $\mathbb{I}_t$.

Lightning Review: An M-Theoretic Derivation of the 4d Pure AGT Correspondence for Compact Groups

The relevant spacetime (quarter) BPS states on the LHS of (15) and (16) are captured by a gauged sigma model on instanton moduli space, and are spanned by

$$\bigoplus_m \mathrm{IH}^*_{U(1)^2 \times T} \mathcal{U}(\mathcal{M}_{G,m}), \tag{17}$$

while those on the RHS of (15) and (16) are captured by a gauged chiral WZW model on the I-brane $\mathcal{C}$ in the equivalent IIA frame, and are spanned by

$$\widehat{\mathcal{W}}({}^L\mathfrak{g}_{\mathrm{aff}}). \tag{18}$$

The physical duality of the compactifications in (15) and (16) will mean that (17) is equivalent to (18), i.e.

$$\boxed{\bigoplus_m \mathrm{IH}^*_{U(1)^2 \times T} \mathcal{U}(\mathcal{M}_{G,m}) = \widehat{\mathcal{W}}({}^L\mathfrak{g}_{\mathrm{aff}})} \tag{19}$$

The 4d Nekrasov instanton partition function is given by

$$Z_{\text{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}) = \sum_m \Lambda^{2mh^\vee_{\mathfrak{g}}} Z_{\text{BPS},m}(\epsilon_1, \epsilon_2, \vec{a}, \beta \to 0), \tag{20}$$

where Λ can be interpreted as the inverse of the observed scale of the $\mathbb{R}^4|_{\epsilon_1,\epsilon_2}$ space on the LHS of (15), and $Z_{\text{BPS},m}$ is a 5d worldvolume index.

Thus, since $Z_{\text{BPS},m}$ is a weighted count of the states in $\mathcal{H}^{\Omega}_{\text{BPS},m} = \text{IH}^*_{U(1)^2\times T}\,\mathcal{U}(\mathcal{M}_{G,m})$, it would mean from (20) that

$$Z_{\text{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}) = \langle \Psi | \Psi \rangle, \tag{21}$$

where $|\Psi\rangle = \bigoplus_m \Lambda^{mh^\vee_{\mathfrak{g}}} |\Psi_m\rangle \in \bigoplus_m \text{IH}^*_{U(1)^2\times T}\,\mathcal{U}(\mathcal{M}_{G,m})$.

In turn, the duality (19) and the consequential observation that $|\Psi\rangle$ is a sum over 2d states of all energy levels m, mean that

$$|\Psi\rangle = |q, \Delta\rangle, \tag{22}$$

where $|q, \Delta\rangle \in \widehat{\mathcal{W}}({}^L\mathfrak{g}_{\mathrm{aff}})$ is a *coherent state*, and from (21),

$$\boxed{Z_{\mathrm{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a}) = \langle q, \Delta | q, \Delta\rangle} \tag{23}$$

Since the LHS of (23) is defined in the $\beta \to 0$ limit of the LHS of (15), $|q, \Delta\rangle$ and $\langle q, \Delta|$ ought to be a state and its dual associated with the puncture at $z = 0, \infty$ on $\mathcal{C}$, respectively (as these are the points where the $\mathbf{S}^1_n$ fiber has zero radius). This is depicted in fig. 1 and 2.

Incidentally, Σ_{SW} in fig. 1 and 2 can also be interpreted as the Seiberg-Witten curve which underlies $Z_{\mathrm{inst}}(\Lambda, \epsilon_1, \epsilon_2, \vec{a})$!

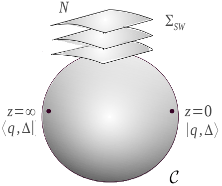


Figure 1: Σ_{SW} as an N-fold cover of $\mathcal{C}$

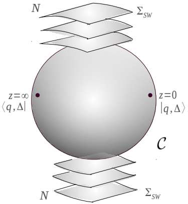


Figure 2: Σ_{SW} as a $2N$-fold cover of $\mathcal{C}$

Let us now extend our derivation of the pure AGT correspondence to include matter.

For illustration, we shall restrict ourselves to the A-type superconformal quiver gauge theories described by Gaiotto in [8].

To this end, first note that our derivation of the pure 4d AGT correspondence is depicted in fig. 3.

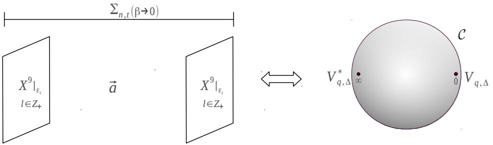
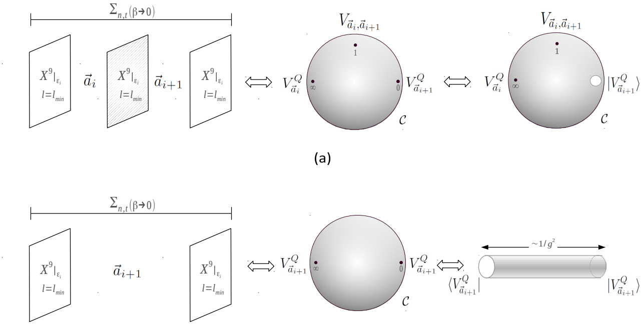


Figure 3: A pair of M9-branes in the original compactification in the limit $\beta \to 0$ and the corresponding CFT on $\mathcal{C}$ in the dual compactification in our derivation of the 4d pure AGT correspondence.

Lightning Review: An M-Theoretic Derivation of the 4d AGT Correspondence with Matter

This suggests that we can use the following building blocks in fig. 4 for our derivation of the 4d AGT correspondence with matter.



(a)

(b)

Figure 4: Building blocks with "minimal" M9-branes for our derivation of the AGT correspondence with matter

Consider a conformal necklace quiver of n $SU(N)$, $N \geq 2$.

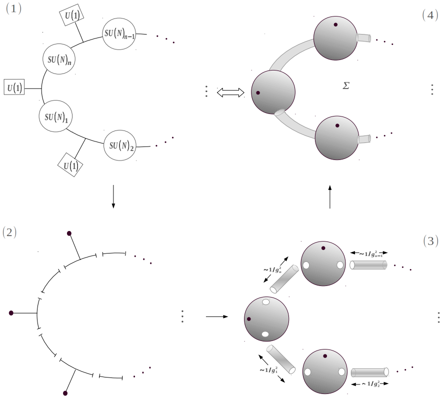


Figure 5: The necklace quiver diagram and the various steps that lead us to the overall Riemann surface Σ on which our 2d CFT lives.

Lightning Review: An M-Theoretic Derivation of the 4d AGT Correspondence with Matter

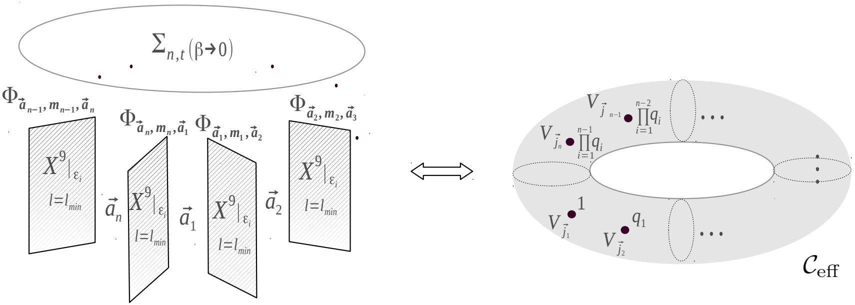


Figure 6: The effective 4d-2d correspondence

In the case of a necklace quiver of n $SU(N)$ gauge groups,

$$Z_{\text{inst}}^{\text{neck}} \sim \left\langle V_{\vec{j}_1}(1) V_{\vec{j}_2}(q_1) \ldots V_{\vec{j}_n}(q_1 q_2 \ldots q_{n-1}) \right\rangle_{\mathbf{T}^2} \tag{24}$$

where $V_{\vec{j}_i}(z)$ is a primary vertex operator of the Verma module $\widehat{\mathcal{W}}({}^L\mathfrak{su}(N)_{\text{aff}})$ with highest weight

$$\vec{j}_s = \frac{-i\vec{m}_{s-1}}{\sqrt{\epsilon_1 \epsilon_2}} \quad \text{for} \quad s = 1, 2, \ldots, n \tag{25}$$

and conformal dimension

$$u_s^{(2)} = \frac{\vec{j}_s^2}{2} - \frac{\vec{j}_s \cdot i\vec{\rho}(\epsilon_1 + \epsilon_2)}{\sqrt{\epsilon_1 \epsilon_2}}, \quad \text{where} \quad s = 1, 2, \ldots, n \tag{26}$$

A pure $U(1)$ theory can also be interpreted as the $m \to \infty$, $e^{2\pi i \tau'} \to 0$ limit of a $U(1)$ theory with an adjoint hypermultiplet matter of mass m and complexified gauge coupling τ', where $me^{2\pi i \tau'} = \Lambda$ remains fixed. This means from fig. 5 (with $n = 1$) that the 5d Nekrasov instanton partition function for pure $U(1)$ can be expressed as

$$Z^{\text{pure}, 5d}_{\text{inst}, U(1)}(\epsilon_1, \epsilon_2, \beta, \Lambda) = \langle \emptyset | \Phi_{m \to \infty}(1) | \emptyset \rangle_{\mathbf{S}^2}, \tag{27}$$

where $\Phi_{m\to\infty}(1)$ is the 5d analog of the 4d primary vertex operator $V_{\vec{j}_1}$ in fig. 6 in the $m \to \infty$ limit.

