Analyticity and Crossing Symmetry of Superstring Loop Amplitudes

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Consider a scattering amplitude of n particles with momenta $\{p_a\}$ satisfying on-shell condition

$$p_a^2 + m_a^2 = 0, \qquad m_a > 0$$

(all momenta ingoing, $p_a^2 \equiv -(p_a^0)^2 + \vec{p}_a^{\,2}$)

Define

$$s_{ab} = -(p_a + p_b)^2 = m_a^2 + m_b^2 - 2p_a.p_b$$

In what domain in the complex s_{ab} space is the scattering amplitude analytic?

– useful starting point for studying many other properties, e.g. crossing symmetry, dispersion relations etc.

How is this studied in local QFT?

Consider the off-shell amputated Green's function $G(p_1, \cdots p_n)$

$$S(p_1, \cdots p_n) = G(p_1, \cdots p_n)|_{p_a^2=-m_a^2}$$

We have dropped the Lorentz indices for notational convenience.

Define $P_{(\alpha)} = \sum_{a\in A_\alpha} p_a$, $\qquad A_\alpha \subset \{1, 2, \cdots, n\}$

Based on the locality of the position space Green's function one can show that $G(p_1, \cdots p_n)$ is analytic in the domain

$$\{Im(P_{(\alpha)}) \neq 0, \quad (Im(P_{(\alpha)}))^2 \leq 0\}$$

$$\text{or} \quad \{Im(P_{(\alpha)}) = 0, \; -P_{(\alpha)}^2 < M_\alpha^2\}, \quad \forall A_\alpha$$

M_α: Threshold of production of multi-particle states in the channel A_α

$G(p_1, \cdots p_n)$ is analytic in the domain

$$\{\mathrm{Im}(P_{(\alpha)}) \neq 0, \quad (\mathrm{Im}(P_{(\alpha)}))^2 \leq 0\}$$

$$\text{or} \quad \{\mathrm{Im}(P_{(\alpha)}) = 0,\ -P^2_{(\alpha)} < M^2_\alpha\}, \quad \forall A_\alpha$$

We shall call this the primitive domain of analyticity

We shall now see that this manifold has
zero intersection with the subspace in which external states are on-shell

$$p_a^2 + m_a^2 = 0 \quad \text{for } a = 1,2,\cdots,n$$

Proof:

$$\{\mathrm{Im}(P_{(\alpha)}) \neq 0, \quad (\mathrm{Im}(P_{(\alpha)}))^2 \leq 0\}$$

$$\text{or} \quad \{\mathrm{Im}(P_{(\alpha)}) = 0, -P^2_{(\alpha)} < M^2_\alpha\}, \quad \forall A_\alpha$$

If $p_a = p_{aR} + i\, p_{aI}$ then

$$-m_a^2 = p_a^2 = p_{aR}^2 - p_{aI}^2 + 2\, i\, p_{aR}.p_{aI}$$

$$p_{aR}.p_{aI} = 0 \;\Rightarrow\; \quad p_{aI}^2 \geq 0 \;\text{ or }\; p_{aR}^2 \geq 0 \;\text{ or both} \geq 0$$

$$m_a^2 = p_{aI}^2 - p_{aR}^2 \quad \Rightarrow \quad p_{aI}^2 > 0$$

– conflict with first condition

$\Rightarrow$ in order to have intersection with the primitive domain of analyticity we must have p_a real.

$$\{\mathrm{Im}(P_{(\alpha)}) \neq 0, \quad (\mathrm{Im}(P_{(\alpha)}))^2 \leq 0\}$$

$$\text{or} \quad \{\mathrm{Im}(P_{(\alpha)}) = 0, \, -P^2_{(\alpha)} < M^2_\alpha\}, \quad \forall A_\alpha$$

All p_a's real $\Rightarrow P_{(\alpha)}$ real

But now $P_{(\alpha)}$ below threshold is impossible to satisfy since we can always produce the incoming states as intermediate states in any channel

– conflict with second condition

Apparent conclusion: Analyticity of G in the primitive domain is useless for exploring analyticity properties of S-matrix

The way out:

Suppose $f(z_1, \cdots z_n)$ is function of multiple complex variables, known to be analytic in some domain D.

Without knowing anything else about f, we can often prove that the function is analytic in an extended domain D'.

D' depends only on D and not on f.

