

$\frac{\infty}{2}$ -Hodge structures:
from BCOV theory to Seiberg-Witten geometry

Si Li

YMSC, Tsinghua University

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$\frac{\infty}{2}$ -Hodge structure

$\frac{\infty}{2}$ -Hodge structure has its origin from **K. Saito**'s theory of higher residues and primitive form in his study of period maps for isolated singularities. This is generalized and systematically developed in Calabi-Yau geometry by **Barannikov-Kontsevich**.

In this talk, we explain the role of $\frac{\infty}{2}$ -Hodge structure in

1. B-twisted topological string field theory (BCOV theory)
2. Seiberg-Witten geometry realized via singularities.

Outline

Singularity and $\frac{\infty}{2}$ -Hodge Structure

BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

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BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

We motivate $\frac{\infty}{2}$ -HS from K.Saito's original version.

- ▶ de Rham theory: let X be a compact oriented n -manifold

$$\int : \Omega_X^\bullet \rightarrow \mathbb{R}$$
$$\alpha \in \Omega^n(X) \rightarrow [\alpha] \in H_{dR}^n(X) \simeq \mathbb{R}.$$

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$\implies (\Omega^\bullet(X), d)$ gives an algebraic model for integration.

- ▶ K. Saito's theory of primitive forms can be viewed as an algebraic model for understanding complex oscillatory integral

$$\int_{\Gamma} e^{f/t}(\dots)$$

Isolated singularity

Let

$$f : X = (\mathbb{C}^{n+1}, 0) \rightarrow \Delta = (\mathbb{C}, 0)$$

be the germ of an isolated singularity. We consider the quotient

$$\Omega_f := \Omega_X^{n+1} / df \wedge \Omega_X^n.$$

With a choice of holomorphic volume form $d\mathbf{x} = dx_0 \wedge \cdots \wedge dx_n$

$$\Omega_f = \text{Jac}(f) d\mathbf{x}$$

where $\text{Jac}(f) = \mathbb{C}\{x_0, \dots, x_n\} / (\partial_i f)$ is the Jacobian algebra of f at 0. There is a non-degenerate pairing on Ω_f given by residue

$$\text{Res}_f : \Omega_f \otimes \Omega_f \rightarrow \mathbb{C}.$$

Brieskorn lattice

The space Ω_f is the leading part of the **Brieskorn lattice**

$$\mathcal{H}_f^{(0)} := \Omega_X^{n+1} / df \wedge d\Omega_X^{n-1}.$$

There is a well-defined operator, denoted by a formal variable t ,

$$t : \mathcal{H}_f^{(0)} \rightarrow \mathcal{H}_f^{(0)}.$$

Given $\alpha \in \mathcal{H}_f^{(0)}$, there exists n -form β such that $\alpha = d\beta$, then

$$t \cdot \alpha := -df \wedge \beta \in \mathcal{H}_f^{(0)}.$$

Symbolically,

$$t = -\frac{df}{d} : \alpha \rightarrow -df \wedge d^{-1}\alpha.$$

Descendant forms

Given $\omega \in \mathcal{H}_f^{(0)}$ and $k \geq 0$, we define its k -th descendant form

$$\boxed{\omega^{(-k)} := (-t)^k \omega} \in \mathcal{H}_f^{(0)}.$$

Descendant forms give natural semi-infinite filtrations

$$\dots \subset \mathcal{H}_f^{(-k)} \subset \mathcal{H}_f^{(-k+1)} \subset \dots \subset \mathcal{H}_f^{(-1)} \subset \mathcal{H}_f^{(0)}$$

The formal completion of $\mathcal{H}_f^{(0)}$ w.r.t. this t -adic topology identifies

$$\boxed{\hat{\mathcal{H}}_f^{(0)} = \Omega_X^{n+1}[[t]] / (td + df)\Omega_X^n[[t]]}.$$