In the 5d case where $\beta \nrightarrow 0$, states on $\mathcal{C}$ are no longer localized to a point but are projected onto a circle of radius β. This results in the contribution of higher excited states which were decoupled in the 2d CFT of chiral fermions when the states were defined at a point. Consequently, we can compute that

$$Z^{\mathrm{pure},\,5d}_{\mathrm{inst},\,U(1)}(\epsilon_1, \epsilon_2, \beta, \Lambda) = \langle G_{U(1)} | G_{U(1)} \rangle \tag{28}$$

with

$$|G_{U(1)}\rangle = \exp\left(- \sum_{n>0} \frac{1}{n} \frac{(\beta\Lambda)^n}{1 - t^n}\, a_{-n} \right) |\emptyset\rangle, \tag{29}$$

where the deformed Heisenberg algebra

$$[a_p, a_n] = p \frac{1 - t^{|p|}}{1 - q^{|p|}} \delta_{p+n,0}, \quad a_{p>0} |\emptyset\rangle = 0 \tag{30}$$

and

$$t = e^{-i\beta\sqrt{\epsilon_1 \epsilon_2}}, \quad q = e^{-i\beta(\epsilon_1 + \epsilon_2 + \sqrt{\epsilon_1\epsilon_2})}. \tag{31}$$

According to fig. 1, the 2d CFT in the $SU(N)$ case is just an N-tensor product of the 2d CFT in the $U(1)$ case. In other words, we have

$$\boxed{Z^{\text{pure}, 5d}_{\text{inst}, SU(N)}(\epsilon_1, \epsilon_2, \vec{a}, \beta, \Lambda) = \langle G_{SU(N)} | G_{SU(N)} \rangle} \tag{32}$$

$$|G_{SU(N)}\rangle = \left(\otimes_{i=1}^{n} e^{-\sum_{n_i>0} \frac{1}{n_i} \frac{(\beta\Lambda)^{n_i}}{1-t^{n_i}} a_{-n_i}}\right) \cdot \ \left(\otimes_{i=1}^{n} |\emptyset\rangle_i\right) \tag{33}$$

where

$$[a_{m_k}, a_{n_k}] = m_k \frac{1-t^{|m_k|}}{1-q^{|m_k|}} \delta_{m_k+n_k,0}, \quad a_{m_k>0}|\emptyset\rangle_k = 0 \tag{34}$$

and

$$t = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}, \quad q = e^{-i\beta(\epsilon_1+\epsilon_2+\sqrt{\epsilon_1\epsilon_2})} \tag{35}$$

Note that (33)–(35) means that $|G_{SU(N)}\rangle$ is a *coherent state* state in a level N module of a Ding-Iohara algebra [9], which, according to [10], means that

$$\boxed{|G_{SU(N)}\rangle \in \widehat{\mathcal{W}^q}({}^L\mathfrak{su}(N)_{\mathrm{aff}})} \tag{36}$$

is a *coherent state* in the Verma module of $\mathcal{W}^q({}^L\mathfrak{su}(N)_{\mathrm{aff}})$, the q-deformed affine $\mathcal{W}$-algebra associated with ${}^L\mathfrak{su}(N)_{\mathrm{aff}}$.

The relations (32) and (36) define a 5d pure AGT correspondence for the A_{N-1} groups.

Therefore, according to [11, 12, 13], with regard to the 2d CFT's on the RHS of (23) and (32), we have the diagram

$$
\begin{array}{ccc}
\underbrace{\widehat{\mathbf{Y}}(\mathfrak{gl}(1)_{\text{aff},1}) \otimes \cdots \otimes \widehat{\mathbf{Y}}(\mathfrak{gl}(1)_{\text{aff},1})}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathcal{W}}(\mathfrak{su}(N)_{\text{aff},k}) \\
\beta \to 0 \Big\uparrow \Big\downarrow \beta \nrightarrow 0 & & \beta \to 0 \Big\uparrow \Big\downarrow \beta \nrightarrow 0 \\
\underbrace{\widehat{\mathbf{U}_q}(\mathbf{L}\mathfrak{gl}(1)_{\text{aff},1}) \otimes \cdots \otimes \widehat{\mathbf{U}_q}(\mathbf{L}\mathfrak{gl}(1)_{\text{aff},1})}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathcal{W}^q}(\mathfrak{su}(N)_{\text{aff},k})
\end{array}
\qquad (37)
$$

where $\widehat{\mathbf{Y}}(\mathfrak{gl}(1)_{\text{aff},1})$ and $\widehat{\mathbf{U}_q}(\mathbf{L}\mathfrak{gl}(1)_{\text{aff},1})$ are level one modules of the Yangian and quantum toroidal algebras, respectively, and the level $k(N, \epsilon_{1,2})$.

Let $\epsilon_3 = 0$, i.e. turn off Omega-deformation on 2d side. This ungauges the chiral WZW model on $\mathcal{C}$. Then, conformal invariance, and the remarks above (28), mean that we have the following diagram

$$\begin{array}{ccc}
\underbrace{\widehat{\mathfrak{gl}(1)}_{\text{aff},1} \otimes \cdots \otimes \widehat{\mathfrak{gl}(1)}_{\text{aff},1}}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathfrak{su}(N)}_{\text{aff},1} \\
\beta \to 0 \Big\uparrow\Big\downarrow \beta \nrightarrow 0 & & \beta \to 0 \Big\uparrow\Big\downarrow \beta \nrightarrow 0 \\
\underbrace{\widehat{\mathbf{L}\mathfrak{gl}(1)}_{\text{aff},1} \otimes \cdots \otimes \widehat{\mathbf{L}\mathfrak{gl}(1)}_{\text{aff},1})}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathbf{L}\mathfrak{su}(N)}_{\text{aff},1}
\end{array} \tag{38}$$

From diagrams (37) and (38), turning on Omega-deformation on the 2d side effects (i) $\mathfrak{gl}(1)_{\text{aff},1} \to \mathbf{Y}(\mathfrak{gl}(1)_{\text{aff},1})$ and $\mathfrak{su}(N)_{\text{aff},1} \to \mathcal{W}(\mathfrak{su}(N)_{\text{aff},k})$; (ii) $\mathbf{L}\mathfrak{gl}(1)_{\text{aff},1} \to \mathbf{U}_q(\mathbf{L}\mathfrak{gl}(1)_{\text{aff},1})$ and $\mathbf{L}\mathfrak{su}(N)_{\text{aff},1} \to \mathcal{W}^q(\mathfrak{su}(N)_{\text{aff},k})$.

Now, with $\epsilon_3 = 0$ still, let $n = 2$ in (15), i.e. $G = SO(N+1)$. Then, we have, on the 2d side, the following diagram

$$\begin{array}{ccc}
\underbrace{\widehat{\mathfrak{gl}(1)}^{(2)}_{\text{aff},1} \otimes \cdots \otimes \widehat{\mathfrak{gl}(1)}^{(2)}_{\text{aff},1}}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathfrak{su}(N)}^{(2)}_{\text{aff},1} \\
\beta \to 0 \Big\uparrow\Big\downarrow \beta \nrightarrow 0 & & \beta \to 0 \Big\uparrow\Big\downarrow \beta \nrightarrow 0 \\
\underbrace{\widehat{\mathbf{L}\mathfrak{gl}(1)}^{(2)}_{\text{aff},1} \otimes \cdots \otimes \widehat{\mathbf{L}\mathfrak{gl}(1)}^{(2)}_{\text{aff},1})}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathbf{L}\mathfrak{su}(N)}^{(2)}_{\text{aff},1}
\end{array} \tag{39}$$

Now, turn on Omega-deformation on 2d side, i.e. $\epsilon_3 \neq 0$. According to the remarks below (38), we have

$$\begin{array}{ccc}
\underbrace{\widehat{\mathbf{Y}}(\mathfrak{gl}(1)^{(2)}_{\text{aff},1}) \otimes \cdots \otimes \widehat{\mathbf{Y}}(\mathfrak{gl}(1)^{(2)}_{\text{aff},1})}_{N \text{ times}} & \longleftrightarrow & \widehat{\mathcal{W}}(\mathfrak{su}(N)^{(2)}_{\text{aff},k}) \\
\beta \to 0 \Big\uparrow \Big\downarrow \beta \nrightarrow 0 & & \beta \to 0 \Big\uparrow \Big\downarrow \beta \nrightarrow 0 \\
\underbrace{\widehat{\mathbf{U}}_q(\mathbf{L}\mathfrak{gl}(1)^{(2)}_{\text{aff},1}) \otimes \cdots \otimes \widehat{\mathbf{U}}_q(\mathbf{L}\mathfrak{gl}(1)^{(2)}_{\text{aff},1})}_{N \text{ times}} & \longleftrightarrow & \widehat{\mathcal{W}^q}(\mathfrak{su}(N)^{(2)}_{\text{aff},k})
\end{array} \tag{40}$$

Comparing the bottom right-hand corner of (40) with the bottom right-hand corner of (37) for the A groups, and bearing in mind the isomorphism $\mathfrak{su}(N)^{(2)}_{\mathrm{aff}} \cong {}^L\mathfrak{so}(N+1)_{\mathrm{aff}}$, it would mean that we ought to have

$$\boxed{Z^{\mathrm{pure,\,5d}}_{\mathrm{inst},\,SO(N+1)}(\epsilon_1, \epsilon_2, \vec{a}, \beta, \Lambda) = \langle G_{SO(N+1)} | G_{SO(N+1)} \rangle} \tag{41}$$

where the *coherent state*

$$\boxed{|G_{SO(N+1)}\rangle \in \widehat{\mathcal{W}^q}({}^L\mathfrak{so}(N+1)_{\mathrm{aff}})} \tag{42}$$

The relations (41) and (42) define a 5d pure AGT correspondence for the $B_{N/2}$ groups.

