T-duality methods in topological matter

Guo Chuan Thiang

University of Adelaide
Institute for Geometry and its Applications

String and M-Theory: The New Geometry of the 21st Century
IMS, NUS, Singapore

12/12/18

Outline

Topological insulator $\leftrightarrow$ D-brane analogy (K-theory charge).

(T)-duality concept from string theory is useful and natural in solid state physics.

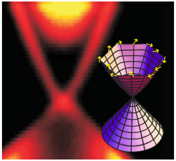
Bulk-edge correspondence heuristic: analytic boundary zero modes detect bulk topology $\to$ useful for formulating new index theorems.

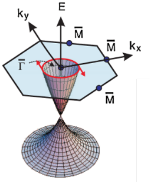
Torsion invariants, antiunitary symmetries, and "super-ness" are extremely important.

Whole zoo of new crystallographic topological T-dualities, involving K-theories with *graded* equivariant twists.

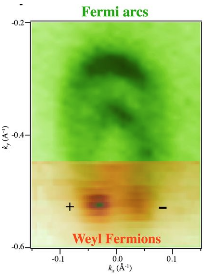
Solid state phys $\to$ new maths $\to$ string/M-theory?

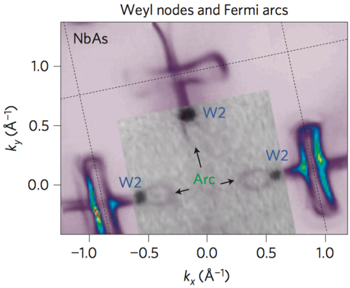
Experimentally verified topological matter and torsion





Experiment: Hsieh et al, PRL **103** (2009). Theory (2005+): Fu–Kane–Mele $\mathbb{Z}/2$ invariant (e.g. in *KR*-theory)



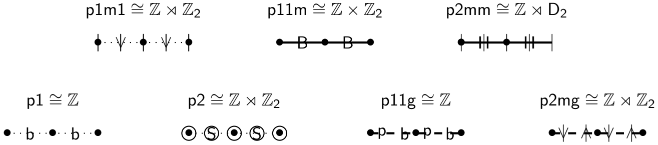


Experiment: (L) Xu et al, Science **349** 613 (2015); (R) [—] Nature Phys. **11** 748 (2015).
Theory: [T+Sato+Gomi, Nucl. Phys. B **923** (2017)], $\mathbb{Z}/2$ monopole and Dirac strings

Why graded symmetry groups?

A *super*, or *graded group* will just be a group $\mathscr{G} \xrightarrow{c} \mathbb{Z}_2$.
E.g. we generate $\mathbb{Z} \xrightarrow{c} \mathbb{Z}_2$ this we walk — L/R switching data.

In crystallography, this is called the *frieze group* p11g.



There are 7 frieze groups (2D patterns with 1D translation symmetry). Higher dimension analogue are called *subperiodic groups*, e.g. 75 *rod groups*, 80 *layer groups*.

Baum–Connes conjecture and (T)-duality

Let $\mathscr{G}$ be a discrete (ungraded) group. *Baum–Connes conjecture*:

$$\mu_{\mathscr{G}} : K_\bullet(\underbrace{\underline{B}\mathscr{G}}_{\text{class. space}}) \xrightarrow{\cong} K_\bullet(\underbrace{C_r^*(\mathscr{G})}_{\text{group algebra}}).$$

RHS is hard; here $C_r^*(\mathscr{G})$ is "nonabelian Fourier transform", generalising E.g. $C_r^*(\mathbb{Z}) = C(\mathbb{T})$.
LHS is computable with algebraic topology!

No counterexamples! I will use a concrete physics model to motivate a super-BC conjecture, and relate it to crystallographic T-duality. This is a "good" duality in the following sense:

$$\begin{tikzcd}
\boxed{\text{Easy in POV 1}} \arrow[rr, leftrightarrow, "\sim", "\text{Duality}"'] \arrow[d, leftrightarrow, "\oplus"'] & & \boxed{\text{Hard in POV 2}} \arrow[d, leftrightarrow, "\oplus"] \\
\boxed{\text{Hard in POV 1}} \arrow[rr, leftrightarrow, "\sim", "\text{Duality}"'] & & \boxed{\text{Easy in POV 2}}
\end{tikzcd}$$