Example

Suppose $f(z_1, z_2)$ is known to be analytic in $D = D_1 \cup D_2$:

$$D_1 = \{1 - \epsilon < |z_1| < 1,\ |z_2| < 1\}, \quad D_2 = \{1 - \epsilon < |z_2| < 1,\ |z_1| < 1\}$$

Claim: $f(z_1, z_2)$ is analytic in

$$D' : \quad |z_1| < 1, \quad |z_2| < 1$$

Proof: Given $(z_1, z_2) \in D'$, define

$$g(z_1, z_2) = \int_C \frac{d\zeta_1}{\zeta_1 - z_1} f(\zeta_1, z_2), \quad C \in 1 - \epsilon < |\zeta_1| < 1$$

g agrees with f when (z_1, z_2) lie in D_2.

$\Rightarrow$ g is the analytic continuation of f in D'.

Using this kind of argument we can extend the domain of analyticity of $G(p_1, \cdots p_n)$ beyond the primitive domain.

Jost, Lehmann; Dyson; Bros, Messiah, Stora; $\cdots$

The extended domain includes points satisfying $p_a^2 + m_a^2 = 0$

– can be used to prove interesting results for $2 \to 2$ scattering

1. Crossing symmetry: Existence of an analytic continuation relating

Bros, Epstein, Glaser

$$A + B \to C + D \quad \Rightarrow \quad A + \bar{C} \to \bar{B} + D$$

2. Analyticity of the elastic forward scattering amplitude (t=0) in the full complex s-plane

see Itzykson-Zuber

etc.

What about in superstring theory?

To follow the same approach we need off-shell Green's function

– use superstring field theory

Our goal is to prove analyticity of $G(p_1, \cdots p_n)$ in the primitive domain D:

$$\{\mathrm{Im}(P_{(\alpha)}) \neq 0, \quad (\mathrm{Im}(P_{(\alpha)}))^2 \leq 0\}$$

$$\text{or} \quad \{\mathrm{Im}(P_{(\alpha)}) = 0, -P^2_{(\alpha)} < M^2_\alpha\}, \quad \forall A_\alpha$$

We do not have the position space Green's functions as the starting point.

Strategy: Try to prove these properties directly in the momentum space using Feynman diagrams.

Superstring field theory has infinite number of fields $\{\phi_\alpha\}$.

Gauge fixed action:

$$\int \frac{d^D k}{(2\pi)^D} K_{\alpha\beta}(k)\phi^\alpha(k)\phi^\beta(-k)$$
$$+\sum_n \int \frac{d^D k_1}{(2\pi)^D}\cdots\frac{d^D k_1}{(2\pi)^D}\,(2\pi)^D\delta^{(D)}(k_1+\cdots+k_n)$$
$$V^{(n)}_{\alpha_1\cdots\alpha_n}(k_1,\cdots k_n)\,\phi^{\alpha_1}(k_1)\cdots\phi^{\alpha_n}(k_n)$$

Kinetic term $K_{\alpha\beta}$: quadratic in momentum k

$V^{(n)}$ has exponential suppression factors that makes it suppressed as k_i^0's approach $\pm i\infty$ and k_n^i's approach $\pm\infty$.

Each Feynman diagram is manifestly UV finite as long as the ends of k_s^0 integration contours are at $\pm i\infty$ and the ends of k_s^i integration contours are at $\pm\infty$.

Infrared issues

In the presence of massless states S-matrix has infrared singularities

– need subtraction / regulation

Subtraction:

Replace each internal propagator by (1-P)

P: projection operator to massless fields.

Analyticity will be analyzed for this amplitude

The effect of massless internal states will have to be taken care of separately.

Regulation:

Add mass term for massless particles in the gauge fixed action.

The regulated amplitude is free from IR singularities, and one can analyze analyticity of this amplitude.

For D $>$ 4 the regulated amplitude approaches the actual amplitude for small mass.

From now on we shall proceed by assuming that there are no strictly zero mass particles in the theory.

We begin by analyzing $G(p_1, \cdots p_n)$ at $p_a = 0$ for $a = 1, \cdots n$.

Take the loop energy integration contour along imaginary axis and the integration contour for spatial components along real axis.

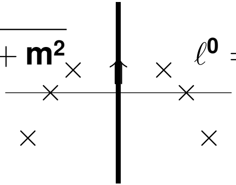
For an internal propagator with momentum ℓ, ℓ^0 is imaginary and ℓ^i are real for $i \geq 1$.

$\Rightarrow \ell^2 + m^2 = -(\ell^0)^2 + \vec{\ell}^{\,2} + m^2$ is strictly positive, and the integrals are well defined.