Now, with $\epsilon_3 = 0$, let $n = 1$, 2 or 3 in (16), i.e. $G = SO(2N)$, $USp(2N-2)$ or G_2 (with $N = 4$). This ungauges the chiral WZW model on $\mathcal{C}$. Then, conformal invariance, and the remarks above (28), mean that we have, on the 2d side, the following diagram

$$
\begin{array}{ccc}
\underbrace{\widehat{\mathfrak{so}(2)}^{(n)}_{\mathrm{aff},1} \otimes \cdots \otimes \widehat{\mathfrak{so}(2)}^{(n)}_{\mathrm{aff},1}}_{N\,\mathrm{times}} & \longleftrightarrow & \widehat{\mathfrak{so}(2N)}^{(n)}_{\mathrm{aff},1} \\
\beta \to 0 \;\Big\uparrow\Big\downarrow\; \beta \nrightarrow 0 & & \beta \to 0 \;\Big\uparrow\Big\downarrow\; \beta \nrightarrow 0 \\
\underbrace{\widehat{\mathbf{L}\mathfrak{so}(2)}^{(n)}_{\mathrm{aff},1} \otimes \cdots \otimes \widehat{\mathbf{L}\mathfrak{so}(2)}^{(n)}_{\mathrm{aff},1})}_{N\,\mathrm{times}} & \longleftrightarrow & \widehat{\mathbf{L}\mathfrak{so}(2N)}^{(n)}_{\mathrm{aff},1}
\end{array}
\tag{43}
$$

Now, turn on Omega-deformation on 2d side, i.e. $\epsilon_3 \neq 0$. According to the remarks below (38), we have

$$
\begin{array}{ccc}
\underbrace{\widehat{\mathbf{Y}}(\mathfrak{so}(2)^{(n)}_{\text{aff},1}) \otimes \cdots \otimes \widehat{\mathbf{Y}}(\mathfrak{so}(2)^{(n)}_{\text{aff},1})}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathcal{W}}(\mathfrak{so}(2N)^{(n)}_{\text{aff},k'}) \\
\beta \to 0 \Big\uparrow \Big\downarrow \beta \nrightarrow 0 & & \beta \to 0 \Big\uparrow \Big\downarrow \beta \nrightarrow 0 \\
\underbrace{\widehat{\mathbf{U}}_q(\mathbf{L}\mathfrak{so}(2)^{(n)}_{\text{aff},1}) \otimes \cdots \otimes \widehat{\mathbf{U}}_q(\mathbf{L}\mathfrak{so}(2)^{(n)}_{\text{aff},1})}_{N\,\text{times}} & \longleftrightarrow & \widehat{\mathcal{W}^q}(\mathfrak{so}(2N)^{(n)}_{\text{aff},k'})
\end{array}
\tag{44}
$$

Comparing the bottom right-hand corner of (44) with the bottom right-hand corner of (37) for the A groups, and bearing in mind the isomorphism $\mathfrak{so}(2N)^{(n)}_{\textrm{aff}} \cong {}^L\mathfrak{g}_{\textrm{aff}}$, it would mean that we ought to have

$$\boxed{Z^{\textrm{pure, 5d}}_{\textrm{inst}, G}(\epsilon_1, \epsilon_2, \vec{a}, \beta, \Lambda) = \langle G_G | G_G \rangle} \tag{45}$$

where the *coherent state*

$$\boxed{|G_G\rangle \in \widehat{\mathcal{W}^q}({}^L\mathfrak{g}_{\textrm{aff}})} \tag{46}$$

The relations (45) and (46) define a 5d pure AGT correspondence for the C_{N-1}, D_N and G_2 groups.

By starting with M-theory on K3 with $G = E_{6,7,8}$ and F_4 singularity and its string-dual type IIB on the same K3 (in the presence of fluxbranes), one can, from the principle that the relevant BPS states in both frames ought to be equivalent, obtain, in the limit $\epsilon_1 = h = -\epsilon_2$, the relation

$$\mathrm{IH}^*_{U(1)_h \times U(1)_{-h} \times T}(\mathcal{M}^G_{\mathbf{R}^4}) = \widehat{{}^L\mathfrak{g}}_{\mathrm{aff},1}, \tag{47}$$

Then, repeating the arguments that took us from (21) to (23), we have

$$Z^{\mathrm{pure,\,4d}}_{\mathrm{inst},\,G}(h, \vec{a}, \Lambda) = \langle \mathrm{coh}_h | \mathrm{coh}_h \rangle. \tag{48}$$

In turn, according to the remarks above (28), we find that

$$\boxed{Z_{\text{inst}, G}^{\text{pure, 5d}}(h, \vec{a}, \Lambda) = \langle \text{cir}_h | \text{cir}_h \rangle} \quad (49)$$

where

$$\boxed{|\text{cir}_h\rangle \in \widehat{\mathbf{L}^L\mathfrak{g}_{\text{aff},1}}} \quad (50)$$

and $\mathbf{L}^L\mathfrak{g}_{\text{aff},1}$ is a Langlands dual toroidal Lie algebra given by the loop algebra of ${}^L\mathfrak{g}_{\text{aff},1}$.

Together, (49) and (50) define a 5d pure AGT correspondence for the $E_{6,7,8}$ and F_4 groups in the topological string limit. They are consistent with (39) and (43).

The analysis for $\epsilon_3 \neq 0$ is more intricate via this approach. Left for future work.

Consider the non-anomalous case of a conformal linear quiver $SU(N)$ theory in 6d. As explained in [14, §5.1], we have

$$\boxed{Z^{\text{lin, 6d}}_{\text{inst}, SU(N)}(q_1, \epsilon_1, \epsilon_2, \vec{m}, \beta, R_6) = \langle\, \tilde{\Phi}^{\mathbf{w}}_{\mathbf{v}}(z_1)\, \tilde{\Phi}^{\mathbf{v}}_{\mathbf{u}}(z_2)\, \rangle_{\mathbf{T}^2}} \tag{51}$$

where β and R_6 are the radii of $\mathbf{S}^1$ and $\mathbf{S}^1_t$ in $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1_t$, $\beta \gg R_6$, and the 6d vertex operators $\tilde{\Phi}(z)$ have a projection onto two transverse circles C_β and C_{R_6} in $\mathbf{T}^2$ of radius β and R_6, respectively, which intersect at the point z. Here, $\mathbf{w}, \mathbf{v}, \mathbf{u}$ are related to the matter masses.

In the same way that we arrived at (32) and (33), we have

$$\tilde{\Phi}^{\mathbf{c}}_{\mathbf{d}} : \tilde{\mathcal{F}}_{d_1} \otimes \tilde{\mathcal{F}}_{d_2} \otimes \cdots \otimes \tilde{\mathcal{F}}_{d_N} \longrightarrow \tilde{\mathcal{F}}_{c_1} \otimes \tilde{\mathcal{F}}_{c_2} \otimes \cdots \otimes \tilde{\mathcal{F}}_{c_N}, \tag{52}$$

where $\tilde{\mathcal{F}}_{c,d}$ is a module over the elliptic Ding-Iohara algebra [15] defined by

$$\boxed{[\tilde{a}_m, \tilde{a}_n] = m(1 - v^{|m|})\frac{1 - t^{|m|}}{1 - q^{|m|}}\delta_{m+n,0}, \quad \tilde{a}_{m>0}|\tilde{\emptyset}\rangle = 0} \tag{53}$$

$$\boxed{[\tilde{b}_m, \tilde{b}_n] = \frac{m(1 - v^{|m|})}{(tq^{-1}v)^{|m|}}\,\frac{1 - t^{|m|}}{1 - q^{|m|}}\delta_{m+n,0}, \qquad \tilde{b}_{m>0}|\tilde{\emptyset}\rangle = 0} \tag{54}$$

where $[\tilde{a}_m, \tilde{b}_n] = 0$, and

$$\boxed{t = e^{-i\beta\sqrt{\epsilon_1\epsilon_2}}, \quad q = e^{-i\beta(\epsilon_1+\epsilon_2+\sqrt{\epsilon_1\epsilon_2})}, \quad v = e^{-\frac{1}{R_6}}} \tag{55}$$

In other words,

$$\boxed{\tilde{\Phi}^{\mathbf{c}}_{\mathbf{d}} : \widehat{\mathcal{W}_{q,v}}({}^L\mathfrak{su}(N)_{\text{aff}}) \to \widehat{\mathcal{W}_{q,v}}({}^L\mathfrak{su}(N)_{\text{aff}})} \tag{56}$$

where $\widehat{\mathcal{W}_{q,v}}$ is a Verma module over $\mathcal{W}_{q,v}({}^L\mathfrak{su}(N)_{\text{aff}})$, an elliptic affine $\mathcal{W}({}^L\mathfrak{su}(N)_{\text{aff}})$-algebra.