Fourier transform and T-duality

The Fourier transform is the prototypical "good" duality,and involves a Pull-Convolve-Push construction:

$$L^2(\mathbb{R}) \ni \{f : x \mapsto f(x)\} \longleftrightarrow \{\hat{f} : p \mapsto \int_{\mathbb{R}} f(x)e^{ipx}\} \in L^2(\widehat{\mathbb{R}})$$

There is a geometric version, called the *Fourier–Mukai transform*, or *topological T-duality*:

$$\mathrm{T} : K^\bullet(T) \xrightarrow{\cong} K^{\bullet-1}(\mathbb{T})$$

$$[\mathcal{E}] \mapsto \hat{p}_*[(p^*\mathcal{E}) \otimes \mathcal{P}]$$

$$\begin{tikzcd} & \mathcal{P} \arrow[d] & \\ & T \times \mathbb{T} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ T & & \mathbb{T} \end{tikzcd}$$

Here, the "kernel" $\mathcal{P}$ is the Poincaré line bundle over the "correspondence space" $T \times \mathbb{T}$.

T-duality and Baum–Connes

T-duality is closely related to BC for the group $\mathbb{Z}$, because there are two *different* circles associated to $\mathbb{Z}$:

(1) the classifying space $T = B\mathbb{Z} = R/\mathbb{Z}$,

(2) Pontryagin dual (irreps) $\mathbb{T} = \mathrm{Hom}(\mathbb{Z}, \mathrm{U}(1)) \Rightarrow C_r^*(\mathbb{Z}) \overset{\mathrm{FT}}{\cong} C(\mathbb{T})$.

Then T-duality is BC for $\mathbb{Z}$ with some further identifications,

$$K^\bullet(T) \overset{PD}{\to} K_{1-\bullet}(T = B\mathbb{Z}) \xrightarrow{BC} K_{1-\bullet}(C_r^*(\mathbb{Z})) \overset{FT}{\to} K^{\bullet-1}(\mathbb{T})$$

Both T^d and $\mathbb{T}^d$ appear naturally in solid state physics, as the *unit cell* and *Brillouin zone* respectively!

Crystallography $\Leftrightarrow$ extra finite point group action (with twists).

Why $K_\bullet(C_r^*(\mathscr{G}))$ relates to topological phases?

Free electron has Hamiltonian $H = -\nabla^2$ on $L^2(R)$. Euclidean invariance broken to $\mathbb{Z}$ by periodic potential V from crystal atoms.

Think of $L^2(R) \cong L^2(\mathcal{F}) \otimes \ell^2_{\mathrm{reg}}(\mathbb{Z})$, where $\mathcal{F} = R/\mathbb{Z}$ is the position space "unit cell".

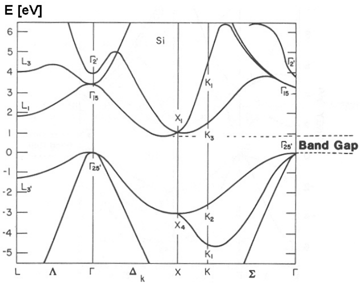
In the *Bloch–Floquet transform*, $\ell^2_{\mathrm{reg}}(\mathbb{Z})$ part is Fourier transformed to $L^2(\mathbb{T})$ where $\mathbb{T}$ is momentum space "Brillouin zone".

$H = -\nabla^2 + V$ decomposes into Bloch Hamiltonians H_k acting on k-quasiperiodic functions (Bloch waves)

$$\mathcal{E}_k = \{\psi : \psi(x+1) = \psi(x)e^{\mathrm{i}k}\} \sim L^2(\mathcal{F}; \mathcal{E}_k), \quad k \in [0, 2\pi) \cong \mathbb{T}.$$

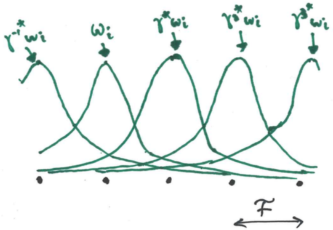
Schrödinger's equation for H_k on *compact* $\mathcal{F} \rightarrow$ discrete spectrum.

Abelian Bloch–Floquet transform



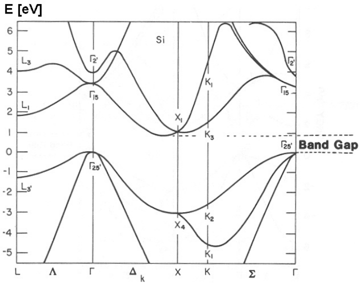
$\leftarrow$ For $E < E_{\mathrm{Fermi}}$ get Hermitian vector eigenbundle $\mathcal{E}_{\mathrm{Fermi}} \to \mathbb{T}$.

