Wu Chern-Simons theories and string theory

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Outline

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Summary

I will describe certain topological field theories in dimension $d = 4k + 3$ involving abelian $2k + 1$-form gauge fields.

They can be thought of as "abelian quadratic Chern-Simons theories at half-integer level".

In dimension 3, they are (abelian) spin Chern-Simons theories, involving 1-form gauge fields.

Spin Chern-Simons theories are for instance used as an effective description of the quantum Hall effect, see [Belov, Moore hep-th/0505235].

Summary

Unlike spin Chern-Simons theories, which require a spin structure on spacetime, the higher dimensional theories require *Wu structures*, which are suitable generalizations of spin structures.

We therefore call them Wu Chern-Simons (WCS) theories.

Summary

Given an anomalous quantum field theory in dimension d, all its anomalies can be encoded into a (very simple) quantum field theory in dimension $d + 1$, its *anomaly field theory*. [Freed 1404.7224], [Moore Klein's Lectures, 2012], [S.M. 1410.7442]

This is a useful point of view to study the cancellation of global anomalies.

The appearance of WCS theories in string theory comes from the fact that they are a component of the anomaly field theories of chiral p-forms.

Summary

At the end of the talk, I will describe two applications involving 7-dimensional WCS theories.

1. A WCS theory is a crucial ingredient in the construction of the anomaly field theories of 6-dimensional $(2, 0)$ superconformal field theories.
2. The Green-Schwarz term of a 6d supergravity theory can be seen as a vector in the state space of a WCS theory. This point of view is necessary to understand global anomaly cancellations in 6d supergravity.

In dimension 11, there should be an analogous theory describing the anomalies of the RR fields of type IIB supergravity, involving differential KO-theory, but its construction is not understood yet.

Outline

Differential cocycles

To do concrete computations in field theory, we need an *algebraic* model of the fields.

Gauge fields are connections, i.e. not algebraic objects. In the physics literature, they are often modelled algebraically as differential 1-forms.

But this is accurate only if the gauge bundle is trivial.

A general and accurate algebraic model for abelian gauge fields is provided by differential cocycles. [Cheeger, Simons 1985]

Differential cocycles also model appropriately higher degree abelian gauge fields, as well as gauge fields with shifted flux quantization.

[Dijkgraaf, Witten 1990], [Hopkins, Singer math/0211216], [Freed hep-th/0011220]

Differential cocycles

A degree p differential cochain is a triplet $\check{A} = (f, A, F)$ where

- f is a degree p $\mathbb{Z}$-valued cochain, the *characteristic*;
- A is a degree $p-1$ $\mathbb{R}$-valued cochain, the *connection*;
- F is a degree p differential form, the *curvature*;

We define a differential by

$$d(f, A, F) = (df, F - f - dA, dF) \ .$$

Elements of $\ker(d)$ are *differential cocycles*, satisfying

$$df = 0 \ , \quad f + dA - F = 0 \ , \quad dF = 0 \ .$$

Differential cocycles

Gauge transformations are given by pairs (g, B) composed of a degree $p-1$ $\mathbb{Z}$-valued cochain g and a degree $p-2$ $\mathbb{R}$-valued cochain B:

$$(f, A, F) \to (f + dg, A - g - dB, F) = (f, A, F) + d(g, B, 0)$$

Gauge equivalence classes are *differential cohomology classes*.

The physical interpretation is as follows:

- Differential cocycles are (higher abelian) $U(1)$ gauge fields.
- F is the gauge field strength.
- A is essentially the gauge connection.
- f encodes the gauge field topology, including torsion fluxes.
- Large gauge transformations have $g \neq 0$.

DIFFERENTIAL COCYCLES

In the topologically trivial case, we can set $f = 0$ so $F = dA$. The cochain A can be realized as a differential form. $\Rightarrow$ We recover the naive model in terms of differential forms.

- $p = 1$ corresponds to $U(1)$-valued scalar fields, including the topologically non-trivial configurations.
- $p = 2$ corresponds to ordinary (degree 1) $U(1)$-valued gauge fields: one can show that degree 2 differential cocycles modulo the gauge transformations are in bijection with isomorphism classes of $U(1)$-bundles with connections (unlike differential 1-forms).
- $p > 2$ corresponds to higher degree $U(1)$-valued gauge fields.