Higher residue

K. Saito defines a sesqui-linear **higher residue pairing**

$$K_f : \hat{\mathcal{H}}_f^{(0)} \times \hat{\mathcal{H}}_f^{(0)} \rightarrow t^n \mathbb{C}[[t]].$$

whose leading term coincides with the residue pairing on

$$\boxed{\Omega_f = \mathcal{H}_f^{(0)} / t\mathcal{H}_f^{(0)}}.$$

We can further extend K_f to

$$\boxed{\hat{\mathcal{H}}_f := \Omega_X^{n+1}((t)) / (td + df) = \hat{\mathcal{H}}_f^{(0)}[t^{-1}]}$$

and write

$$K_f : \hat{\mathcal{H}}_f \times \hat{\mathcal{H}}_f \rightarrow \mathbb{C}((t)).$$

$\frac{\infty}{2}$ -Hodge structure

A $\frac{\infty}{2}$ -Hodge structure of weight n consists of $(\mathcal{H}, \mathcal{E}, \nabla, K)$ where

1. $\mathcal{H}$ is a finite dim vector space over $\mathbb{C}((t))$;
2. $\mathcal{E}$ is a $\mathbb{C}[[t]]$ -lattice;
3. ∇ is a meromorphic connection on the formal disk such that

$$\nabla_{\frac{\partial}{\partial t}} \mathcal{E} \subset t^{-2} \mathcal{E}$$

4. $K : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}((t))$ a ∇ -compatible pairing (weight n) s.t.
 - ▶ $K(v(t)\alpha, \beta) = K(\alpha, v(-t)\beta) = v(t)K(\alpha, \beta)$;
 - ▶ $K(\alpha, \beta) = (-1)^n K(\beta, \alpha)^*$ where $*$ -operator takes $t \rightarrow -t$;
 - ▶ $K : \mathcal{E} \times \mathcal{E} \rightarrow t^n \mathbb{C}[[t]]$ with non-degenerate leading pairing

$$t^{-n} K : \mathcal{E}/t\mathcal{E} \times \mathcal{E}/t\mathcal{E} \rightarrow \mathbb{C}$$

$(\hat{\mathcal{H}}_f, \hat{\mathcal{H}}_f^{(0)}, \nabla, K_f)$ forms a weight n $\frac{\infty}{2}$ -HS, which varies along deformations of f exhibiting good properties.

Oscillatory integral and Period map

Given an element $\omega \in \mathcal{H}_f^{(0)}$, we can consider the oscillatory integral

$$\int_{\Gamma} e^{f/t} \omega.$$

Under Laplace transformation, this is related to the period map

$$\boxed{\int_{\gamma} \frac{\omega}{df}}.$$

In Seiberg-Witten **curve** geometry, we have a 2-form ω with $\omega = d\lambda$. Then the SW period map is related to the period map of the first descendant of ω

$$\int_{\gamma} \lambda = \int_{\gamma} \frac{df \wedge d^{-1}\omega}{df} = - \int_{\gamma} \frac{\omega^{(-1)}}{df}.$$

As we will see, this observation allows us to obtain SW differential arising from higher dimensional geometry (in particular 3-fold fibration) via different choices of the descendants.

Primitive form

Let $\mathcal{M}$ be the miniversal deformation space of $f(x)$, represented by a universal unfolding $F(x^i, \lambda^\alpha)$, $\lambda \in \mathcal{M}$. Using $\frac{\infty}{2}$ -HS, K. Saito [1982] constructed a special family of holomorphic volume forms $\xi(x, \lambda) = \varphi(x, \lambda) d\mathbf{x}$, called **primitive form**. It determines a set of **flat coordinates** $\{\tau^\alpha\}$ on $\mathcal{M}$ such that

$$\left(t \frac{\partial}{\partial \tau^\alpha} \frac{\partial}{\partial \tau^\beta} - A_{\alpha\beta}^\gamma(\tau) \frac{\partial}{\partial \tau^\gamma} \right) \int e^{F/t} \xi = 0.$$

$A_{\alpha\beta}^\gamma$ is nowadays the Yukawa coupling of 2d LG B-models.

On Calabi-Yau geometries, the analogue of primitive form is called $\frac{\infty}{2}$ -period map. In the context of Gromov-Witten theory, this is related to Givental's J-function.