If $\mathcal{E}_{\mathrm{Fermi}}$ is trivialisable, take (cts/smooth) O.N. frame $\{\phi_i\}_{i=1,\ldots,n}$. Inverse transforms of ϕ_i are *Wannier wavefunctions* $w_i \in L^2(R)$ with orthonormal translates, and decay condition.



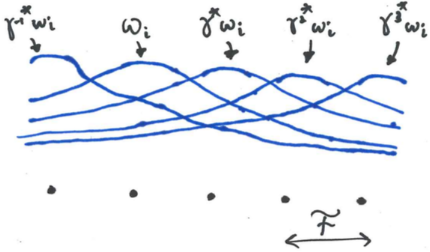
"Atomic limit":
$L^2(\mathcal{E}_{\mathrm{Fermi}}) \cong \ell^2_{\mathrm{reg}}(\mathbb{Z}) \otimes \mathbb{C}^n$
via *localised* basis of wavefunctions.

Abelian Bloch–Floquet transform



$\leftarrow$ For $E < E_{\mathrm{Fermi}}$ get Hermitian vector eigenbundle $\mathcal{E}_{\mathrm{Fermi}} \to \mathbb{T}$.

Choose (cts/smooth) O.N. frame $\{\phi_i\}_{i=1,\ldots,n}$. Inverse transforms of ϕ_i are *Wannier wavefunctions* $w_i \in L^2(R)$ with orthonormal translates, and decay condition.



For $\mathscr{G} = \mathbb{Z}^2$, the first Chern class obstructs trivialisation of $\mathcal{E}_{\mathrm{Fermi}}$, so no "atomic limit". [Brouder, Panati, Monaco,..., 2007$^+$]

$K_0(C^*_r(\mathscr{G}))$ as obstruction to "atomic limit"

[Theorem: Ludewig+T '18/19] Let $\mathscr{G}$ (nonabelian generally), act on Riemannian X and $S \subset L^2(X)$ be a spectral subspace of a $\mathscr{G}$-invariant Hamiltonian. Under mild assumptions, the subspace S_0 of "good wavefunctions" in S form a f.g.p. module for $C^*_r(\mathscr{G})$.

$K_0(C^*_r(\mathscr{G}))$ measures failure of S_0 to be a free module (i.e. generated by translates of a good wavefunction) $\Rightarrow$ obstruction to existence of "atomic limit" description.

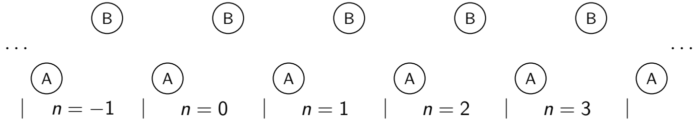
Bulk-topology-imposed tails should be visible analytically at a suitable boundary as zero modes $\Leftrightarrow$ index theorem

Remark: For $\mathscr{G}$ crystallographic, $K_\bullet(C^*_r(\mathscr{G}))$ classifies "twisted equivariant matter" [Freed–Moore '13]; modelled over $\mathbb{T}^d$.

Example of bulk-boundary correspondence (BBC)

For $\mathscr{G} = \mathbb{Z}$, $K_1(C_r^*(\mathbb{Z})) \cong K^{-1}(\mathbb{T}) = \mathbb{Z}$ characterises another type of (relative) topological phase, which is instructive for BBC.

Consider $\ell^2_{\text{reg}}(\mathbb{Z}) \oplus \ell^2_{\text{reg}}(\mathbb{Z})$, sublattice operator $\mathsf{S} = 1_A \oplus -1_B$.



A *supersymmetric* Hamiltonian $H = H^*$ commutes with $\mathbb{Z}$-translations, but $H\mathsf{S} = -\mathsf{S}H$. Thus H exchanges $A \leftrightarrow B$.

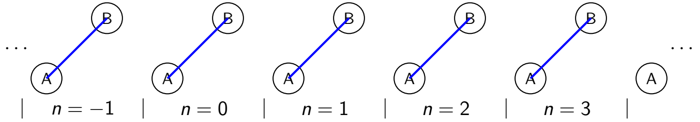
E.g. "Dimers" H_{blue} (intracell) and H_{red} (intercell)

$$\text{General} : H\mathsf{S} = -\mathsf{S}H \quad \overset{\text{Fourier}}{\Longleftrightarrow} \quad H(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}, \quad U(k) \in \mathbb{C}$$

Example of bulk-boundary correspondence (BBC)

For $\mathscr{G} = \mathbb{Z}$, $K_1(C_r^*(\mathbb{Z})) \cong K^{-1}(\mathbb{T}) = \mathbb{Z}$ characterises another type of (relative) topological phase, which is instructive for BBC.