Differential cocycles

There is a cup product $\cup$ on differential cochains. It coincides with the cup and wedge products on the characteristics and on the curvatures.

On the connections it reads

$$\left[\check{A}_1 \cup \check{A}_2\right]_{\text{conn}} := (-1)^{p_1} f_1 \cup A_2 + A_1 \cup F_2 + H_{\cup}^{\wedge}(F_1, F_2)$$

with

$$dH_{\cup}^{\wedge}(x,y) - H_{\cup}^{\wedge}(dx,y) - (-1)^{|x|} H_{\cup}^{\wedge}(x,dy) = x \cup y - x \wedge y$$

a homotopy between the cup and wedge product.

The $H_{\cup}^{\wedge}$ term is needed in order for the cup product to map differential cocycles to differential cocycles.

Outline

Chern-Simons action

We can write the action of level k $U(1)$ Chern-Simons theory as

$$\frac{1}{2\pi}\mathrm{CS}_k(\check{A}) := k\int_M \left[\check{A}\cup\check{A}\right]_{\mathrm{conn}}$$

where $\check{A}$ is a differential cocycle.

In the topologically trivial case, $F = dA, f = 0$ and

$$\frac{1}{2\pi}\mathrm{CS}_k(\check{A}) = k\int_M A\cup dA + H_{\cup}^{\wedge}(dA, dA) = k\int_M A\wedge dA + \text{exact}$$

This is the usual expression of the Chern-Simons action in terms of the gauge differential 1-form.

CHERN-SIMONS ACTION

If $M = \partial W$ and $\check{A}$ extends to W, we have:

$$\begin{aligned}
\frac{1}{2\pi}\mathrm{CS}(\check{A}) &= k\int_W d[\check{A}\cup\check{A}]_{\mathrm{conn}} \\
&= k\int_W \left([\check{A}\cup\check{A}]_{\mathrm{curv}} - [\check{A}\cup\check{A}]_{\mathrm{char}}\right) \\
&= k\int_W (F\wedge F - f\cup f) \\
&= k\int_W F\wedge F \quad \mathrm{mod}\ 1\ .
\end{aligned}$$

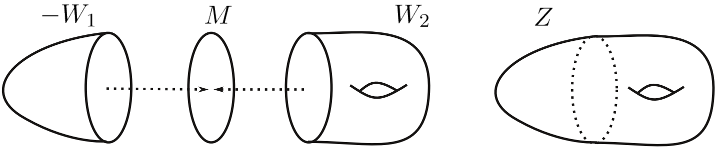
We recover the Chern-Simons action in terms of W.

The definition of the Chern-Simons action carries over verbatim in the case of a $4\ell+3$-dimensional manifold M, with degree $2\ell+2$ differential cocycles modeling degree $2\ell+1$ gauge fields.

Spin Chern-Simons action

Could we have a half-integer level, $k \in \frac{1}{2}\mathbb{Z}$? Consider again

$$\frac{1}{2\pi}\mathrm{CS}(\check{A}) = k \int_W F \wedge F \mod 1 \ ,$$



For $\exp i\mathrm{CS}(\check{A})$ to independent on the choice of W, we need that for any closed 4-manifold Z, $k \int_Z F \wedge F = k \int_Z f \cup f \in \mathbb{Z}$.

This is not the case in general, because while $\int_Z f \cup f \in \mathbb{Z}$, $\int_Z f \cup f \notin 2\mathbb{Z}$.

Spin Chern-Simons action

Recall that the degree p *Wu class* ν_p is a $\mathbb{Z}_2$-valued characteristic class such that on (closed) $2p$-dimensional manifolds Z,

$$\int_Z x \cup x = \int_Z x \cup \nu_p \quad \text{for all } x \in H^p(Z; \mathbb{Z}_2) \ .$$

On orientable manifolds, the degree 2 Wu class coincides with w_2, the second Stiefel-Whitney class.

Spin Chern-Simons action

Therefore if Z is spin, $w_2 = 0$ and we *do* have $\int_Z f \cup f \in 2\mathbb{Z}$.

Z is spin if M is spin and the bounding manifolds $W_{1,2}$ are required to extend the spin structure on M.

