

Aspects of T-duality: Recent progress and open questions

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Talk based on:

PB, Mark Bugden, Ctirad Klimčík and Kyle Wright,
Hidden Isometry of “T-duality without Isometry”
JHEP 08 (2017) 116, arXiv:1705.09254

Mark Bugden, “A Tour of T-duality – Geometric and Topological Aspects of T-dualities”, PhD Thesis 2018

Kyle Wright, “Generalised Geometries and Lie Algebroid Gauging in String Theory”, PhD Thesis 2018

The 2D (non-linear) sigma model

A 2D non-linear sigma model describes maps X from a 2-dimensional surface ('worldsheet') Σ to an N -dimensional manifold M ('target'), equipped with additional structure

For example

$$S[X] = \frac{1}{2} \int_{\Sigma} G_{ij}(X) dX^i \wedge \star dX^j + B_{ij}(X) dX^i \wedge dX^j$$

Symmetries of sigma model

Given a set of vector fields $v_a(X) = v_a^i(X)\partial_i$ forming a Lie algebra \mathfrak{g}

$$[v_a, v_b] = C^c{}_{ab}v_c$$

Consider the infinitesimal transformations

$$\delta_\epsilon X^i = v_a^i(X) \epsilon^a$$

we have

$$\delta_\epsilon \mathcal{S} = \int_\Sigma \epsilon^a \left((\mathcal{L}_{v_a} G)_{ij} dX^i \wedge \star dX^j + (\mathcal{L}_{v_a} B)_{ij} dX^i \wedge dX^j \right)$$

The sigma model action is invariant under these transformations if

$$\mathcal{L}_{v_a} G = 0, \quad \mathcal{L}_{v_a} B = 0$$

If this is the case, we can *gauge* the model by promoting the global symmetry to a local one (i.e. take $\epsilon \in C^\infty(\Sigma, \mathfrak{g})$)

The gauged action

Introducing gauge fields $A \in \Omega^1(\Sigma, \mathfrak{g})$ the gauged action is given by

$$S[X, A] = \frac{1}{2} \int_{\Sigma} G_{ij}(X) DX^i \wedge \star DX^j + B_{ij}(X) DX^i \wedge DX^j$$

where

$$DX^i = dX^i - v_a^i A^a$$

are the covariant derivatives.

Gauge invariance

The gauged action $S[X, A]$ is invariant with respect to the following (local) gauge transformations:

$$\delta_\epsilon X^i = v_a^i \epsilon^a$$

$$\delta_\epsilon A = d\epsilon + [A, \epsilon] = (d\epsilon^a + C^a_{bc} A^b \epsilon^c) T_a$$

where T_a is a basis of \mathfrak{g} .

Now suppose we want the gauged sigma model to be equivalent to the the ungauged model. Then we need to 'fix the gauge'

Introduce the curvature $F \in \Omega^2(\Sigma, \mathfrak{g})$

$$F = dA + A \wedge A = (dA^a + \frac{1}{2}C^a_{bc} A^b \wedge A^c) T_a = F^a T_a$$

and an 'auxiliary field' $\widehat{X} \in C^\infty(\Sigma, \mathfrak{g}^*)$, with infinitesimal transformation rules

$$\delta_\epsilon F^a = C^a_{bc} F^b \epsilon^c$$

$$\delta_\epsilon \widehat{X}_a = -C^c_{ab} \widehat{X}_c \epsilon^b$$

Consider the action

$$S[X, A, \hat{X}] = \frac{1}{2} \int_{\Sigma} \left(G_{ij}(X) DX^i \wedge \star DX^j + B_{ij}(X) DX^i \wedge DX^j \right) + \int_{\Sigma} \hat{X}_a F^a$$

The equation of motion for \hat{X}_a gives $F^a = 0$.