To derive $\mathcal{W}$-algebra identities which underlie a Langlands duality, let us specialize our discussion of the 4d AGT correspondence to the $\mathcal{N} = 4$ or massless $\mathcal{N} = 2^*$ case, so that we can utilize S-duality. From fig. 6 (and its straightforward generalization to include an OM5-plane), we have the dual compactifications

$$\underbrace{\mathbf{R}^4|_{\epsilon_1,\epsilon_2} \times \mathbf{T}^2_\sigma}_{N \text{ M5-branes}} \times \mathbf{R}^5|_{\epsilon_3;\, x_{6,7}} \Longleftrightarrow \mathbf{R}^5|_{\epsilon_3;\, x_{4,5}} \times \underbrace{\mathbf{T}^2_\sigma \times TN_N^{R\to 0}|_{\epsilon_3;\, x_{6,7}}}_{1 \text{ M5-branes}}, \tag{57}$$

and

$$\underbrace{\mathbf{R}^4|_{\epsilon_1,\epsilon_2} \times \mathbf{T}^2_\sigma}_{N \text{ M5-branes} + \text{OM5-plane}} \times \mathbf{R}^5|_{\epsilon_3;\, x_{6,7}} \Longleftrightarrow \mathbf{R}^5|_{\epsilon_3;\, x_{4,5}} \times \underbrace{\mathbf{T}^2_\sigma \times SN_N^{R\to 0}|_{\epsilon_3;\, x_{6,7}}}_{1 \text{ M5-branes}}, \tag{58}$$

where $\mathbf{T}^2_\sigma = \mathbf{S}^1_t \times \mathbf{S}^1_n$.

Recall from earlier that

$$\bigoplus_m \mathrm{IH}^*_{U(1)^2\times T}\,\mathcal{U}(\mathcal{M}_{G,m}) = \widehat{\mathcal{W}_{\mathrm{aff},{}^L\kappa}}({}^L\mathfrak{g}), \qquad {}^L\kappa + {}^Lh_{\mathfrak{g}} = -\frac{\epsilon_2}{\epsilon_1}. \tag{59}$$

Let $n = 1$. From the symmetry of $\epsilon_1 \leftrightarrow \epsilon_2$ in (57) and (58), and ${}^L\mathfrak{g}_{\mathrm{aff}} \cong \mathfrak{g}_{\mathrm{aff}}$ for simply-laced case, we have, from the RHS of (59),

$$\boxed{\mathcal{W}_{\mathrm{aff},k}(\mathfrak{g}) = \mathcal{W}_{\mathrm{aff},{}^Lk}({}^L\mathfrak{g}), \quad \text{where} \quad r^\vee(k+h^\vee) = ({}^Lk + {}^Lh^\vee)^{-1}} \tag{60}$$

$r^\vee = n$ is the lacing number, and $\mathfrak{g} = \mathfrak{su}(N)$ or $\mathfrak{so}(2N)$.

Let $n = 2$ or 3. Effect a modular transformation $\tau \to -1/r^\vee \tau$ of $\mathbf{T}^2_\sigma$ in (57) and (58) which effects an S-duality in the 4d gauge theory along the directions ortohgonal to it. As the LHS of (59) is derived from a topological sigma model on $\mathbf{T}^2_\sigma$ that is hence invariant under this transformation, it would mean from (59) that

$$\mathcal{W}_{\mathrm{aff},k}(\mathrm{g}) = \mathcal{W}_{\mathrm{aff},{}^L k}({}^L\mathrm{g}), \quad \text{where} \quad r^\vee(k+h) = ({}^L k + {}^L h)^{-1}; \qquad (61)$$

$h = h(\mathrm{g})$ and ${}^L h = h({}^L\mathrm{g})$ are Coxeter numbers; and

$\mathrm{g} = {}^L\mathfrak{so}(2M+1)$, ${}^L\mathfrak{usp}(2M)$ or ${}^L\mathfrak{g}_2$.

In order to obtain an identity for $\mathrm{g} = \mathfrak{g}$, i.e. the Langlands dual of (61), one must exchange the roots and coroots of the Lie algebra underlying (61). This also means that h must be replaced by its dual $h^\vee$. In other words, from (61), one also has

$$\boxed{\mathcal{W}_{\mathrm{aff},k}(\mathfrak{g}) = \mathcal{W}_{\mathrm{aff},{}^Lk}({}^L\mathfrak{g}), \quad \text{where} \quad r^\vee(k + h^\vee) = ({}^Lk + {}^Lh^\vee)^{-1}} \tag{62}$$

and $\mathfrak{g} = \mathfrak{so}(2M+1)$, $\mathfrak{usp}(2M)$ or $\mathfrak{g}_2$.

Clearly, (60) and (62), define a quantum geometric Langlands duality for G as first formulated by Feigin-Frenkel [16].

From the relations (34) and (35), it would mean that we can write the algebra on the RHS of (36) as a two-parameter algebra

$$\mathcal{W}^{q,t}_{\mathrm{aff},k}(\mathfrak{su}(N)). \tag{63}$$

Note that as $\mathfrak{so}(2)_{\mathrm{aff},1}$ in diagram (44) is also a Heisenberg algebra like $\mathfrak{gl}(1)_{\mathrm{aff},1}$, it would mean that $\mathbf{U}_q(\mathbf{L}\mathfrak{so}(2)_{\mathrm{aff},1})$ therein is also a Ding-Iohara algebra at level 1 (with an extra reality condition) that can be defined by the relations (34) and (35). Hence, we also have a two-parameter algebra

$$\mathcal{W}^{q,t}_{\mathrm{aff},k}(\mathfrak{so}(2N)). \tag{64}$$

Note that the change $(\epsilon_1, \epsilon_2) \to (-\epsilon_2, -\epsilon_1)$ is a symmetry of our physical setup, and if we let $p = q/t = e^{-i\beta(\epsilon_1+\epsilon_2)}$, then, the change $p \to p^{-1}$ which implies $q \leftrightarrow t$, is also a symmetry of our physical setup. Then, the last two paragraphs together with $k + h^\vee = -\epsilon_2/\epsilon_1$ mean that

$$\boxed{\mathcal{W}^{q,t}_{\text{aff},k}(\mathfrak{g}) = \mathcal{W}^{t,q}_{\text{aff},{}^Lk}({}^L\mathfrak{g}), \quad \text{where} \quad r^\vee(k + h^\vee) = ({}^Lk + {}^Lh^\vee)^{-1}} \tag{65}$$

and $\mathfrak{g} = \mathfrak{su}(N)$ or $\mathfrak{so}(2N)$.

Identity (65) is just Frenkel-Reshetikhin's result in [17, §4.1] which defines a quantum q-geometric Langlands duality for the simply-laced groups!

The nonsimply-laced case requires a modular transformation of $\mathbf{T}^2_\sigma$ which effects the swop $\mathbf{S}^1_n \leftrightarrow \mathbf{S}^1_t$, where in 5d, $\mathbf{S}^1_n$ is a *preferred* circle as states are projected onto it. So, (65) doesn't hold, consistent with Frenkel-Reshetikhin's result.

Similarly, from (53) and (54), we can express $\mathcal{W}^{q,v}(\mathfrak{su}(N)_{\mathrm{aff},k})$ on the RHS of (56) as a three-parameter algebra

$$\mathcal{W}^{q,t,v}_{\mathrm{aff},k}(\mathfrak{su}(N)). \tag{66}$$

Repeating our arguments, we have

$$\boxed{\mathcal{W}^{q,t,v}_{\mathrm{aff},k}(\mathfrak{g}) = \mathcal{W}^{t,q,v}_{\mathrm{aff},{}^Lk}({}^L\mathfrak{g}), \quad \text{where} \quad r^\vee(k + h^\vee) = ({}^Lk + {}^Lh^\vee)^{-1}} \tag{67}$$

and $\mathfrak{g} = \mathfrak{su}(N)$ or $\mathfrak{so}(2N)$.

Clearly, identity (67) defines a quantum q, v-geometric Langlands duality for the simply-laced groups!

The nonsimply-laced case should reduce to that for the 5d one, but since the latter does not exist, neither will the former.