We can then define the *spin Chern-Simons* action by

$$\frac{1}{2\pi}\mathrm{CS}(\check{A}) = k\int_W F \wedge F \mod 1 \ , \quad k \in \frac{1}{2}\mathbb{Z} \ .$$

The spin condition on M and W restricts the choices of bounding manifolds W in precisely the right way to make the half-integer level action well-defined.

Spin Chern-Simons action

Given any spin 3-manifold M, there is always a spin 4-manifold W admitting M as a boundary on which $\check{A}$ extends. $\Rightarrow$ The definition above of the spin Chern-Simons action is completely general.

What about the Chern-Simons theories in dimension $4\ell + 3$?

- If M and W are degree $2\ell + 2$ Wu manifolds, i.e. manifolds such that $\nu_{2\ell+2} = 0$, we can repeat the arguments above.
- However, while we automatically have $\nu_{2\ell+2} = 0$ on M, it may not be possible to find a Wu-manifold W bounded by M.
- In fact, it may not even be possible to find a non-Wu manifold W bounded by M on which $\check{A}$ extends.

$\Rightarrow$ We need to find a genuine $4\ell + 3$-dimensional action.

Outline

Wu Chern-Simons action

Consider our original Chern-Simons action as a candidate:

$$\frac{1}{2\pi}\mathrm{WCS}^{\mathrm{naive}}(\check{A}) := k\int_M \left[\check{A}\cup\check{A}\right]_{\mathrm{conn}}\ , \quad k\in\frac{1}{2}\mathbb{Z}$$

Under a gauge transformation by $(g,B,0)$, the action changes by

$$\Delta_{(g,B,0)}\frac{1}{2\pi}\mathrm{WCS}^{\mathrm{naive}}(\check{A}) = k\int_M(-g\cup f - f\cup g - g\cup dg)\notin\mathbb{Z}$$

so $\exp i\mathrm{WCS}^{\mathrm{naive}}(\check{A})$ is *not* gauge invariant: it can change by a sign under large gauge transformations.

Having a spin/Wu structure on M does not help. We are missing a fundamental new idea.

Wu Chern-Simons action

The $4\ell + 3$-dimensional Chern-Simons action was constructed in two steps:

1. Use the differential cocycle $\check{A}$ representing the gauge field to construct a degree $4\ell + 3$ $\mathbb{R}/\mathbb{Z}$-valued cocycle $k[\check{A} \cup \check{A}]_{\text{conn}}$ mod 1, the Lagrangian.
   The Lagrangian up to gauge transformations determines a class in $H^{4\ell+3}(M; \mathbb{R}/\mathbb{Z})$.
2. Apply integration map/pairing with the fundamental homology class to the Lagrangian to obtain the value of the action, in $\mathbb{R}/\mathbb{Z}$.

We saw above that in the spin/Wu case, there is no way to construct a Lagrangian inducing a class in $H^{4\ell+3}(M; \mathbb{R}/\mathbb{Z})$.

WU CHERN-SIMONS ACTION

Idea: Replace the ordinary cohomology $H^{4\ell+3}(M;\mathbb{R}/\mathbb{Z})$ by a generalized cohomology.

1. The Lagrangian will be a "cocycle" in a model for a generalized cohomology theory.
   The Lagrangian up to gauge transformations will define a generalized cohomology class.
2. The action will be given by the "integral" in the generalized cohomology theory of the Lagrangian. (I.e. the spacetime manifold will have to carry an orientation for the generalized cohomology, which will turn out to be exactly a spin/Wu structure.)
   The fact that the Lagrangian defines a well-defined generalized cohomology class ensures gauge invariance.

Generalized cohomology

Cohomology theories are functors assigning abelian groups to (pairs of) topological spaces, satisfying the five Eilenberg-Steenrod axioms.

They ensure in particular the homotopy invariance of cohomology and the additivity under disjoint unions.

The 5th axiom, known as the *dimension axiom*, says that the G-valued cohomology of the point is:

$$H^p(*;G) = \begin{cases} G & \text{if } p = 0\ , \\ 0 & \text{else.} \end{cases}$$

Generalized cohomology theories satisfy the same axioms except for the dimension axiom. Examples include K-theories, cobordism theories, ...