Singularity and $\frac{\infty}{2}$ -Hodge Structure

BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

B-model string field theory

- ▶ Let (X, Ω_X) be a compact Calabi-Yau manifold.
- ▶ [Bershadsky-Cecotti-Ooguri-Vafa, 1994]: B-twisted topological string field theory on Calabi-Yau 3-fold

→ Kodaira-Spencer gravity

which describes deformations of Ricci-flat metrics in terms of complex structures (Calabi-Conjecture/Yau-Theorem).

- ▶ [Costello-L, 2012] An extension of Kodaira-Spencer gravity in the sense of Zwiebach is formulated on arbitrary Calabi-Yau

→ BCOV theory.

This can be coupled with Witten's open string field theory (HCS) in terms of a cyclic version of Kontsevich's Formality.

Deformation of Calabi-Yau structure

The pair deformation (X, Ω_X) can be specified by a pair (μ, ρ)

$$\mu \in \mathcal{A}^{0,1}(X, T_X^{1,0}), \quad \rho \in C^\infty(X)$$

where μ specifies the **new complex structure**, and ρ specifies the **new Calabi-Yau volume form**. They satisfy the equations

$$\begin{cases} \bar{\partial}\mu + \frac{1}{2}[\mu, \mu] = 0 \\ d(e^\rho e^\mu \lrcorner \Omega_0) = 0 \end{cases}$$

where Ω_0 is the CY form on X_0 . This equation is equivalent to

$$Q\tilde{\mu} + \frac{1}{2}[\tilde{\mu}, \tilde{\mu}] = 0$$

where

$$\tilde{\mu} = \mu + t\rho, \quad Q = \bar{\partial} + t\partial.$$

∂ is the divergence operator w.r.t. the CY volume form Ω_0 .

BCOV theory: fields

We define the fields of BCOV theory on Calabi-Yau X by

$$\boxed{\mathcal{E} := \mathrm{PV}(X)[[t]]}, \quad \mathrm{PV}(X) = \bigoplus_{i,j} \Omega^{0,j}(X, \wedge^i T_X).$$

$\mathcal{E}$ has a differential $\boxed{Q := \bar{\partial} + t\partial}$ (∂ is the divergence operator) and Schouten-Nijenhuis bracket $[-, -]$. The associated Maurer-Cartan equation describes (extended) [deformation of Calabi-Yau structure](#). This is used by Barannikov-Kontsevich to obtain Frobenius manifold structure for compact Calabi-Yau.

$\mathcal{E}$ carries the analogue of higher residue pairing

$$\boxed{K(f(t)\alpha, g(t)\beta) := f(t)g(-t) \int \alpha \wedge \beta} \in \mathbb{C}[[t]].$$

BCOV theory: equation of motion

- ▶ The equation of motion describes **deforming CY structure**

$$Q\mu + \frac{1}{2}[\mu, \mu] = 0, \quad \mu \in \mathrm{PV}(X)[[t]].$$

This is a resolution of BCOV's Kodaira-Spencer gravity on divergence free polyvectors $\ker \partial \subset \mathrm{PV}(X)$.

- ▶ To generalize Kodaira-Spencer gravity, we need to find an action functional whose variation gives the above equation. Unfortunately **such action does not seem exist**.
- ▶ **Solution** [Costello-L, 2012]: After we perform a nonlinear

L_∞ transformation

of the above equation, we can write down a local interaction whose variation gives the transformed equation. We call this **BCOV interaction**.