To solve this equation we need to lift the action of \mathfrak{g} to an action of the group G ($\mathfrak{g} = \text{Lie } G$)

Example: Group manifold

Let $g : \Sigma \rightarrow G$

$$S[g] = \frac{1}{2} \int_{\Sigma} (g^{-1} dg \wedge *g^{-1} dg)_G$$

Invariant under left action of $h \in G$

$$S[hg] = S[g]$$

while

$$S[gh] = \frac{1}{2} \int_{\Sigma} (\text{Ad}(h^{-1})g^{-1} dg \wedge \text{Ad}(h^{-1}) * g^{-1} dg)_G$$

So, invariant under right action of G if G is Ad-invariant (Killing form)

Example: Gauged

In that case we can gauge in the standard way, and obtain the gauged model (with F -term)

$$S[g, A, \hat{X}] = \frac{1}{2} \int_{\Sigma} (g^{-1} Dg \wedge *g^{-1} Dg)_G + \int_{\Sigma} \langle \hat{X}, F \rangle$$

where

$$\begin{aligned} g^{-1} Dg &= g^{-1} dg - A \\ F &= dA + A \wedge A \end{aligned}$$

and gauge symmetry, for $h \in G$

$$\begin{aligned} g &\rightarrow gh \\ A &\rightarrow h^{-1} Ah + h^{-1} dh \\ \hat{X} &\rightarrow \text{Ad}^*(h^{-1})\hat{X} \end{aligned}$$

Example: Gauged model

Solving $F = 0$ gives $A = -dkk^{-1}$ for $k \in C^\infty(\Sigma, G)$, and substituting

$$g^{-1}Dg \rightarrow g^{-1}dg + dkk^{-1} = k((gk)^{-1}d(gk))k^{-1}$$

i.e.

$$S[g, A = -dkk^{-1}] = S[gk]$$

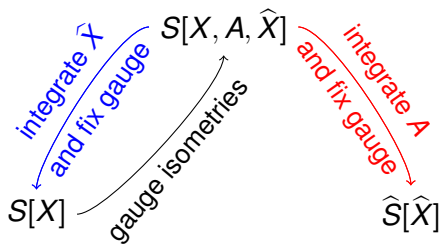
so after 'fixing the gauge' we recover the ungauged model.

On the other hand, first solving the equation of motion for A , and then fixing the gauge, gives dual model

$$\widehat{S}[\widehat{X}] = \frac{1}{2} \int_{\Sigma} \widehat{G}^{ab}(\widehat{X}) d\widehat{X}_a \wedge \star d\widehat{X}_b$$

with dual metric

$$\widehat{G}^{-1}_{ab} = G_{ab} - C^c_{ab} \widehat{X}_c$$



Gauging without isometries?

The existence of global symmetries is a very stringent requirement. A generic metric will not have any Killing vectors.

Question: Is it possible to follow the same procedure when the vector fields are not Killing vectors?

Kotov and Strobl¹ introduced a method of gauging a sigma model without requiring the model to possess isometries.

¹[1403.8119]

Their method uses Lie algebroids, and generalises the standard gauging in two notable ways:

- The structure constants of the Lie algebra are promoted to structure functions:

$$[v_a, v_b] = C^c{}_{ab}(X) v_c$$

- The gauge invariance of the gauged action doesn't require the original vector fields to be isometries:

$$\mathcal{L}_{v_a} G \neq 0 \quad \mathcal{L}_{v_a} B \neq 0$$

Gauging without isometry

The set-up involves a Lie algebroid Q , a map $X : \Sigma \rightarrow M$,

$$\begin{array}{ccccc} X^*Q & \longrightarrow & Q & \xrightarrow{\rho} & TM \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma & \xrightarrow{X} & M & \xrightarrow{\cong} & M \end{array}$$

together with a gauge field

$$A \in \Omega^1(\Sigma, X^*Q)$$

a connection ∇ on Q

$$\nabla : \Gamma(Q) \rightarrow \Gamma(T^*M \otimes Q) = \Omega^1(M) \otimes \Gamma(Q)$$

and infinitesimal gauge parameter $\epsilon \in C^\infty(\Sigma, X^*Q)$.

Non-isometric T-duality

Upon choosing a basis e_a of sections of Q , and defining matrix-valued one-forms $\omega^b{}_a$ by

$$\nabla e_a = \omega^b{}_a e_b$$

the conditions on G and B become

$$\mathcal{L}_{v_a} G = \omega^b{}_a \vee \iota_{v_b} G$$

$$\mathcal{L}_{v_a} B = \omega^b{}_a \wedge \iota_{v_b} B$$

where $v_a = \rho(e_a)$.