E-THEORY

The generalized cohomology of interest is *E-theory* and satisfies

$$E^p_\ell(*) = \begin{cases} \mathbb{R}/\mathbb{Z} & \text{if } p = 0\ , \\ \mathbb{Z}_2 & \text{if } p = 2\ell+1 \\ 0 & \text{else.} \end{cases}$$

It fits in the long exact sequence

$$\begin{aligned} &\ldots \to H^p(M;\mathbb{R}/\mathbb{Z}) \to E^p(M) \to \\ &H^{p-2\ell-1}(M;\mathbb{Z}_2) \overset{\mathrm{Sq}^{2\ell+2}\bullet + \bullet\cup\nu_{2\ell+2}}{\to} H^{p+1}(M;\mathbb{R}/\mathbb{Z}) \to \ldots \end{aligned}$$

Degree n E-theory classes on n-dimensional manifolds endowed with a degree $2\ell+2$ Wu structure can be integrated to obtain an element of $\mathbb{R}/\mathbb{Z}$. We write the integration map $\int^{\mathrm{E}}$.

Cochain model

- Degree p E-cochains on M are elements

$$\bar{c} = (x, a) \in C^p(M; \mathbb{R}/\mathbb{Z}) \times C^{p-2\ell-1}(M; \mathbb{Z}_2)$$

- Addition is *non-commutative* and given by

$$(x_1, a_1) \boxplus (x_2, a_2) = \left(x_1 + x_2 + \frac{1}{2} da_1 \cup_{p-2\ell-1} a_2 + \frac{1}{2} a_1 \cup_{p-2\ell-2} a_2, \right.$$
$$\left. a_1 + a_2 \right)$$

- The differential is

$$d(x, a) = \left(dx + \frac{1}{2} a \cup_{p-2\ell-2} da + \frac{1}{2} a \cup_{p-2\ell-3} a + \frac{1}{2} a \cup \nu_{2\ell+2}, da\right)$$

Closed degree p E-cochains up to exact ones form a group isomorphic to $E^p_\ell(M)$.

The Wu Chern-Simons action

Let $\check{A} = (f, A, F)$ a differential cocycle, and write $\rho_2(f) := f \bmod 2$.

1. The Lagrangian

$$\bar{l}_k(\check{A}) := \left(k[\check{A} \cup (\check{A} + \check{\nu})]_{\mathrm{conn}}, \rho_2(2kf)\right)$$

   is an E-cocycle. Up to gauge transformations, it defines a class in $E_\ell^{4\ell+3}(M)$.

2. The action

$$\frac{1}{2\pi}\mathrm{WCS}_k(\check{A}) = \int^{\mathrm{E}} \bar{l}_k(\check{A}) \ , \quad k \in \frac{1}{2}\mathbb{Z}$$

   is gauge invariant.

Generalization

The whole story so far generalizes to Chern-Simons theories with arbitrary abelian gauge groups and couplings.

Recall that an abelian Chern-Simons theory is fully specified by an *even* lattice Λ. The gauge group is the torus $G = (\Lambda \otimes \mathbb{R})/\Lambda$ and the couplings are encoded in the integral pairing on Λ.

The simplest way to define the action is by

$$\frac{1}{2\pi}\mathrm{CS}(\check{A}) = \frac{1}{2}\int_W F \wedge F$$

where F is the extension of the field strength, with Λ-valued periods, to a $4\ell + 4$ manifold.

For instance, if $\Lambda = (\mathbb{Z}, (2k))$, we recover the $U(1)$ Chern-Simons theory at level k.

GENERALIZATION

If the spacetime manifold carries a degree $2\ell + 2$ Wu structure, we can define a Wu Chern-Simons theory associated to *any* integral lattice Λ by means of the $4\ell + 3$-dimensional action

$$\frac{1}{2\pi}\mathrm{WCS}(\check{A}) = \int^{\mathrm{E}} \bar{l}(\check{A}) \;, \quad k \in \frac{1}{2}\mathbb{Z}$$

$$\bar{l}(\check{A}) := \left(\frac{1}{2}[\check{A} \cup (\check{A} + \check{\nu})]_{\mathrm{conn}}, \rho_2(f)\right)$$

involving a certain E-theory depending on both ℓ and Λ.

Properties of the action

The Wu Chern-Simons action has interesting properties.

Suppose $\check{A}_i = (f_i, A_i, 0)$, $i = 1, 2$ are flat differential cocycles. Then $f_i = dA_i$, so $[f_i] \in H^{2\ell+2}(M;\Lambda)$ are torsion classes.