BCOV interaction via period map

The nonlinear transformation that allows us to write down local interaction is given by the cochain period map as follows. Let

$$S(X) := \text{PV}(X)((t))$$

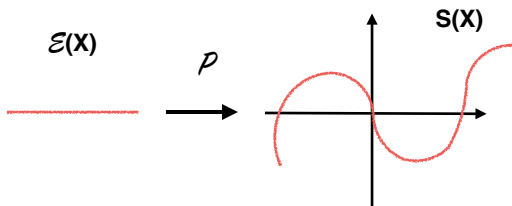
equipped with a symplectic pairing

$$\omega(f(t)\alpha, g(t)\beta) = \text{Res}_{t=0}(f(t)g(-t)dt) \text{Tr}(\alpha \wedge \beta).$$

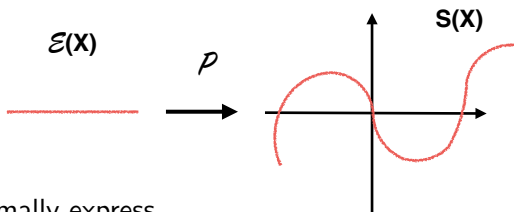
Consider the following period map (strictly speaking, this is a formal map between two functors on Artinian rings)

$$\mathcal{P} : \mathcal{E}(X) = \text{PV}(X)[[t]] \rightarrow S(X),$$

$$\mu \rightarrow t \left(e^{\mu/t} - 1 \right)$$



$\mathcal{P}$ embeds $\mathcal{E}(X)$ as a formal **lagrangian** submanifold of $S(X)$!



We can formally express

$$S(X) \quad " = " \quad T^*\mathcal{E}(X).$$

Let $\mathbf{I}_0$ be the generating functional on $\mathcal{E}(X)$ for the image of period map

$$\text{im } \mathcal{P} = \text{Graph}(d\mathbf{I}_0).$$

Theorem (Costello-L)

The functional $\mathbf{l}_0$ is local and given by

$$\mathbf{l}_0(\mu) := \int \langle e^\mu \rangle_0$$

where

$$\left\langle t^{k_1} \mu_1, \dots, t^{k_n} \mu_n \right\rangle_0 := \binom{n-3}{k_1, \dots, k_n} \mu_1 \wedge \dots \wedge \mu_n.$$

Moreover, $\mathcal{E}(X)$ has a (-1) -shifted degenerate Poisson structure inducing a (degenerate) BV bracket $\{-, -\}$. Then $\mathbf{l}_0$ satisfies the classical BV master equation

$$Q\mathbf{l}_0 + \frac{1}{2} \{\mathbf{l}_0, \mathbf{l}_0\} = 0.$$

Remark: Note that the leading **cubic** part of $\mathbf{l}_0$ is the interaction of Kodaira-Spencer gravity. A.Losev has a similar construction for a finite dimensional model of $PV(X)$.

The L_∞ transformation $\mathcal{P}_+$

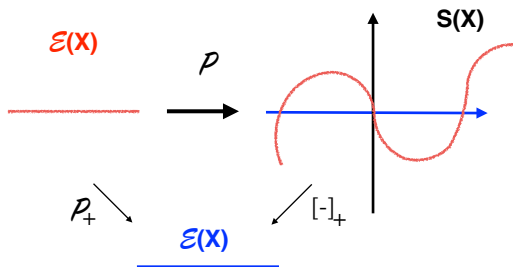
The cohomological vector field $Q + \{\mathbf{l}_0, -\}$ defines a (local) L_∞ -structure on $\mathcal{E}(X)$. The nonlinear transformation

$$\mathcal{P}_+ : \mathcal{E}(X) \rightarrow \mathcal{E}(X)$$

$$\mu = \sum_{k \geq 0} t^k \mu_k \rightarrow \left[te^{\mu/t} - t \right]_+$$

identifies

$$\text{solutions of } Q\mu + \frac{1}{2}[\mu, \mu] = 0 \xrightarrow{\mathcal{P}_+} \text{zero locus of } Q + \{\mathbf{l}_0, -\}$$



Quantum master equation

Start with a solution of classical master equation, there is a standard quantization procedure in the BV-formalism. It amounts to find quantum corrected functional

$$I_0 \rightarrow I = I_0 + I_1 \hbar + \dots$$

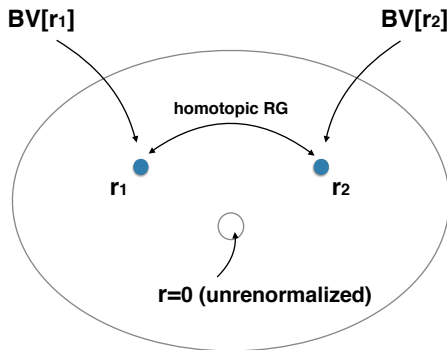
solving the quantum master equation

$$QI + \hbar \Delta I + \frac{1}{2} \{I, I\} = 0.$$

Here Δ is the BV-operator associated to the shifted Poisson structure above. However, $PV(X)$ is infinite dimensional and the above equation is not well-defined suffering from UV divergence.