Consider $\ell^2_{\text{reg}}(\mathbb{Z}) \oplus \ell^2_{\text{reg}}(\mathbb{Z})$, sublattice operator $\mathsf{S} = 1_A \oplus -1_B$.



A *supersymmetric* Hamiltonian $H = H^*$ commutes with $\mathbb{Z}$-translations, but $H\mathsf{S} = -\mathsf{S}H$. Thus H exchanges $A \leftrightarrow B$.

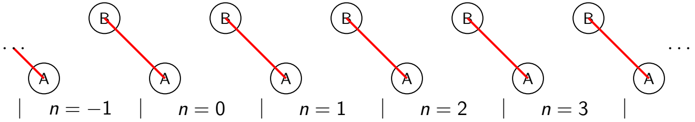
E.g. "Dimers" H_{blue} (intracell) and H_{red} (intercell)

$$\text{General} : H\mathsf{S} = -\mathsf{S}H \quad \overset{\text{Fourier}}{\Longleftrightarrow} \quad H(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}, \quad U(k) \in \mathbb{C}$$

Example of bulk-boundary correspondence (BBC)

For $\mathscr{G} = \mathbb{Z}$, $K_1(C_r^*(\mathbb{Z})) \cong K^{-1}(\mathbb{T}) = \mathbb{Z}$ characterises another type of (relative) topological phase, which is instructive for BBC.

Consider $\ell^2_{\mathrm{reg}}(\mathbb{Z}) \oplus \ell^2_{\mathrm{reg}}(\mathbb{Z})$, sublattice operator $\mathsf{S} = 1_A \oplus -1_B$.



A *supersymmetric* Hamiltonian $H = H^*$ commutes with $\mathbb{Z}$-translations, but $H\mathsf{S} = -\mathsf{S}H$. Thus H exchanges $A \leftrightarrow B$.

E.g. "Dimers" H_{blue} (intracell) and H_{red} (intercell)

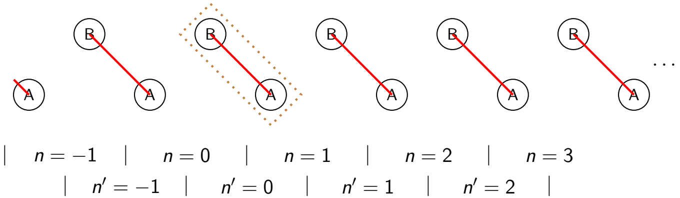
$$\text{General}: H\mathsf{S} = -\mathsf{S}H \quad \overset{\text{Fourier}}{\Longleftrightarrow} \quad H(k) = \begin{pmatrix} 0 & U(k) \\ U(k)^* & 0 \end{pmatrix}, \quad U(k) \in \mathbb{C}$$

SSH model

Gap condition: $0 \notin \mathrm{spec}(H) \Leftrightarrow U : \mathbb{T} \to GL(1)$.
Wind(U) $\leftrightarrow$ gapped top. phases of $\mathbb{Z}$-invariant, supersymmetric H!

$$H_{\text{blue}}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \text{Wind}{=}0, \quad H_{\text{red}}(k) = \begin{pmatrix} 0 & e^{ik} \\ e^{-ik} & 0 \end{pmatrix} \to \text{Wind}{=}1.$$

Actually $H_{\text{blue}} \sim_{\text{unitary}} H_{\text{red}}$, so Wind($U$) has no bulk meaning.



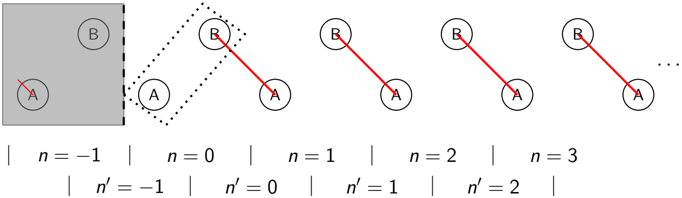
A boundary "fixes the gauge", and also cuts a red link, leaving behind one "dangling zero mode" of A-type.