Summary: M-Theoretic Realization of $\mathcal{W}$-algebras and Higher Geometric Langlands Duality

In summary, by considering various limits, we have

$$
\begin{array}{ccc}
\boxed{\mathcal{W}_{\text{aff},k}(\mathrm{g}) = \mathcal{W}_{\text{aff},{}^Lk}({}^L\mathrm{g})} & \underset{\epsilon_2 \nrightarrow 0}{\overset{\epsilon_2 \to 0}{\rightleftarrows}} & \boxed{Z(U(\hat{\mathrm{g}})_{\text{crit}}) = \mathcal{W}_{\text{cl}}({}^L\mathrm{g})} \\
\beta \to 0 \uparrow\downarrow \beta \nrightarrow 0 & & \beta \to 0 \uparrow\downarrow \beta \nrightarrow 0 \\
\boxed{\mathcal{W}^{q,t}_{\text{aff},k}(\mathfrak{g}) = \mathcal{W}^{t,q}_{\text{aff},{}^Lk}({}^L\mathfrak{g})} & \underset{\epsilon_2 \nrightarrow 0}{\overset{\epsilon_2 \to 0}{\rightleftarrows}} & \boxed{Z(U_q(\hat{\mathfrak{g}})_{\text{crit}}) = \mathcal{W}^q_{\text{cl}}({}^L\mathfrak{g})} \\
R_6 \to 0 \uparrow\downarrow R_6 \nrightarrow 0 & & R_6 \to 0 \uparrow\downarrow R_6 \nrightarrow 0 \\
\boxed{\mathcal{W}^{q,t,v}_{\text{aff},k}(\mathfrak{g}) = \mathcal{W}^{t,q,v}_{\text{aff},{}^Lk}({}^L\mathfrak{g})} & \underset{\epsilon_2 \nrightarrow 0}{\overset{\epsilon_2 \to 0}{\rightleftarrows}} & \boxed{Z(U_{q,v}(\hat{\mathfrak{g}})_{\text{crit}}) = \mathcal{W}^{q,v}_{\text{cl}}({}^L\mathfrak{g})}
\end{array}
\tag{68}
$$

where g is arbitrary while $\mathfrak{g}$ is simply-laced.

From the fact that in the low energy sector of the worldvolume theories in (57) and (58) that is relevant to us, the worldvolume theory is topological along $\mathbf{R}^4$, we have

$$\underbrace{D_{R,\epsilon_1} \times D_{R,\epsilon_2} \times \Sigma_1}_{N \text{ M5-branes}} \quad \text{and} \quad \underbrace{D_{R,\epsilon_1} \times D_{R,\epsilon_2} \times \Sigma_1}_{N \text{ M5-branes + OM5-plane}} \;, \tag{69}$$

where $\Sigma_1 = \mathbf{S}^1_t \times \mathbf{S}^1_n$ is a Riemann surface of genus one with zero punctures.

Macroscopically at low energies, the curvature of the cigar tips is not observable. Therefore, we can simply take (69) to be

$$\underbrace{\mathbf{T}^2_{\epsilon_1,\epsilon_2} \times \mathbf{I}_1 \times \mathbf{I}_2 \times \Sigma_1}_{N \text{ M5-branes}} \quad \text{and} \quad \underbrace{\mathbf{T}^2_{\epsilon_1,\epsilon_2} \times \mathbf{I}_1 \times \mathbf{I}_2 \times \Sigma_1}_{N \text{ M5-branes + OM5-plane}} \;, \tag{70}$$

where $\mathbf{T}^2_{\epsilon_1,\epsilon_2} = \mathbf{S}^1_{\epsilon_1} \times \mathbf{S}^1_{\epsilon_2}$ is a torus of rotated circles, and $\mathbf{I}_{1,2} = \mathbf{R}_+$.

Clearly, the relevant BPS states are captured by the remaining uncompactified 2d theory on $\mathbf{I}_1 \times \mathbf{I}_2$ which we can regard as a sigma model which descended from the $\mathcal{N} = 4$, G theory over $\mathbf{I}_1 \times \mathbf{I}_2 \times \Sigma_1$, so

$$\mathcal{H}^{\sigma}_{\mathbf{I}_1 \times \mathbf{I}_2}(X_G^{\Sigma_1})_{\mathcal{B}} = \widehat{\mathcal{W}_{\mathrm{aff},{}^Lk}}({}^L\mathfrak{g})_{\Sigma_1}, \quad {}^Lk + {}^Lh = -\frac{\epsilon_2}{\epsilon_1}. \tag{71}$$

Now consider

$$\underbrace{\tilde{\mathbf{T}}^2_{\epsilon_1,\epsilon_2} \times \mathbf{I}_1 \times \mathbf{I}_2 \times \tilde{\Sigma}_1}_{N \text{ M5-branes}}, \quad \text{and} \quad \underbrace{\tilde{\mathbf{T}}^2_{\epsilon_1,\epsilon_2} \times \mathbf{I}_1 \times \mathbf{I}_2 \times \tilde{\Sigma}_1}_{N \text{ M5-branes} + \text{OM5-plane}}, \tag{72}$$

where $\tilde{\mathbf{T}}^2_{\epsilon_1,\epsilon_2}$ and $\tilde{\Sigma}_1$ are $\mathbf{T}^2_{\epsilon_1,\epsilon_2}$ and Σ_1 with the one-cycles swopped.

So, in place of (71), we have

$$\mathcal{H}^{{}^L\sigma}_{\mathbf{I}_1 \times \mathbf{I}_2}(X_{{}^LG}^{\Sigma_1})_{{}^L\mathcal{B}} = \widehat{\mathcal{W}_{\mathrm{aff},k}}(\mathfrak{g})_{\Sigma_1}, \quad r^{\vee}(k + h) = -\frac{\epsilon_1}{\epsilon_2}. \tag{73}$$

Since (70) and (72) are equivalent from the viewpoint of the worldvolume theory, we have

$$\begin{array}{ccc}
\mathcal{H}^{\sigma}_{\mathbf{I}_1\times\mathbf{I}_2}(X_G^{\Sigma_1})_{\mathcal{B}} & \xleftrightarrow{\text{String Duality}} & {}^{L}\widehat{\mathcal{W}_{\mathrm{aff},\kappa}}(\mathfrak{g})_{\Sigma_1} \\
S\text{-duality}\ \Big\updownarrow\ \tau \to -\frac{1}{r^{\vee}\tau} & r^{\vee}(\kappa + h) = ({}^{L}\kappa + {}^{L}h)^{-1} & \Big\updownarrow\ \mathcal{W}\text{-duality} \\
\mathcal{H}^{{}^{L}\sigma}_{\mathbf{I}_1\times\mathbf{I}_2}(X_{{}^{L}G}^{\Sigma_1})_{{}^{L}\mathcal{B}} & \xleftrightarrow{\text{String Duality}} & {}^{L}\widehat{\mathcal{W}_{\mathrm{aff},{}^{L}\kappa}}({}^{L}\mathfrak{g})_{\Sigma_1}
\end{array} \qquad (74)$$

where ${}^{L}\mathcal{W}_{\mathrm{aff},\kappa}(\mathfrak{g})$ is the “Langlands dual” of $\mathcal{W}_{\mathrm{aff},\kappa}(\mathfrak{g})$, an affine $\mathcal{W}$-algebra of level κ labeled by the Lie algebra $\mathfrak{g}$, and $\kappa + h = -\epsilon_2/\epsilon_1$.

A Quantum Geometric Langlands Correspondence as an S-duality and a Quantum $\mathcal{W}$-algebra Duality

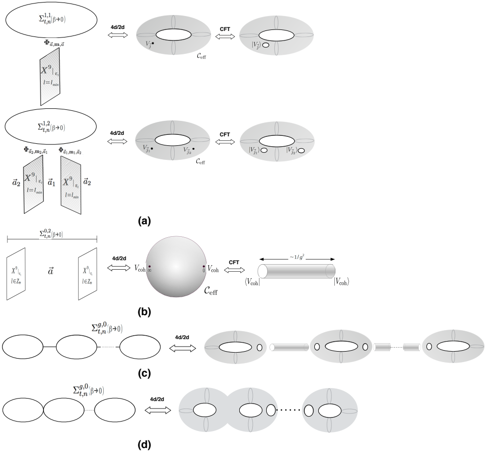


Figure 7: Building blocks relevant to our discussion on the connection between the gauge-theoretic realization and algebraic CFT formulation of the geometric Langlands correspondence.

(a). Dual M-theory compactifications which realize the AGT correspondence for $\mathcal{N} = 2^*$ $SU(N)$ theory and $\mathcal{N} = 2$ necklace quiver theory with two $SU(N)$ gauge groups, where β denotes the size of Σ.

(b). Dual M-theory compactifications which realize the AGT correspondence for $\mathcal{N} = 2$ pure $SU(N)$ theory.

(c). Gluing together g copies of the (a)-compactifications via the (b)-compactifications.

(d). Finally, a single compactification on $\Sigma^{g,0}_{t,n}$ without M9-planes on the 4d side, corresponding to a genus g surface with no punctures on the 2d side.