Therefore $G(p_1, \cdots p_n)$ is analytic in the neighborhood of $\{p_a = 0\}$

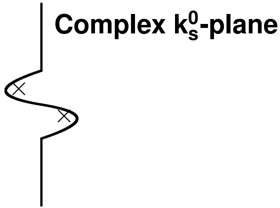
Distribution of poles in a complex loop energy (k^0) plane

$$\ell^0 = -\sqrt{\vec{\ell}^{\,2} + m^2} \qquad \ell^0 = \sqrt{\vec{\ell}^{\,2} + m^2}$$



As we deform the external momenta $\{p_a\}$ away from 0 the pole positions move.

If the poles approach the integration contour, we move the contour away from the poles to avoid singularities, but keep the ends fixed for UV finiteness.



When a contour is ‘pinched’ by poles from two sides so that we cannot deform the contour, the integral becomes singular.

Our goal will be to show that this does not happen inside the primitive domain D.

We have been able to prove a limited version of the result in which Im(p_a)'s lie in the 0-1 plane

– sufficient to prove all the known analyticity properties of S-matrix proved for local QFT

We carry out the analysis in two steps.

1. Keep $\mathrm{Im}(p_a^1) = 0$ and deform all other components of all external momenta from 0 to the desired value remaining inside D.

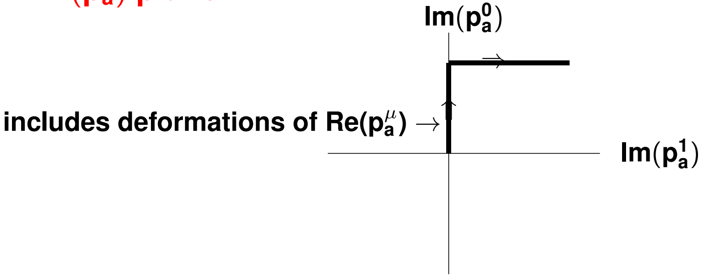
Only p_a^0's have imaginary part.

Show that we do not encounter any pinch singularity during this deformation.

2. Deform $\mathrm{Im}(p_a^1)$ from 0 to the desired value keeping all other components fixed, and remaining inside D.

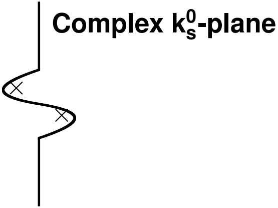
Show that we do not encounter a pinch singularity during this deformation.

A schematic representation of the deformation in $\mathrm{Im}(p_a)$ plane:



Step 1: Deform all components of p_a other than $\mathrm{Im}(p_a^1)$ to the desired values, keeping $\mathrm{Im}(p_a^1)$ fixed at 0.

During this deformation we continue to integrate the spatial components of loop momenta $\{k_s^i\}$ along the real axis, but allow the k_s^0 integration contours to be deformed, keeping their ends fixed at $\pm i\infty$.



Goal: Show that the k_s^0 contours are never pinched.

To prove this we assume that the k_s^0 contour is pinched and show that there is a contradiction.

First time we hit a pinch singularity, certain internal propagators are forced to be on-shell due to pinch.

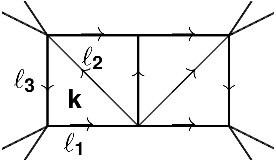
Associate with the pinch a reduced diagram, obtained by shrinking to point all propagators not forced to be on-shell at that point.

For an on-shell propagator carrying momentum ℓ, we have

$$\ell^0 = \pm\sqrt{\vec{\ell}^{\,2} + m^2}$$

Draw an arrow along ℓ if $\ell^0 = \sqrt{\vec{\ell}^{\,2} + m^2}$ and opposite to ℓ if $\ell^0 = -\sqrt{\vec{\ell}^{\,2} + m^2}$ at the pinch.

Claim: The arrows in a reduced diagram cannot form a closed loop

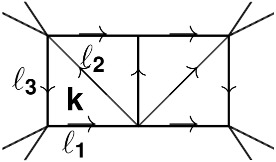


Let k be the momentum along such a loop and ℓ_r ($r \in \mathcal{B}$) be the momenta carried along the arrow by individual propagators.

$\ell_r = k + L_r$ with L_r determined by external momenta and other loop momenta.

For $\{p_a = 0\}$, the pole at $\ell_r^0 = \sqrt{\vec{\ell}^{\,2} + m^2}$ is to the right of k^0 integration contour.

This cannot change during the deformation since the poles do not cross the integration contour.

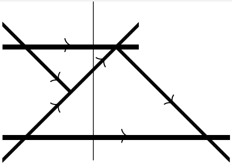


All the poles in k^0 plane are to the right of k^0 integration contour at the pinch.

$\Rightarrow$ k^0 integration contour is not pinched.