SSH model

Gap condition: $0 \notin \mathrm{spec}(H) \Leftrightarrow U : \mathbb{T} \to \mathrm{GL}(1)$.
$\mathrm{Wind}(U) \leftrightarrow$ gapped top. phases of $\mathbb{Z}$-invariant, supersymmetric H!

$$H_{\mathrm{blue}}(k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \to \mathrm{Wind}{=}0, \quad H_{\mathrm{red}}(k) = \begin{pmatrix} 0 & e^{\mathrm{i}k} \\ e^{-\mathrm{i}k} & 0 \end{pmatrix} \to \mathrm{Wind}{=}1.$$

Actually $H_{\mathrm{blue}} \sim_{\mathrm{unitary}} H_{\mathrm{red}}$, so $\mathrm{Wind}(U)$ has no bulk meaning.



A boundary "fixes the gauge", and also cuts a red link, leaving behind one "dangling zero mode" of A-type.

SSH model and index theorem

Truncation to $n \geq 0 \Leftrightarrow$ *Hardy space* $\ell^2(\mathbb{N}) \cong \mathcal{H}^2 \subset L^2(\mathbb{T}) \cong \ell^2(\mathbb{Z})$.

$$H = \begin{pmatrix} 0 & U \\ U^\dagger & 0 \end{pmatrix} \overset{\text{half}-\text{space}}{\mapsto} \widetilde{H} = \begin{pmatrix} 0 & T_U \\ T_U^\dagger & 0 \end{pmatrix}.$$

T_U is Toeplitz operator on $l^2(\mathbb{N}) \otimes \mathbb{C}^N$ with invertible symbol $U \in C(\mathbb{T}, \mathrm{GL}(N))$ representing a $K^{-1}(\mathbb{T}) \cong \mathbb{Z}$ class.

F. Noether index theorem (1921): T_U is Fredholm iff U invertible, and $\mathrm{Ind}(T_U) = -\mathrm{Wind}(U) = \int_{\mathbb{T}} \mathrm{ch}(U)$.

Analytic Fredholm $\mathrm{Ind}(T_U) = \#B - \#A$ zero modes of $\widetilde{H}$, which is topological because of index theorem!

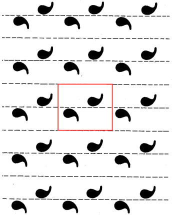
$$H_{\mathrm{blue}}(k) = \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathrm{Wind}=0} \mapsto \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{\mathrm{Ind}=0}, \quad H_{\mathrm{red}}(k) = \underbrace{\begin{pmatrix} 0 & e^{\mathrm{i}k} \\ e^{-\mathrm{i}k} & 0 \end{pmatrix}}_{\mathrm{Wind}=1} \mapsto \underbrace{\begin{pmatrix} 0 & T_{e^{\mathrm{i}k}} \\ T_{e^{-\mathrm{i}k}} & 0 \end{pmatrix}}_{\mathrm{Ind}=-1}.$$

Crystallographic BBC and mod-2 index theorem

Run the story in reverse: Physically reasonable BBC heuristic suggests the index theorem justifying the hueristic.

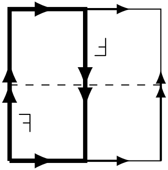
The torsion-free wallpaper group pg is the fundamental group of the Klein bottle. Baum–Connes magic gives easy computation

$$K_1(C_r^*(\mathrm{pg})) \cong K_1(B\mathrm{pg}) \equiv K_1(\mathrm{Klein}) \cong H_1(\mathrm{Klein}) \cong \mathbb{Z} \oplus \mathbb{Z}/2.$$

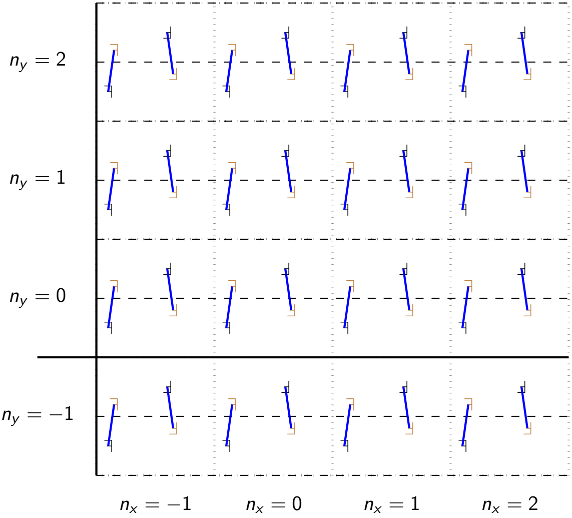


$\Rightarrow \mathbb{Z}/2$ topological phase for pg-invariant, supersymmetric H. Next few slides will pictorially justify why there is a $\mathbb{Z}/2$-index map

$$\underbrace{K_1(C_r^*(\mathrm{pg}))}_{\text{bulk phases}} \to \underbrace{K_0^{\mathrm{gr}}(C_r^*(\mathrm{p11g}))}_{\text{boundary zero modes}} \cong \mathbb{Z}/2.$$