1. $\frac{1}{2\pi}\mathrm{WCS}(\check{A}_i) = q([f_i])$
2. q is a quadratic refinement of the linking pairing L on $H^{2\ell+2}_{\mathrm{tor}}(M;\Lambda)$:

$$q([f_1]+[f_2]) = q([f_1]) + q([f_2]) + L([f_1],[f_2])$$

$$q(n[f_1]) = n^2 q([f_1]) \ .$$

PROPERTIES OF THE ACTION

Suppose $M = \partial W$ and that $\check{A} = (f, A, F)$ extends to a $4\ell + 4$-dimensional W.

3 $\mathrm{WCS}(\check{A})$ has a simple expression on W in terms of differential forms:

$$\mathrm{WCS}(\check{A}) = \frac{1}{2} \int_W F \wedge (F + \lambda) \ .$$

where 2λ is a closed differential form on W vanishing on M and whose periods mod 2 coincide with the periods of the Wu class $\nu_{2\ell+2}(W)$.

Properties of the action

The action admits a discrete symmetry.

Suppose that $\check{A}' = (f', A', 0)$ is a flat differential cocycle such that A' is valued in $\Lambda^* \subset \Lambda \otimes \mathbb{R}$.

4 Then $\mathrm{WCS}(\check{A} + \check{A}') = \mathrm{WCS}(\check{A})$.

These transformations are actually gauge transformations whenever A' is valued in Λ, so they form a finite discrete group up to gauge transformations.

This discrete symmetry can be gauged [S.M. - 1607.01396].

Summary

WCS is a genuinely $4\ell + 3$-dimensional action, defined on spacetime manifolds endowed with a degree $2\ell + 2$ Wu structure.

WCS defines classical (higher) abelian Chern-Simons theories for all possible abelian gauge groups and couplings, not only those associated to even lattices.

The Lagrangian of the Wu Chern-Simons theory is not a differential form or a an ordinary cohomology cocycle, but rather a cocycle representative of a class in a generalized cohomology theory.

A (non-abelian) cochain model enables concrete computations.

Outline

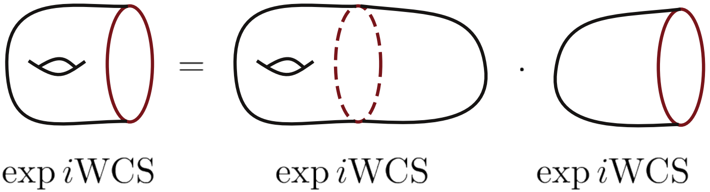
Prequantum theory

We now explain how to construct field theory functors (cf. Atiyah-Segal) out of the WCS action.

One can readily construct the *prequantum field theory functor* $\mathrm{WCS}^{\mathrm{pq}}$ associated to WCS:

The partition function on a closed $4\ell+3$-dimensional manifold $(M, \check{A})$ is

$$\mathrm{WCS}^{\mathrm{pq}}(M, \check{A}) := \exp i\mathrm{WCS}(M, \check{A}) \in U(1) \subset \mathbb{C}$$



Quantum theory

(Details need to be worked out.)

The quantum theory $\mathrm{WCS}^{\mathrm{q}}$ is obtained by integrating out $\check{A}$ which is now seen as a dynamical field.

1. Gauss' law requires $\check{A}$ to be flat.
2. Up to one-loop determinants and volume terms, the path integral should reduce to a discrete sum over the classes $[f] \in H^{2\ell+2}_{\mathrm{tor}}(M;\Lambda)$ associated to the flat gauge field $\check{A}_{[f]} = (f, A, 0)$.

We deduce that

$$\begin{aligned}\mathrm{WCS}^{\mathrm{q}}(M) &= N(M) \sum_{[f]\in H^{2\ell+2}_{\mathrm{tor}}(M;\Lambda)} \exp i\mathrm{WCS}(M,\check{A}_{[f]}) \\ &= N(M) \sum_{[f]\in H^{2\ell+2}_{\mathrm{tor}}(M;\Lambda)} \exp 2\pi i q([f]) \\ &= N'(M) \exp 2\pi i \mathrm{Arf}(q)\end{aligned}$$

Where $\mathrm{Arf}(q) \in \frac{1}{8}\mathbb{Z}/\mathbb{Z}$ is the Arf invariant of q.