So, we can effectively replace Σ_1 with Σ_g in (70), (72), and thus in (71), (73), whence we can do the same in (74), and

$$\mathcal{H}^{\sigma}_{\mathbf{I}_1\times\mathbf{I}_2}(X_G^{\Sigma_g})_{\mathcal{B}} = \mathcal{H}^{\mathrm{A}}_{\mathbf{I}\times\mathbf{R}_+}(\mathcal{M}_H(G,\Sigma_g))_{\mathcal{B}_{d.c.},\mathcal{B}_\alpha} = \mathrm{D}^{\mathrm{mod}}_{\mathcal{L}^{\Psi-h^\vee}}(\mathrm{Bun}_G(\Sigma_g)) \tag{75}$$

where $\mathcal{M}_H(\mathscr{G},\Sigma_g)$ and $\mathrm{Bun}_{\mathscr{G}}(\Sigma_g)$ are the moduli space of $\mathscr{G}$ Hitchin equations and $\mathscr{G}_{\mathbb{C}}$-bundles on Σ_g [2, 3], so in place of (74), we have

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{mod}}_{\mathcal{L}^{\Psi-h^\vee}}(\mathrm{Bun}_G(\Sigma_g)) & \xleftrightarrow{\text{String Duality}} & {}^{L}\widehat{\mathcal{W}_{\mathrm{aff},\kappa}}(\mathfrak{g})_{\Sigma_g} \\
S\text{-duality} \Big\updownarrow \; {}^{L}\Psi = -\frac{1}{r^\vee \Psi} & & ({}^{L}\kappa + {}^{L}h) = \frac{1}{r^\vee(\kappa+h)} \; \Big\updownarrow \text{FF-duality} \\
\mathrm{D}^{\mathrm{mod}}_{\mathcal{L}^{{}^{L}\Psi-{}^{L}h^\vee}}(\mathrm{Bun}_{{}^{L}G}(\Sigma_g)) & \xleftrightarrow{\text{String Duality}} & {}^{L}\widehat{\mathcal{W}_{\mathrm{aff},{}^{L}\kappa}}({}^{L}\mathfrak{g})_{\Sigma_g}
\end{array} \tag{76}$$

Let $\epsilon_1 = 0$ whence we would also have $\kappa = \infty$ and $\Psi = 0$. Then, (76) becomes

$$
\begin{array}{ccc}
\mathrm{D}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g)) & \overset{\text{String Duality}}{\longleftrightarrow} & \mathrm{M}^{\mathrm{flat}}_{^L G_{\mathbb{C}}}(\Sigma_g) \\
{\scriptstyle S\text{-duality}}\ \Big\updownarrow\ {\scriptstyle \text{KW realization}} & & {\scriptstyle \text{BD formulation}}\ \Big\updownarrow\ {\scriptstyle \text{FF-duality}} \\
\mathrm{D}^{\mathrm{flat}}_{^L G_{\mathbb{C}}}(\Sigma_g) & \overset{\text{String Duality}}{\longleftrightarrow} & \mathrm{M}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g))
\end{array}
\tag{77}
$$

Adding boundary M2-branes which realize line operators in the gauge theory and performing the chain of dualities would replace (57) and (58) with

$$\underbrace{\mathring{\mathbf{R}}^4|_{0,\epsilon_2} \times \mathring{\mathbf{S}}^1_n \times \mathbf{S}^1_t}_{N \text{ M5 + M2 on } \circ} \times \mathring{\mathbf{R}}^5|_{\epsilon_2;\, x_{6,7}} \Longleftrightarrow \mathbf{R}^5|_{\epsilon_2;\, x_{4,5}} \times \underbrace{\mathbf{S}^1_t \times \mathring{\mathbf{S}}^1_n \times TN_N^{R\to 0}|_{\epsilon_2;\, x_{6,7}}}_{1 \text{ M5-branes + M0 on } \circ}, \tag{78}$$

and

$$\underbrace{\mathring{\mathbf{R}}^4|_{0,\epsilon_2} \times \mathring{\mathbf{S}}^1_n \times \mathbf{S}^1_t}_{N \text{ M5 + OM5 + M2 on } \circ} \times \mathring{\mathbf{R}}^5|_{\epsilon_2;\, x_{6,7}} \Longleftrightarrow \mathbf{R}^5|_{\epsilon_2;\, x_{4,5}} \times \underbrace{\mathbf{S}^1_t \times \mathring{\mathbf{S}}^1_n \times SN_N^{R\to 0}|_{\epsilon_2;\, x_{6,7}}}_{1 \text{ M5 + M0 on } \circ}, \tag{79}$$

Here, the M0-brane will become a D0-brane when we reduce M-theory on a circle to type IIA string theory [18].

As such, in place of (70) and (72), we have

$$\underbrace{\mathbf{T}^2_{0,\epsilon_2} \times \mathbf{I} \times \overbrace{\mathbf{R}_+ \times \Sigma_g}^{\text{M2 on } \mathbf{R}_+ \times \mathbf{S}^1_{n,g}}}_{N \text{ M5-branes}} \quad \text{and} \quad \underbrace{\mathbf{T}^2_{0,\epsilon_2} \times \mathbf{I} \times \overbrace{\mathbf{R}_+ \times \Sigma_g}^{\text{M2 on } \mathbf{R}_+ \times \mathbf{S}^1_{n,g}}}_{N \text{ M5-branes + OM5-plane}} \tag{80}$$

and

$$\underbrace{\tilde{\mathbf{T}}^2_{0,\epsilon_2} \times \mathbf{I} \times \overbrace{\mathbf{R}_+ \times \tilde{\Sigma}^{g,0}_{t,n}}^{\text{M2 on } \mathbf{R}_+ \times \tilde{\mathbf{S}}^1_{n,g}}}_{N \text{ M5-branes}} \quad \text{and} \quad \underbrace{\tilde{\mathbf{T}}^2_{0,\epsilon_2} \times \mathbf{I} \times \overbrace{\mathbf{R}_+ \times \tilde{\Sigma}^{g,0}_{t,n}}^{\text{M2 on } \mathbf{R}_+ \times \tilde{\mathbf{S}}^1_{n,g}}}_{N \text{ M5-branes + OM5-plane}} \tag{81}$$

Here, $\mathbf{S}^1_{n,g}$ is a disjoint union of a g number of $\mathbf{S}^1_n$ one-cycles of Σ_g.

Similarly, Σ_1 on the RHS of (71), (73) will now be $\Sigma_g^{\text{loop}} - \Sigma_g$ with a loop operator that is a disjoint union of g number of loop operators around its g number of $\mathbf{S}^1_n$ one-cycles, each corresponding to a worldoop of a D0-brane.

A Geometric Langlands Correspondence as an S-duality and a Classical $\mathcal{W}$-algebra Duality

$$\begin{array}{ccc}
\mathrm{D}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g))_{\mathcal{B}_{\text{`t-Hooft}}} & \xleftrightarrow{\text{String Duality}} & \mathrm{M}^{\mathrm{flat}}_{{}^LG_{\mathbb{C}}}(\Sigma_g)_{\widehat{\mathcal{W}}({}^L\mathfrak{g})_{\text{``Wilson''}}} \\
S\text{-duality} \Big\updownarrow \text{KW realization} & & \text{BD formulation} \Big\updownarrow \text{FF-duality} \\
\mathrm{D}^{\mathrm{flat}}_{{}^LG_{\mathbb{C}}}(\Sigma_g)_{{}^L\mathcal{B}_{\mathrm{Wilson}}} & \xleftrightarrow{\text{String Duality}} & \mathrm{M}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g))_{\widehat{\mathcal{W}}(\mathfrak{g})_{\text{```t-Hooft''}}}
\end{array} \tag{82}$$

There is a correspondence in the actions of 4d line operators and 2d loop operators:

$$\mathcal{B}_{\text{`t-Hooft}} \Longleftrightarrow \widehat{\mathcal{W}}(\mathfrak{g})_{\text{``'t-Hooft''}} \tag{83}$$

$${}^L\mathcal{B}_{\mathrm{Wilson}} \Longleftrightarrow \widehat{\mathcal{W}}({}^L\mathfrak{g})_{\text{``Wilson''}} \tag{84}$$

$\mathcal{B}_{\text{`t-Hooft}}$: $\mathbf{m}_0 \to \mathbf{m}_0 + \xi({}^L R)$, where magnetic flux $\mathbf{m}_0$ and $\xi({}^L R)$ are characteristic classes that classify the topology of G-bundles over Σ_g and $\mathbf{S}^2$, respectively. Thus, the 't Hooft line operator acts by mapping each object in $\mathrm{D}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g))$ labeled by $\mathbf{m}_0$, to another labeled by $\mathbf{m}_0 + \xi({}^L R)$.

On the other hand, $\widehat{\mathcal{W}}(\mathfrak{g})_{\text{``'t-Hooft''}}$ is a monodromy operator which acts on the chiral partition functions of the module $\mathrm{M}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_1))$ as (c.f. [19, §3.2])

$$Z_{\mathfrak{g}}(\mathbf{a}) \to \sum_{\mathbf{p}_k} \lambda_{\mathbf{a},\mathbf{p}}\, Z_{\mathfrak{g}}(\mathbf{p}_k). \tag{85}$$

where $\mathbf{p}_k = \mathbf{a} + \mathrm{b}\mathbf{h}_k$, where $\mathbf{h}_k$ are coweights of a representation R of G; and the $\lambda_{\mathbf{a},\mathbf{p}}$'s and b are constants. Therefore, $\widehat{\mathcal{W}}(\mathfrak{g})_{\text{``'t-Hooft''}}$ maps each state in $\mathrm{M}^{\mathrm{mod}}_{\mathrm{crit}}(\mathrm{Bun}_G(\Sigma_g))$ labeled by $\mathbf{a}$, to another labeled by $\mathbf{a} + \mathbf{h}$, where $\mathbf{h}$ is a weight of a representation ${}^L R$ of ${}^L G$.

${}^L\mathcal{B}_{\text{Wilson}} : \mathbf{e}_0 \to \mathbf{e}_0 + \theta_{^LR}$, where electric flux $\mathbf{e}_0$ and $\theta_{^LR}$ are characters of the center of (the universal cover of) ^LG. Because the $\mathbf{e}_0$-labeled zerobranes are points whence the shift $\mathbf{e}_0 \to \mathbf{e}_0 + \theta_{^LR}$ which twists them is trivial, the Wilson line operator acts by mapping each object in $\mathrm{D}^{\text{flat}}_{^LG_{\mathbb{C}}}(\Sigma_g)$ to itself.