Costello's homotopic renormalization

One rigorous approach to the above quantum master equation is achieved by Costello's homotopic renormalization method.



Higher genus B-model

After solving quantum master equation, we obtain a generating function $\mathbf{F}_g^B$ on the zero modes

$$H^\bullet(\mathcal{E}(X), Q) \overset{\text{splitting of Hodge}}{\simeq} H^\bullet(X, \wedge^\bullet T_X)[[t]]$$

by collecting the g -loop Feynman diagrams. This gives **higher genus B-model invariants** in our generalized BCOV theory which are conjectured mirror to higher genus Gromov-Witten invariants (with descendants) . Its dependence on the choice of splitting of the Hodge filtration gives rise to **holomorphic anomaly equation**.

The geometric interpretation of quantum BV master equation is that it defines a Q -closed element

$$Q|F\rangle = 0$$

in the Fock representation of the Weyl algebra that quantizes the dg symplectic space $(S(X), Q, \omega)$.

Application: an explanation for integrable hierarchy

Counting curves on CY geometry always leads to (KdV-type) **integrable hierarchy**. We propose an explanation in the B-model in terms of **quantum master equation** of BCOV theory. ([L, 2017], also work in progress with He and Yoo):

Let X be a Calabi-Yau geometry, we iterate B-model by considering the product $X \times \mathbb{C}$ (again Calabi-Yau).

B-model (BCOV theory) on

$$X \times \mathbb{C}$$

integration out over X

$$\downarrow$$

Effective 2d chiral theory on

$$\mathbb{C}$$

$$\boxed{\text{BV master equation on } \mathbb{C}} \implies \boxed{\text{integrable hierarchy associated to } X}$$

More precisely, there is an (∞ -dim) abelian Lie algebra

$$H^*(X)[[t]] \otimes \text{translation on } \mathbb{C}$$

acting as a symmetry of the effective theory on $\mathbb{C}$, whose Noether currents give commuting currents. It can be shown that at genus 0, such commuting currents lead to **Dubrovin's dispersionless integrable hierarchy** associated to Frobenius manifolds.

B-model open-closed string field theory

BCOV theory can be coupled with Witten's holomorphic CS theory

$$HCS(A) = \int_X \text{Tr} \left(\frac{1}{2} A \wedge \bar{\partial} A + \frac{1}{3} A^3 \right) \wedge \Omega_X, \quad A \in \Omega^{0,*}(X, \mathfrak{g})[1],$$

to give open-closed string field theory in the B-model. It is required to satisfy Zwiebach's open-closed BV master equation.

Q: How to put HCS and KS together to get open-closed theory?

Theorem (Costello-L)

The *infinitesimal deformation* of holomorphic Chern-Simons theory for $\mathfrak{g} = \mathfrak{gl}_N$ (consistent with $N \rightarrow \infty$) by single trace operators is isomorphic to $PV(X)[[t]]$.

This is a version of **Tsygan-Loday-Quillen** theorem on the cyclic homology and **Hochschild-Kostant-Rosenberg** isomorphism. Here's an example of 1st order deformation

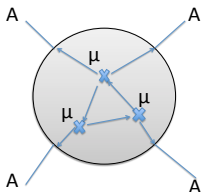
$$\sum \int_X \text{Tr} (\mu^{i_1 \cdots i_k} (A \wedge \partial_{i_1} A \wedge \cdots \wedge \partial_{i_k} A)) \wedge \Omega_X,$$

for $\mu = \sum \mu^{i_1 \cdots i_k} \partial_{i_1} \wedge \cdots \wedge \partial_{i_k} \in PV(X)$, $A \in \Omega^{0,*}(X, \mathfrak{g})$.