Conclusion: A reduced diagram cannot have any loop along which the arrows are unidirectional.

$\Rightarrow$ the vertices admit partial ordering so that arrows flow from left to right.



$\mathcal{A} \equiv$ **a set of propagators cut by a vertical line.**

Total momentum flowing across the vertical line: some $P_{(\alpha)}$

$$\vec{P}_{(\alpha)} = \sum_{r \in \mathcal{A}} \vec{\ell}_r, \quad P^0_{(\alpha)} = \sum_{r \in \mathcal{A}} \sqrt{\vec{\ell}_r^{\,2} + m_r^2}$$

Note: $P_{(\alpha)}$ is real and is above the threshold since it is the sum of momenta carried by on-shell particles in the set $\mathcal{A}$.

– contradicts the condition for the domain D

$\Rightarrow$ we do not encounter a singularity during step 1.

Step 2: Deform $\text{Im}(p_a^1)$'s from 0 to the desired values keeping all other components of $\{p_a\}$ fixed.

We choose the integration contours for loop momenta $\{k_s\}$ as follows:

1. $\{k_s^2, \cdots k_s^{D-1}\}$ are integrated along the real axis.

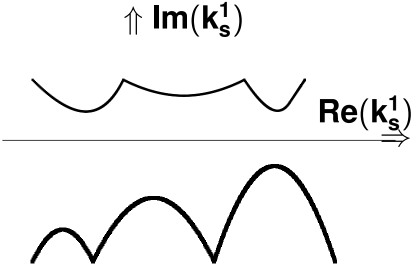
2. $\{k_s^1\}$ contours are allowed to be deformed away from real axis with their ends kept fixed at $\pm\infty$.

3. The integration contours of $\{k_s^0\}$ are taken to be the same as at the end of step 1 for $\{\text{Re}(k_s^1), k_s^2, \cdots k_s^{D-1}\}$.

Our goal will be to show that by appropriate choice of $\{k_s^1\}$ contours, we can avoid pinch.

Consider the configuration at the end of first step.

Plot in the complex k_s^1 plane the location of the poles with lowest $|Im(k_s^1)|$ for given $Re(k_s^1)$ as we vary all other loop momenta along their chosen contours.



– singular loci

$\mathcal{B}_s$: set of propagators through which k_s flows.

ℓ_r: momenta carried by individual propagators in the set $\mathcal{B}_s$

$$\ell_r = k_s + L_r, , \; r \in \mathcal{B}_s$$

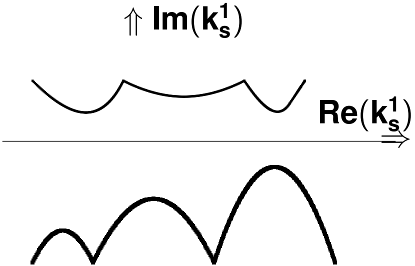
L_r are linear combinations of external and other loop momenta.

The poles are at

$$\ell_r^1 = \pm i \sqrt{\ell_{r\perp}^2 - (\ell_r^0)^2 + m_r^2} = \pm i \sqrt{(k_{s\perp} + L_{r\perp})^2 - (k_s^0 + L_r^0)^2 + m_r^2}$$

$$\ell_\perp \equiv (\ell^2, \cdots \ell^{D-1})$$

At the poles, k_s^1 and $\ell_r^1 = k_s^1 + L_r^1$ have same imaginary parts since L_r^1 are all real at the end of step 1.



On the upper curve, Im(ℓ_r^1) $>$ 0 since Im(k_s^1) $>$ 0.

On the lower curve, Im(ℓ_r^1) $<$ 0 since Im(k_s^1) $<$ 0.

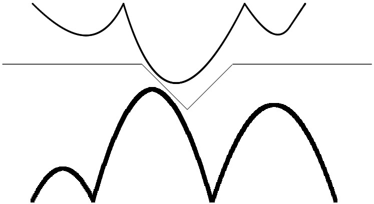
We shall now deform Im(p_a^1) keeping other components fixed, and see what happens to the singular loci.

The poles are at

$$\ell_r^1 = \pm i\sqrt{\ell_{r\perp}^2 - (\ell_r^0)^2 + m_r^2} = \pm i\sqrt{(k_{s\perp} + L_{r\perp})^2 - (k_s^0 + L_r^0)^2 + m_r^2}$$

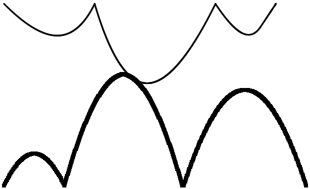
In the k_s^1 plane they are at $k_s^1 = \ell_r^1 - L_r^1$

While ℓ_r^1 values at the poles remain unchanged during step 2, the k_s^1 values shift vertically since the L_r^1's acquire imaginary parts.