On the other hand, $\widehat{\mathcal{W}}(^L\mathfrak{g})_{\text{“Wilson”}}$ is a monodromy operator which acts on the chiral partition functions of the module $\mathrm{M}^{\text{flat}}_{^LG_{\mathbb{C}}}(\Sigma_1)$ as (c.f. [19, Appendix D])

$$Z_{^L\mathfrak{g}}(\mathbf{a}^\vee) \to \lambda_{\mathbf{a}^\vee}\, Z_{^L\mathfrak{g}}(\mathbf{a}^\vee), \tag{86}$$

where the highest coweight vector $\mathbf{a}^\vee$ of $^L\mathfrak{g}$ labels a submodule, and $\lambda_{\mathbf{a}^\vee}$ is a constant. Thererfore, $\widehat{\mathcal{W}}(^L\mathfrak{g})_{\text{“Wilson”}}$ maps each state in $\mathrm{M}^{\text{flat}}_{^LG_{\mathbb{C}}}(\Sigma_g)$ to itself.

In the 5d case where $\beta \nrightarrow 0$, in place of (80), we have

$$\underbrace{\mathbf{T}^2_{0,\epsilon_2} \times \mathbf{R}_+ \times \mathbf{I} \times \Sigma_g^{\mathrm{S}^1}}_{N \text{ M5-branes}} \text{ and } \underbrace{\mathbf{T}^2_{0,\epsilon_2} \times \mathbf{R}_+ \times \mathbf{I} \times \Sigma_g^{\mathrm{S}^1}}_{N \text{ M5-branes + OM5-plane}} \tag{87}$$

where $\Sigma_g^{\mathrm{S}^1}$ is the compactified Riemann surface Σ_g (where $g > 1$) with an S^1 loop of radius β over every point.

Then,

$$\mathcal{H}^{\mathrm{A}}_{\mathbf{I}\times\mathbf{R}_+}(\mathcal{M}_H^{\mathrm{S}^1}(G,\Sigma_g))_{\mathcal{B}^\beta_{c.c.},\,\mathcal{B}^\beta_\alpha} = \widehat{\mathcal{W}^q_{\mathrm{cl}}}({}^L\mathfrak{g})_{\Sigma_g}, \tag{88}$$

or

$$\mathrm{C}^{\mathrm{mod}}_{\mathcal{O}_\hbar}(\mathcal{M}^{\mathrm{S}^1}_{\mathrm{H.S.}}(G,\Sigma_g)) = \mathrm{M}^{\mathrm{S}^1}_{{}^L G}(\Sigma_g)_{\mathrm{flat}}, \tag{89}$$

($\mathcal{O}_\hbar$ is a noncommutative algebra of holomorphic functions), so

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\mathrm{S}^1}_{\mathrm{H.S.}}(G,\Sigma_g))\text{-module}} \Longleftrightarrow \boxed{\text{circle-valued flat } {}^L G\text{-bundle on } \Sigma_g} \tag{90}$$

Clearly, this defines a q-geometric Langlands correspondence for simply-laced G!

Note that nonsingular $\widehat{G}$-monopoles on a flat three space M_3 can also be regarded as well-behaved G-instantons on $\hat{\mathrm{S}}^1 \times M_3$ in [20], while nonsingular G-monopoles on $M_3 = \mathrm{S}^1 \times \Sigma$ correspond to S^1-valued G Hitchin equations on Σ. Since principal bundles on a flat space with Kac-Moody structure group are also well-defined [20], a consistent $\widehat{G}$ version of (90) would be

$$\mathcal{O}_\hbar(\mathcal{M}^{\mathrm{S}^1}_{\mathrm{H.S.}}(\widehat{G}, \Sigma))\text{-module} \Longleftrightarrow \text{circle-valued flat } \widehat{{}^L G}\text{-bundle on } \Sigma \tag{91}$$

or equivalently,

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\hat{\mathrm{S}}^1 \times \mathrm{S}^1}_{\mathrm{H.S.}}(G, \Sigma))\text{-module}} \Longleftrightarrow \boxed{\text{circle-valued flat } \widehat{{}^L G}\text{-bundle on } \Sigma} \tag{92}$$

where $\Sigma = \mathbf{R} \times \mathbf{S}^1$. This defines a $\widehat{G}$ version of the q-geometric Langlands correspondence for simply-laced G.

In light of the fact that a $\widehat{\mathscr{G}}$-bundle can be obtained from a $\mathscr{G}$-bundle by replacing the underlying Lie algebra $\mathbf{g}$ of the latter bundle with its Kac-Moody generalization $\hat{\mathbf{g}}$, from (68), it would mean that we now have,

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}_{\mathrm{H.S.}}^{\hat{\mathrm{S}}^1\times\mathrm{S}^1}(G,\Sigma)) \iff \mathcal{T}_{\mathrm{xxz}}(\widehat{G},\Sigma)} \tag{93}$$

which relates the quantization of an elliptic-valued G Hitchin system on Σ to the transfer matrices of a $\widehat{G}$-type XXZ spin chain on Σ!
This also means that

$$\boxed{x \in \mathcal{M}_{\mathrm{H.S.}}^{\hat{\mathrm{S}}^1\times\mathrm{S}^1}(G,\Sigma) \iff \chi_q(\hat{V}_i) = \hat{T}_i(z), \quad \hat{V}_i \in \mathrm{Rep}\,[U_q^{\mathrm{aff}}(\hat{\mathfrak{g}})_\Sigma]} \tag{94}$$

where $i = 0, \ldots, \mathrm{rank}(\mathfrak{g})$, $\hat{T}_i(z)$ is a polynomial whose degree depends on $\hat{V}_i$, and $U_q^{\mathrm{aff}}(\hat{\mathfrak{g}})$ is the quantum toroidal algebra of $\mathfrak{g}$.

A Realization of Nekrasov-Pestun-Shatashvili's Results for 5d, $\mathcal{N}=1$ G ($\widehat{G}$)-Quiver $SU(K_i)$ Gauge Theories

Consider instead of the theories in figure (7), an $n=1$ linear quiver theory; then the present version of (90) and (68) imply

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},y_1,y_2}_{\mathrm{S^1\text{-}mono},\mathbf{k}} \Longleftrightarrow \chi_q(V_i) = T_i(z), \quad V_i \in \mathrm{Rep}\,[U_q^{\mathrm{aff}}(\mathfrak{g})_{\{\mathrm{C_x}\}_{z_1,z_2}}]} \tag{95}$$

where $\mathrm{C_x} = \mathbf{R} \times \mathbf{S}^1$, $i \in I_\Gamma$, the G Dynkin vertices.

Note that (94) also means that

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},k}_{\hat{\mathrm{S}}^1\times\mathrm{S}^1\text{-inst}} \Longleftrightarrow \chi_q(\hat{V}_i) = \hat{T}_i(z), \quad \hat{V}_i \in \mathrm{Rep}\,[U_q^{\mathrm{aff}}(\hat{\mathfrak{g}})_{\mathrm{C_x}}]} \tag{96}$$

where $\mathrm{C_x} = \mathbf{R} \times \mathbf{S}^1$, $i \in \hat{I}_\Gamma$, the affine-G Dynkin vertices.

Can argue via momentum around $\mathbf{S}^1_n$ (counted by D0-branes) $\leftrightarrow$ 2d CFT energy level correspondence that degree of T_i ($\hat{T}_i$) is K_i (aK_i).

(95)/(96) are Nekrasov-Pestun-Shatashvili's main result in [5, §1.3] which relates the moduli space of the 5d $G/\widehat{G}$-quiver gauge theory to the representation theory of $U_q^{\mathrm{aff}}(\mathfrak{g})/U_q^{\mathrm{aff}}(\hat{\mathfrak{g}})$!

In our derivation of the 6d AGT $\mathcal{W}$-algebra identity in diagram (68), the 2d CFT is defined on a torus $\mathbf{S}^1 \times \mathbf{S}^1_t$ with two punctures at positions $z_{1,2}$ [14, §5.1]. i.e. $\Sigma_{1,2}$. Here, $\mathbf{S}^1$ corresponds to the decompactified fifth circle of radius $\beta \nrightarrow 0$, while $\mathbf{S}^1_t$ corresponds to the sixth circle formed by gluing the ends of an interval $\mathbf{I}_t$ of radius R_6 much smaller than β. So, we effectively have a *single* decompactification of circles, like in the 5d case, although the 2d CFT states continue to be projected onto two circles of radius β and R_6, whence in place of (90), we have

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\mathrm{S}^1}_{\mathrm{H.S.}}(G, \Sigma_{1,2}))\text{-mod}} \Longleftrightarrow \boxed{\text{elliptic-valued flat } {}^L G\text{-bundle on } \Sigma_{1,2}} \tag{97}$$

Clearly, this defines a q, v-geometric Langlands correspondence for simply-laced G!