We propose the **classical open-closed B-model interaction** by

$$I_0(\mu, A) = \sum_{m,n} \int_X \int_{C_{m,n}} \mathcal{L}_{m,n}(\mu, A), \quad \mu \oplus A \in \mathrm{PV}(X)[[t]] \oplus \Omega^{0,*}(X, \mathfrak{g})[1]$$

where $C_{m,n}$ is the configuration space of the disk with m points on the boundary and n points on the interior. The lagrangian density $\mathcal{L}_{m,n}$ is of the form of **Kontsevich's graph formula** of deformation quantization



Theorem (Costello-L)

The coupled HCS-KS action (or open-closed BCOV) satisfies a version of classical master equation.

- ▶ Mathematically, this is essentially equivalent to Kontsevich's formality theorem (the cyclic version by Willwacher-Calaque):
- ▶ This gives a concrete realization of Zwiebach's master equation for open-closed string field theory.

$Q = \text{BV bracket in open sector} + \text{BV bracket in closed sector}$

Classical BV-master equation

Quantization of open-closed BCOV theory

The first step toward the understanding of the quantization of open-closed BCOV theory is the following local version:

Theorem (Costello-L)

On $X = \mathbb{C}^n$, n odd, there exists a canonical perturbative quantization (solving quantum BV-master equation) of open-closed BCOV theory with gauge group $gl(N|N)$ that is consistent with $N \rightarrow \infty$.

Singularity and $\frac{\infty}{2}$ -Hodge Structure

BCOV theory and quantum B-model

Singularity and Seiberg-Witten geometry

Seiberg-Witten geometry

We are interested in 4D $N = 2$ SCFT. Seiberg-Witten discovered that for many theories the low energy effective theory on the Coulomb branch could be described by a Seiberg-Witten curve fibered over the moduli space:

$$F(x, z; \lambda_\alpha) = 0$$

Here λ_α 's are the parameters including coupling constants, mass parameters, and expectation values for Coulomb branch operators. The period integral of an appropriate 1-form over the Riemann surface $F(x, z; \lambda_\alpha) = 0$ determines the low energy photon coupling.

Singularity and 4d $N = 2$ SCFT

We consider type IIB string theory on the following background:

$$\mathbb{R}^{3,1} \times X$$

Here X is a **3-fold weighted homogeneous isolated singularity**. It is argued to define 4d $N = 2$ SCFT (Shapere, Vafa, 99; D. Xie, Yau, 15). The SW solution is associated to a three-fold fibration

$$F(x_1, x_2, x_3, x_4; \lambda_\alpha) = 0$$

which may or may not be reduced to the curve geometry. This suggests that the more general SW solution could be three-fold fibrations rather than curves. Our goal is to figure out the corresponding SW differential.

3-fold Singularity

Consider an isolated weighted homogeneous 3-fold singularity:

$$f : \mathbb{C}^4 \rightarrow \mathbb{C}, \quad f(\lambda^{q_i} x^i) = \lambda f(x^i), \quad q_i > 0, \quad \lambda \in \mathbb{C}^*.$$

The rational number

$$\hat{c}_f = \sum_i (1 - 2q_i)$$

is the central charge of the 2d (2,2) SCFT defined by LG model with superpotential f . To define a 4d SCFT we require

$$\sum_i q_i > 1 \iff \hat{c}_f < 2.$$

Coulomb branch and miniversal deformations

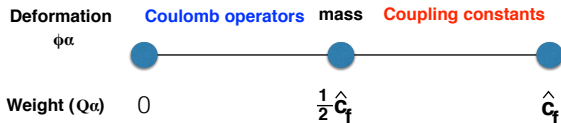
The Coulomb branch of the associated $N=2$ SCFT is described by the local moduli of miniversal deformations of f :

$$F(x_i, \lambda_\alpha) = f(x_i) + \sum_{\alpha=1}^{\mu} \lambda_\alpha \phi_\alpha$$

where $\{\phi_\alpha\}$ is a basis of the Jacobi algebra $\text{Jac}(f)$. Define

$$\boxed{[\lambda_\alpha] = \frac{1 - Q_\alpha}{\sum_i q_i - 1}}, \quad Q_\alpha = \text{homogeneous weight of } \phi_\alpha.$$