We can deform the k_s^1 contour and make it pass through the gap.

Singularity appears when the contour is pinched.



Recall that on the upper curve $\mathrm{Im}(\ell_{\mathbf{r}}^{1}) > 0$ and on the lower curve $\mathrm{Im}(\ell_{\mathbf{r}'}^{1}) < 0$ with $\mathbf{r}, \mathbf{r}' \in \mathcal{B}_{\mathbf{s}}$.

This shows that in order to hit a singularity in the k_{s}^{1} integration contour, at least a pair of propagators along the loop must have opposite values of $\mathrm{Im}(\ell_{\mathbf{r}}^{1})$ at the pinch.

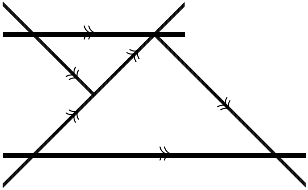
Must hold for every loop.

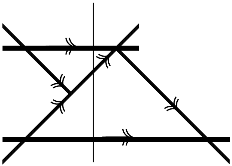
At the pinch, at least a pair of propagators along any loop must have opposite values of $\mathrm{Im}(\ell_r^1)$ at the pinch.

We now associate to each propagator of a reduced diagram a double arrow, showing the direction of $\mathrm{Im}(\ell_r^1)$.

We cannot traverse a loop by moving along the double arrows.

$\Rightarrow$ we can assign partial ordering to the vertices so that $\mathrm{Im}(\ell_r^1)$ flows from left to right.





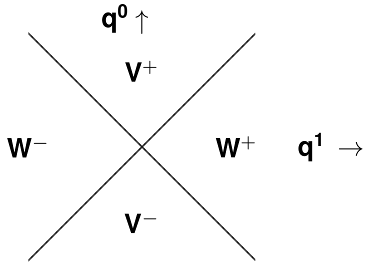
If now draw a vertical cut through propagators carrying momenta ℓ_r ($r \in \mathcal{B}$), and if $P_{(\alpha)}$ is the total momenta entering the graph from right, we have

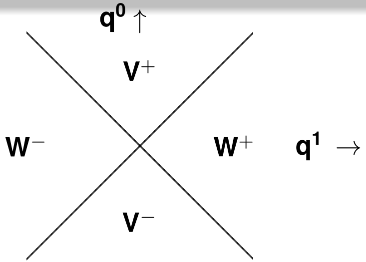
$$P_{(\alpha)} = \sum_{r\in\mathcal{B}} \ell_r, \qquad \ell_r^2 + m_r^2 = 0, \quad \mathrm{Im}(\ell_r^1) > 0$$

Simple kinematic analysis now shows that this is incompatible with $\{p_a\} \in D$.

Define four quadrants in 0-1 plane parametrized by (q^0, q^1):

$V^+ : q^0 > |q^1|, \quad V^- : q^0 < -|q^1|, \quad W^+ : q^1 > |q^0|, \quad W^- : q^1 < -|q^0|$





$$\mathbf{P}_{(\alpha)} = \sum_{r \in \mathcal{B}} \ell_r, \qquad \ell_r^2 + m_r^2 = 0, \quad \mathrm{Im}(\ell_r^1) > 0$$

Recall that if ℓ_r is on-shell then Im(ℓ_r) must be space-like; hence

$$\mathrm{Im}(\ell_r^1) > 0 \quad \Rightarrow \quad \mathrm{Im}(\ell_r) \in W^+$$

$$\mathrm{Im}(P_{(\alpha)}) \in W^+ \quad \Rightarrow \quad \mathrm{Im}(P_{(\alpha)}) \text{ space-like}$$

– incompatible with $\mathrm{Im}(P_{(\alpha)})^2 < 0$

Conclusion: Our initial assumption must be wrong

$\Rightarrow$ the k_s^1 contour is not pinched.

$\Rightarrow$ the off-shell Green's function is analytic in the domain D.

This proves the analyticity of the off-shell Green's function of string field theory in the primitive domain of analyticity $\cdots$

$\cdots$ with $\mathrm{Im}(p_a)$ lying in a 2-dimensional Lorentzian plane.

References (both talks)

1. Spurious poles: arXiv:1504.00609

2. Superstring field theory: arXiv:1508.05387

3. Action for IIB supergravity: arXiv:1511.08220

4. Analyticity: arXiv:1810.07197