Using results in the theory of Gauss sums, and the fact that when $M = \partial W$, $\mathrm{WCS}(M, \check{A}) = \frac{1}{2}\int_W F \wedge (F + \lambda)$, we can show that

$$\mathrm{WCS}^{\mathrm{q}}(M) = N'(M) \exp 2\pi i \left(\frac{1}{2} \int_W \lambda^2 - \frac{1}{8} \sigma_{W,\Lambda} \right)$$

and

$$\mathrm{WCS}^{\mathrm{pq}}(M, \check{A}) \mathrm{WCS}^{\mathrm{q}}(M) = N'(M) \exp 2\pi i \left(\frac{1}{2} \int_W (F + \lambda)^2 - \frac{1}{8} \sigma_{W,\Lambda} \right)$$

where $\sigma_{W,\Lambda}$ is the signature of $H^{2\ell+2}_{\mathrm{free}}(W, M; \Lambda)$.

Quantum theory

The state space of the quantum theory on a $4\ell + 2$-dimensional manifold U is constructed using a discrete analogue of geometric quantization. [Witten hep-th/9812012], [Gurevich, Hadani 0705.4556], [S.M. - 1607.01396]

- The group $H^{2\ell+1}_{\text{free}}(U; \Lambda^*/\Lambda)$ carries an alternating pairing.
- It admits Lagrangian subgroups, over which the pairing vanishes.
- The state space is up to isomorphism the (finite-dimensional) algebra of functions over a Lagrangian subgroup.
- The subtleties arise in the construction of a canonical representative in this isomorphism class, independent of the choice of Lagrangian subgroup.

If Λ is unimodular, the state space is 1-dimensional.

Outline

Anomaly field theories

The anomalies of a d-dimensional quantum field theory $\mathcal{F}$ are best described using a quantum field theory in dimension $d+1$, the *anomaly field theory* $\mathcal{A}$. [Freed 1404.7224], [Moore, Klein Lectures, 2012]

Informally, the anomalous field theory $\mathcal{F}$ "takes value" in the anomaly field theory $\mathcal{A}$.

Equivalently, $\mathcal{A}$ is the "bulk theory" that makes non-anomalous the system with $\mathcal{A}$ in the bulk and $\mathcal{F}$ on the boundary.

In the Atiyah-Segal framework, $\mathcal{F}$ is a natural transformation $\mathcal{F} : 1 \to \mathcal{A}$ of field theory functors.

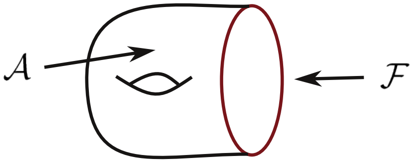
All the familiar properties of anomalies (and more!) can be recovered in this framework, including local and global anomalies, Hamiltonian anomalies, ... [S.M. 1410.7442]

ANOMALY FIELD THEORIES

Example: the quantum Hall effect:

- Chiral fermions on the boundary ($\mathcal{F}$). (Equivalent to chiral scalars/0-forms)
- An effective spin Chern-Simons theory in the bulk ($\mathcal{A}$), with Λ the charge lattice of the chiral fermions.

The boundary gauge variation of the spin Chern-Simons theory cancels the gauge anomalies of the chiral fermions.



What is $\mathcal{A}$ when $\mathcal{F}$ is a theory of chiral 2ℓ-forms in dimension $4\ell + 2$?

ANOMALY FIELD THEORIES

Recall that the local anomalies in dimension $4\ell + 2$ are described by degree $4\ell + 4$ characteristic form. For chiral 2ℓ-forms coupling to a degree $2\ell + 1$ abelian gauge field $\check{A} = (f, A, F)$:

$$c = \left[\frac{1}{8}L(R) - \frac{1}{2}F^2\right]_{(4\ell+4)}$$

More generally, global anomalies in dimension $4\ell + 2$ are described by a geometric invariant $\mathcal{A}$ in dimension $4\ell + 3$, which is nothing but the partition function of the anomaly field theory.