The gauged action

The gauged action

$$S^\omega[X, A] = \frac{1}{2} \int_\Sigma G_{ij} DX^i \wedge \star DX^j + B_{ij} DX^i \wedge DX^j$$

is invariant under the modified (infinitesimal) gauge transformations

$$\delta_\epsilon X^i = v_a^i \epsilon^a$$

$$\delta_\epsilon A^a = d\epsilon^a + C^a_{bc} A^b \epsilon^c + \omega^a_{bi} \epsilon^b DX^i$$

Problems:

- Infinitesimal gauge transformations do not necessarily close
- How to lift this to a global (groupoid) action?

Non-isometric T-duality

Chatzistavrakidis, Deser, and Jonke² apply this non-isometric gauging procedure to T-duality

The curvature is now given by

$$F_{\omega}^a = dA^a + \frac{1}{2} C^a{}_{bc}(X) A^b \wedge A^c - \omega^a{}_{bi} A^b \wedge DX^i$$

and

$$\delta_{\epsilon} \widehat{X}_a = -C^c{}_{ab} \epsilon^b \widehat{X}_c + v_a^i \omega^c{}_{bi} \epsilon^b \widehat{X}_c$$

Problem: In all their examples their ‘non-isometric T-duality’ is equivalent to non-abelian T-duality.

²[1509.01829] and [1604.03739]

A necessary condition for gauge invariance

A necessary condition for gauge invariance of the non-isometrically gauged action with F_ω -term, is that ω^b_a is flat

$$R^b_a = d\omega^b_a + \omega^b_c \wedge \omega^c_a = 0$$

This tells us that ω^b_a is of the form $K^{-1}dK$ for some $K^b_a(X)$.

Using this K , we can perform the following field redefinitions:

$$\tilde{A}^a = K^a_b A^b$$

$$\tilde{\tilde{X}}_a = \hat{X}_b (K^{-1})^b_a$$

$$\tilde{v}_a = v_b^i (K^{-1})^b_a$$

Note that

$$\tilde{D}X^i = dX^i - \tilde{v}_a^i \tilde{A}^a = dX^i - v_a^i A^a = DX^i$$

Equivalence!

The gauged action can now be rewritten in terms of the new fields $(X^i, \tilde{A}^a, \tilde{X}_a)$.

$$\begin{aligned} S^\omega[X, \tilde{A}, \tilde{X}] &= \frac{1}{2} \int_{\Sigma} G_{ij} DX^i \wedge \star DX^j + B_{ij} DX^i \wedge DX^j + \int_{\Sigma} \tilde{X}_a \tilde{F}^a \\ &= S[X, \tilde{A}, \tilde{X}] \end{aligned}$$

where

$$\tilde{F}^a = d\tilde{A}^a + \frac{1}{2} \tilde{C}^a{}_{bc} \tilde{A}^b \wedge \tilde{A}^c$$

The gauge transformations become the usual non-abelian gauge transformations, and a short computation reveals

$$\mathcal{L}_{\tilde{V}_a} G = 0 \quad \mathcal{L}_{\tilde{V}_a} B = 0$$

Finally, gauge invariance of the action also requires that the structure functions $\tilde{C}^c_{ab}(X)$ be constants.

Conclusion: Infinitesimal gauge invariance of the non-isometrically gauged Lie algebroid sigma model implies that the connection ∇^ω is flat, and that there exists a Lie algebra $\mathfrak{g}(Q, \omega)$, with constant structure functions \tilde{C}^a_{bc} which is equivalent to this model upon field redefinition ('change of basis of the Lie algebroid').

Corollary: Non-isometric T-duality is equivalent to non-abelian T-duality.

Example: revisited

If, in our example,

$$S[g] = \frac{1}{2} \int_{\Sigma} (g^{-1} dg \wedge *g^{-1} dg)_G$$

the metric G is not Ad-invariant, then we can still non-isometrically gauge with respect to the right action.

It turns out that by performing the field redefinitions this model is equivalent to isometrically gauging the left action.

THANKS