- ▶ $[\lambda_\alpha] < 1$: Coupling constants
- ▶ $[\lambda_\alpha] = 1$: Mass parameters
- ▶ $[\lambda_\alpha] > 1$: Expectation value of Coulomb branch operators



Period map

Given a family of holomorphic volume forms $\xi(x^i, \lambda_\alpha)$, we consider the period map

$$\mathcal{P} : \mathcal{M} \rightarrow H^3, \quad \{\lambda_\alpha\} \rightarrow \int_\gamma \frac{\xi}{dF}.$$

Here the integration is over vanishing cycles γ in $F^{-1}(0)$ and $H^3 = H^3(F^{-1}(0), \mathbb{C})$ is the dual space. For simplicity, let us consider the case with no mass parameters. Then H^3 has a natural symplectic structure induced dually from the intersection pairing. This allows choices of an electro-magnetic charge lattice from H_3 .

Seiberg-Witten differential

The low energy effective theory of Coulomb branch is described by Seiberg-Witten (SW) geometry.

Question

What is the Seiberg-Witten differential associated to singularities?

Seiberg-Witten differential

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What is the Seiberg-Witten differential associated to singularities?

Naive guess: the SW differential is the family of 3-forms

$$\frac{\xi}{dF}, \quad \text{where} \quad \xi = dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

The $N = 2$ prepotential comes from the corresponding period map. This is indeed true in the case of **ADE** singularities ($\hat{c}_f < 1$).

If we go beyond ADE singularities, this no longer holds.

Seiberg-Witten differential

Solution [L-Xie-Yau, 2018]: for 3-fold singularity

$$\xi \text{ primitive form} \implies \frac{\xi}{dF} \text{ SW differential}$$

where ξ is K. Saito's primitive form.

The main support for the connection between primitive form and SW differential is about the integrability condition

$$\langle d\mathcal{P}, d\mathcal{P} \rangle = 0$$

for the existence of $N = 2$ prepotential arising from SW period map. The verification of such integrability requires a connection between the intersection pairing for vanishing homology and the period map. Primitive form provides precisely such a relationship [K.Saito, 1982].

Curve v.s. 3-fold geometry

This result also generalizes to curve geometry

$$\xi \text{ primitive form} \implies \frac{\xi^{(-1)}}{dF} \text{ SW differential}$$

The SW differential for three-fold geometry picks up ξ instead of $\xi^{(-1)}$ for curve geometry, by the reason of shift of Hodge theory arising from the shift of dimension.

There is also a 5d hypersurface singularity example. The analogue discussion implies that the SW differential is expected to be

$$\frac{\xi^{(1)}}{dF}.$$

Examples of primitive forms: ADE type ($\hat{c}_f < 1$)

We consider

$$f(x) = x_1^2 + x_2^2 + x_3^k + x_4^N, \quad \frac{1}{k} + \frac{1}{N} > \frac{1}{2}.$$

This example can be reduced to curve geometry. The primitive form is trivial in this case and doesn't depend on the deformation parameter

$$\xi = dx_1 \wedge \cdots \wedge dx_4.$$

The 3-fold SW differential is given by

$$\lambda = \frac{\xi}{dF}$$

Simple elliptic singularity ($\hat{c}_f = 1$)

We consider

$$f(x) = x_1^3 + x_2^3 + x_3^3 + x_4^2$$

The miniversal deformation is

$$F = f + \lambda_1 + \lambda_2 x_1 + \lambda_3 x_2 + \lambda_4 x_3 + \lambda_5 x_1 x_2 + \lambda_6 x_2 x_3 + \lambda_7 x_3 x_1 + \lambda_8 x_1 x_2 x_3.$$