On a $4\ell + 3$-dimensional manifold M such that $M = \partial W$, the relation is

$$\frac{1}{2\pi i} \ln \mathcal{A}(M) = \int_W c + \text{ topological terms} \quad \text{mod } 1.$$

So what is $\mathcal{A}(M)$ for chiral 2ℓ-forms? First guess:

$$\frac{1}{2\pi i}\ln\mathcal{A}(M) \stackrel{?}{=} \int_W \left[\frac{1}{8}L(R) - \frac{1}{2}F^2\right]_{(4\ell+4)}$$

The right-hand side is generally not an integer when W is closed $\Rightarrow$ ill-defined. But

$$\frac{1}{2\pi i}\ln\mathcal{A}(M) = \int_W \left[\frac{1}{8}L(R) - \frac{1}{2}(F+\lambda)^2\right]_{(4\ell+4)}$$

is well-defined because

$$\int_W \left[\frac{1}{8}L(R) - \frac{1}{2}\lambda^2\right]_{(4\ell+4)} \quad \text{and} \quad -\frac{1}{2}\int_W F\wedge(F+\lambda)$$

are separately well-defined.

Anomaly field theories

The extra term $-\frac{1}{2}\int_W \lambda \wedge (F+\lambda)$ is indeed topological because λ is a relative form.

Some rewriting yields

$$\frac{1}{2\pi i} \ln \mathcal{A}(M) = \frac{1}{8}\left(\int_W [L(R)]_{(4\ell+4)} - \sigma_{W,\Lambda} \right) - \left(\frac{1}{2}\int_W (F+\lambda)^2 - \frac{1}{8}\sigma_{W,\Lambda} \right)$$

so

$$\mathcal{A}(M) = \left(\exp \frac{\pi i \sigma_\Lambda}{4} \eta_\sigma \right) \left(\mathrm{WCS}^{\mathrm{pq}}(M, \check{A}) \mathrm{WCS}^{\mathrm{q}}(M) \right)^\dagger$$

The anomaly field theory of chiral 2ℓ-form in dimension $4\ell+2$ therefore involves a Wu Chern-Simons theory in dimension $4\ell+3$.

Anomalies 6d SCFT

The anomaly field theory of 6d (2,0) SCFT was determined in the A_n case and conjectured in the D_n and E_n cases in [S.M. 1706.01903].

The A_n theory is constructed as follows:

1. Consider a stack of $n + 1$ M5-branes;
2. Take a suitable decoupling limit;
3. Subtract the free center of mass tensor multiplet.

Assuming all anomalies cancel in M-theory, we can compute the local [Harvey, Minasian, Moore hep-th/9808060] and global [S.M. 1406.4540] anomalies of the 6d SCFT.

Anomalies 6d SCFT

- The chiral 2-forms on the stack have (unimodular) charge lattice $\mathbb{Z}^{n+1}$.
- The chiral 2-form in the center of mass has (non-unimodular) charge lattice $\sqrt{n+1}\mathbb{Z}$.

Remarkably, we have a relation

$$\mathrm{WCS}^{\mathrm{q}}_{\mathbb{Z}^{n+1}} \otimes \left(\mathrm{WCS}^{\mathrm{q}}_{\sqrt{n+1}\mathbb{Z}}\right)^{\dagger} = \mathrm{WCS}^{\mathrm{q}}_{A_n}$$

The anomaly field theory of the 6d SCFT then involves a Wu Chern-Simons theory with charge lattice A_n (+ fermions and Hopf-Wess-Zumino).

Anomalies 6d SCFT

The A_n lattice is not unimodular. $\rightarrow \mathrm{WCS}^{\mathrm{q}}_{A_n}(M_6)$ has dimension > 1.
$\rightarrow$ The 6d SCFT has no single partition function, but a vector of "conformal blocks" [Witten hep-th/9812012].

The anomaly field theory determines completely the transformation law of the conformal blocks under diffeomorphisms and R-symmetry transformations.

It defines analogs of the S and T matrices of 2d CFTs on T^2. These matrices computable to the extent that the partition function of the anomaly field theory is computable on suitable 7-dimensional mapping tori.

The Green-Schwarz mechanism in 6d sugra

Based on [S.M, Moore - 1808.01334], see also [S.M, Moore - 1808.01335] and Greg Moore's talk at Strings'18.

6d supergravities require a version of the Green-Schwarz mechanism to cancel their anomalies.

The existing descriptions of the 6d GS mechanism were valid only in flat space for topologically trivial vectormultiplet gauge bundle.

In our recent work, we describe the Green-Schwarz (GS) mechanism for 6d sugra in topologically non-trivial situations and derive constraints from global anomaly cancellation.