Consequently, from diagram (68), if $\mathcal{T}_{\rm xyz}(G, \Sigma_{1,2})$ is the polynomial algebra of commuting transfer matrices of a G-type XYZ spin chain with $U_{q,v}(\hat{\mathfrak{g}})$ symmetry on $\Sigma_{1,2}$, where $i = 1, \dots, \mathrm{rank}(\mathfrak{g})$, we now have

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}^{\rm S^1}_{\rm H.S.}(G, \Sigma_{1,2})) \iff \mathcal{T}_{\rm xyz}(G, \Sigma_{1,2})} \tag{98}$$

which relates the quantization of a circle-valued G Hitchin system on $\Sigma_{1,2}$ to the transfer matrices of a G-type XYZ spin chain on $\Sigma_{1,2}$!
This also means that

$$\boxed{x \in \mathcal{M}^{\rm S^1}_{\rm H.S.}(G, \Sigma_{1,2}) \iff \chi_{q,v}(V_i) = T_i(z), \quad V_i \in \mathrm{Rep}\left[U^{\rm ell}_{q,v}(\mathfrak{g})_{\Sigma_{1,2}}\right]} \tag{99}$$

and $T_i(z)$ is a polynomial whose degree depends on V_i.

Note that with regard to our arguments leading up to (97), one could also consider unpunctured Σ_1 instead of $\Sigma_{1,2}$ (i.e. consider the massless limit of the underlying linear quiver theory). Consequently, in place of (91), we have

$$\mathcal{O}_\hbar(\mathcal{M}_{\mathrm{H.S.}}^{\mathrm{S}^1}(\widehat{G}, \Sigma_1))\text{-mod} \Longleftrightarrow \text{elliptic-valued flat } \widehat{{}^L G}\text{-bundle on } \Sigma_1 \quad (100)$$

or equivalently,

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}_{\mathrm{H.S.}}^{\hat{\mathrm{S}}^1 \times \mathrm{S}^1}(G, \Sigma_1))\text{-mod}} \Longleftrightarrow \boxed{\text{elliptic-valued flat } \widehat{{}^L G}\text{-bundle on } \Sigma_1} \quad (101)$$

This defines a $\widehat{G}$ version of the q, v-geometric Langlands correspondence for simply-laced G.

Via the same arguments which led us to (93), we have

$$\boxed{\mathcal{O}_\hbar(\mathcal{M}_{\mathrm{H.S.}}^{\hat{\mathrm{S}}^1\times \mathrm{S}^1}(G,\Sigma_1)) \iff \mathcal{T}_{\mathrm{xyz}}(\widehat{G},\Sigma_1)} \tag{102}$$

which relates the quantization of an elliptic-valued G Hitchin system on Σ_1 to the transfer matrices of a $\widehat{G}$-type XYZ spin chain on Σ_1!
This also means that

$$\boxed{x \in \mathcal{M}_{\mathrm{H.S.}}^{\hat{\mathrm{S}}^1\times \mathrm{S}^1}(G,\Sigma_1) \iff \chi_{q,v}(\hat{V}_i) = \hat{T}_i(z), \;\; \hat{V}_i \in \mathrm{Rep}\,[U_{q,v}^{\mathrm{ell}}(\hat{\mathfrak{g}})_{\Sigma_1}]} \tag{103}$$

where $i = 0, \ldots, \mathrm{rank}(\mathfrak{g})$, $\hat{T}_i(z)$ is a polynomial whose degree depends on $\hat{V}_i$, and $U_{q,v}^{\mathrm{ell}}(\hat{\mathfrak{g}})$ is the elliptic toroidal algebra of $\mathfrak{g}$.

Note that (99) also means that

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},y_1,y_2}_{\mathrm{S}^1\text{-mono},\mathbf{k}} \Longleftrightarrow \chi_{q,v}(V_i) = T_i(z), \quad V_i \in \mathrm{Rep}\,[U^{\mathrm{ell}}_{q,v}(\mathfrak{g})_{\{\mathrm{C_x}\}_{z_1,z_2}}]} \tag{104}$$

where $\mathrm{C_x} = \mathbf{S}^1 \times \mathbf{S}^1_t$ and $i \in I_\Gamma$.

Note that (103) also means that

$$\boxed{u \in \mathfrak{M}^{G,\mathrm{C_x},k}_{\hat{\mathrm{S}}^1\times\mathrm{S}^1\text{-inst}} \Longleftrightarrow \chi_{q,v}(\hat{V}_i) = \hat{T}_i(z), \quad \hat{V}_i \in \mathrm{Rep}\,[U^{\mathrm{ell}}_{q,v}(\hat{\mathfrak{g}})_{\mathrm{C_x}}]} \tag{105}$$

where $\mathrm{C_x} = \mathbf{S}^1 \times \mathbf{S}^1_t$ and $i \in \hat{I}_\Gamma$.

Can again argue via momentum around $\mathbf{S}^1_n$ (counted by D0-branes) $\leftrightarrow$ 2d CFT energy level correspondence that degree of T_i ($\hat{T}_i$) is K_i (aK_i).

(104)/(105) are Nekrasov-Pestun-Shatashvili's main result in [5, §1.3] which relates the moduli space of the 6d $G/\widehat{G}$-quiver gauge theory to the representation theory of $U^{\mathrm{ell}}_{q,v}(\mathfrak{g})/U^{\mathrm{ell}}_{q,v}(\hat{\mathfrak{g}})$!

Conclusion

- We furnished purely physical derivations of higher AGT correspondences, $\mathcal{W}$-algebra identities, and higher geometric Langlands correspondences, all within our M-theoretic framework.
- We elucidated the connection between the gauge-theoretic realization of the geometric Langlands correspondence by Kapustin-Witten and its original algebraic CFT formulation by Beilinson-Drinfeld, also within our M-theoretic framework.
- Clearly, M-theory is a very rich and powerful framework capable of providing an overarching realization and generalization of cutting-edge mathematics and mathematical physics.
- At the same time, such corroborations with exact results in pure mathematics also serve as "empirical" validation of string dualities and M-theory, with the former as the "lab".

THANK YOU FOR COMING AND LISTENING!

[1] M.-C. Tan, "Higher AGT Correspondences, $\mathcal{W}$-algebras, and Higher Quantum Geometric Langlands Duality from M-Theory", Adv. Theor. Math. Phys. 22: 429-507, 2018, [arXiv:1607.08330].

[2] A. Kapustin and E. Witten, "*Electric-magnetic duality and the geometric Langlands program*", Comm. Numb. Th. Phys. **1** (2007) 1-236, [arXiv: hep-th/0604151].

[3] A. Kapustin, "A note on quantum geometric Langlands duality, gauge theory, and quantization of the moduli space of flat connections", [arXiv:0811.3264].

[4] A. Beilinson and V. Drinfeld, "*Quantization of Hitchin integrable system and Hecke eigensheaves*", preprint (ca. 1995), http://www.math.uchicago.edu/arinkin/langlands/.

[5] N. Nekrasov, V. Pestun, S. Shatashvili, "Quantum geometry and quiver gauge theories", [arXiv:1312.6689].

[6] S. Reffert, "General Omega Deformations from Closed String Backgrounds", [arXiv:1108.0644].

[7] S. Hellerman, D. Orlando, S. Reffert, "The Omega Deformation From String and M-Theory", [arXiv:1204.4192].

[8] D. Gaiotto, "$\mathcal{N} = 2$ Dualities", [arXiv:0904.2715].

[9] J. Ding, K. Iohara, "Generalization and Deformation of Drinfeld quantum affine algebras", [arXiv:q-alg/9608002].

[10] B. Feigin, A. Hoshino, J. Shibahara, J. Shiraishi, S. Yanagida, "Kernel function and quantum algebras", [arXiv:1002.2485].

[11] D. Maulik and A. Okounkov, "Quantum Groups and Quantum Cohomology", [arXiv:1211.1287].

[12] B. L. Feigin and A. I. Tsymbaliuk, "Equivariant K-theory of Hilbert schemes via shuffle algebra", Kyoto J. Math. 51 (2011), no. 4, 831-854.

[13] N. Guay and X. Ma, "From quantum loop algebras to Yangians", J. Lond. Math. Soc. (2) 86 no. 3, (2012) 683-700.

[14] M.-C. Tan, "An M-Theoretic Derivation of a 5d and 6d AGT Correspondence, and Relativistic and Elliptized Integrable Systems", JHEP **12** (2013) 31, [arXiv:1309.4775].

[15] Y. Saito, "Elliptic Ding-Iohara Algebra and the Free Field Realization of the Elliptic Macdonald Operator", [arXiv:1301.4912].

[16] B. Feigin, E. Frenkel, "Affine Kac-Moody algebras at the critical level and Gelfand-Dikii algebras", Int. J. Mod. Phys. A **7**, Suppl. 1A, 197-215 (1992).

[17] E. Frenkel, N. Reshetikhin, "Deformations of W-algebras associated to simple Lie algebras", [arXiv:q-alg/9708006].

[18] E. Bergshoeff, P. Townsend, "Super D-branes", Nucl. Phys. **B490** (1997) 145-162. [hep-th/9611173].

[19] J. Gomis, B. Le Floch, "'t Hooft Operators in Gauge Theory from Toda CFT", JHEP **11** (2011) 114, [arXiv:1008.4139].

[20] H. Garland and M. K. Murray, “Kac-Moody monopoles and periodic instantons”, Comm. Math. Phys. Volume 120, Number 2 (1988), 335-351.