Primitive form of this example is nontrivial and is not unique. They depend only on the marginal parameter λ_8 described as follows:

$$\xi = \frac{dx_1 \wedge \cdots \wedge dx_4}{P(\lambda_8)}$$

where $P(\lambda_8)$ is a period on the cubic elliptic curve

$$\{x_1^3 + x_2^3 + x_3^3 + \lambda_8 x_1 x_2 x_3 = 0\} \subset \mathbb{P}^2.$$

General singularities ($\hat{c}_f > 1$)

For general singularities with $\hat{c}_f > 1$, there exists a highly nontrivial mixing between relevant and irrelevant deformations. The close formula of primitive form is unknown. Here is one example

$$f = x_1^2 + x_2^2 + x_3^3 + x_4^7.$$

This is type E_{12} of the unimodular exceptional singularities. The miniversal deformation is the following

$$F(x, \lambda) = f + \lambda_1 + \lambda_2 x_4 + \lambda_3 x_4^2 + \lambda_4 x_3 + \lambda_5 x_4^3 + \lambda_6 x_3 x_4 \\ + \lambda_7 x_4^4 + \lambda_8 x_3 x_4^2 + \lambda_9 x_4^5 + \lambda_{10} x_3 x_4^3 + \lambda_{11} x_3 x_4^4 + \lambda_{12} x_3 x_4^5.$$

Here $\lambda_1, \dots, \lambda_{11}$ are relevant deformations, and λ_{12} is an irrelevant deformation.

There is a recursive formula to compute general primitive forms perturbatively [L-Li-Saito, 2015]. For E_{12} , it gives (up to order 10)

$$\zeta = (\varphi(x, \lambda) + O(\lambda^{11})) dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

where

$$\begin{aligned} \varphi(x, \lambda) = & 1 + \frac{4}{3 \cdot 7^2} \lambda_{11} \lambda_{12}^2 - \frac{64}{3 \cdot 7^4} \lambda_{11}^2 \lambda_{12}^4 - \frac{76}{3^2 \cdot 7^4} \lambda_{10} \lambda_{12}^5 + \frac{937}{3^3 \cdot 7^5} \lambda_9 \lambda_{12}^6 + \frac{218072}{3^4 \cdot 5 \cdot 7^6} \lambda_{11}^3 \lambda_{12}^6 \\ & + \frac{1272169}{3^4 \cdot 5 \cdot 7^7} \lambda_{10} \lambda_{11} \lambda_{12}^7 + \frac{28751}{3^4 \cdot 7^7} \lambda_8 \lambda_{12}^8 - \frac{1212158}{3^4 \cdot 7^8} \lambda_9 \lambda_{11} \lambda_{12}^8 - \frac{38380}{3^3 \cdot 7^8} \lambda_7 \lambda_{12}^9 \\ & + \left(\frac{1}{7^2} \lambda_{12}^3 - \frac{101}{5 \cdot 7^4} \lambda_{11} \lambda_{12}^5 + \frac{1588303}{3^4 \cdot 5 \cdot 7^7} \lambda_{11}^2 \lambda_{12}^7 + \frac{378083}{3^4 \cdot 5 \cdot 7^7} \lambda_{10} \lambda_{12}^8 - \frac{108144}{3 \cdot 7^8} \lambda_9 \lambda_{12}^9 \right) x_3 \\ & + \left(\frac{1447}{3^3 \cdot 7^6} \lambda_{12}^7 - \frac{71290}{3^3 \cdot 7^8} \lambda_{11} \lambda_{12}^9 \right) x_4 - \frac{45434}{3^4 \cdot 7^8} \lambda_{12}^{10} x_3 x_4 \\ & - \left(\frac{53}{3^2 \cdot 7^4} \lambda_{12}^6 - \frac{46244}{3^3 \cdot 7^7} \lambda_{11} \lambda_{12}^8 \right) x_3^2 + \frac{22054}{3^4 \cdot 7^7} \lambda_{12}^9 x_3^3. \end{aligned}$$

In particular, the Seiberg-Witten differential of this example is not given by a rescaling of the trivial form and depends on the coupling constants in a nontrivial way.

Thank you!