The Green-Schwarz mechanism in 6d sugra

Let $\mathcal{A}$ the anomaly field theory of the "bare" supergravity theory $\mathcal{F}$ (i.e. without the GS terms). The GS mechanism goes as follows.

1. For any M_6, construct a vector in the state space $\mathcal{A}$ of the anomaly field theory, as the exponential of a term *local* in the fields of $\mathcal{F}$:
$$\exp 2\pi i \int_{M_6} \mathrm{GS} \in \mathcal{A}(M_6)$$
2. Subtract the Green-Schwarz term GS to the action of $\mathcal{F}$, or more precisely, define
$$\mathcal{F}'(M_6) := \mathcal{F}(M_6) \to \mathcal{F}(M_6) \otimes \exp -2\pi i \int_{M_6} \mathrm{GS} \ .$$

The Green-Schwarz mechanism in 6d sugra

The Green-Schwarz mechanism in 6d requires the local anomaly polynomial A_8 of the bare theory to factorize:

$$A_8 = Y \wedge Y$$

where Y is a degree 4 differential form valued in the charge lattice Λ of the chiral 2-forms.

The anomaly field theory $\mathcal{A}$'s partition function on $M_7 = \partial W$ given by

$$\frac{1}{2\pi i} \ln \mathcal{A}(M_7) = \frac{1}{2} \int_W Y_W \wedge Y_W - \frac{\sigma_{W,\Lambda)}}{8} \ .$$

Looks like $\mathcal{A} \simeq \mathcal{A}_{\mathrm{CT}} := \mathrm{WCS}^{\mathrm{pq}}(M_7, \check{A})\mathrm{WCS}^{\mathrm{q}}(M_7)$ for $\check{A}$ with field strength $Y - \lambda$.

The Green-Schwarz mechanism in 6d sugra

Using the cochain model for E-theory reviewed above, we construct construct a GS term in $\mathcal{A}_{\mathrm{CT}}(M^6)$ that reduces to the usual expression in topologically trivial situations.

However:

- $\mathcal{A}(M_7) = \mathcal{A}_{\mathrm{CT}}(M_7)$ holds only on manifolds M_7 that bound.
- In general, $\check{A}$ may differ from $\mathcal{A}_{\mathrm{CT}}$ by a bordism invariant.
- When they do differ, the GS mechanism fails to cancel certain global anomalies.
- For certain supergravity theories with discrete gauge group, we reproduce rather non-trivial properties of the matter representations arising in F-theory compactifications.

The Green-Schwarz mechanism in 6d sugra

We derive other interesting constraints:

1. The Green-Schwarz mechanism cannot work if $\mathcal{A}_{\text{CT}}$ is not invertible (i.e. has a state space of dimension > 1).

$\Rightarrow$ The chiral 2-forms charge lattice Λ has to be unimodular. Derived using independent arguments in [Taylor, Seiberg 1103.0019].

2. Y has the general form

$$Y = \frac{1}{4} a p_1 - \sum_i b_i c_2^i + \frac{1}{2} \sum_{IJ} b_{IJ} c_1^I c_1^J \, .$$

where p_1 is the first Pontryagin form, c_2^i the second Chern forms of the simple factors of the gauge group of $\mathcal{F}$ and c_i^I are the first Chern forms of the abelian factors. $a, b_i, , b_{IJ} \in \Lambda$.

The consistency of the Wu Chern-Simons theory requires that a is a characteristic element of Λ, i.e. $(x, x) = (x, \lambda) \bmod 2 \; \forall x \in \Lambda$.

This is true in F-theory (where a is the canonical class in the degree 2 homology lattice of the base), but a low energy derivation was missing until now.

Conclusion

- Wu Chern-Simons theories are abelian Chern-Simons theories "at half integer level". Their gauge group and level is parametrized by integral lattices, rather than even lattices only.
- The are spin Chern-Simons theories in dimension 3, and generalizes them to dimension $4\ell + 3$, with degree $2\ell + 1$ gauge fields.
- Their construction involves interesting algebraic topology, in particular, their Lagrangian is a generalized cohomology cocycle.
- Their quantization is subtle, but tractable.
- They are a component of the anomaly field theories of chiral 2ℓ-forms, which is how they appear in string theory.