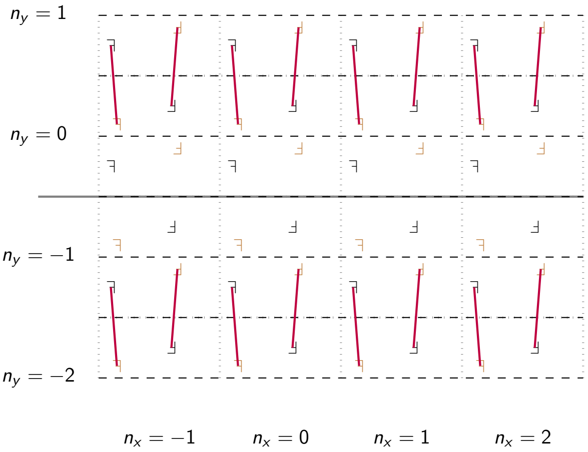


Super-symmetric Hamiltonians: connect Black-Brown



$H_{\rm blue}$ has no zero modes when truncated: trivial phase.

The $\mathbb{Z}/2$ "Klein bottle" phase



Zero modes along glide axis edge have graded p11g symmetry!

$\mathbb{Z}/2$ super-index theorem

Theorem [Gomi+T, Lett. Math. Phys. '18]: A pg supersymmetric H has symbol class in $K^{1+\tau}_{\mathbb{Z}_2}(\mathbb{T}^2)$. Its truncation gives a twisted Toeplitz family over $\mathbb{T}$. The following index maps coincide:

(1) topological Gysin map $K^{1+\tau}_{\mathbb{Z}_2}(\mathbb{T}^2) \overset{\pi_!}{\twoheadrightarrow} K^{0+\tau+c}_{\mathbb{Z}_2}(\mathbb{T}) \cong \mathbb{Z}/2$.

(2) analytic "twisted index bundle", in $K_0^{\mathrm{gr}}(C^*_r(\mathrm{p11g})) \cong \mathbb{Z}/2$.

In physics terms, H_{purple} represents a topologically non-trivial $\mathbb{Z}/2$ phase with pg symmetry. It is detected by p11g-symmetric zero modes induced when sample is cut along a glide axis.

$\mathbb{Z}/2$ index theorems in complex K-theory are very rare! We found this by going to graded K-theory, guided by physics.

Crystallographic T-duality [Gomi+T, 1806:11385]

In string theory, one could transform a T bundle $\mathcal{E} \to X$ into a $\mathbb{T}$ bundle $\widehat{\mathcal{E}} \to X$. Even though $\mathcal{E} \ncong \widehat{\mathcal{E}}$, their *twisted* K-theories coincide, up to a degree shift [Bouwknegt–Mathai–Evslin '04].

Some ad-hoc incorporation of $\mathbb{Z}/2$-actions, e.g. Witten–Atiyah–Hopkins $K_{\pm}$ groups, orientifolds, KR-theory,. . . .

Crystallographic T-duality upgrades these to general finite groups acting on tori (or torus bundles).

In fact, crystallographic space group $\mathscr{G} \Leftrightarrow$ affine F action on T^d!

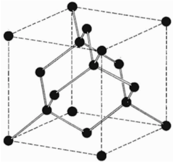
Dually, F acts on $\mathbb{T}^d$ with a twist τ from $\mathscr{G} \not\leftarrow F$.

Crystallographic groups

A crystallographic space group $\mathscr{G}$ is a discrete cocompact subgroup of isometries of Euclidean space R^d.

$$
\begin{tikzcd}
0 \arrow[r] & \mathbb{R}^d \arrow[r] & \mathrm{Euc}(d) \arrow[r] & \mathrm{O}(d) \arrow[r] & 1 \\
0 \arrow[r] & \mathbb{Z}^d \arrow[r] \arrow[u] & \mathscr{G} \arrow[r] \arrow[u] & F \arrow[r] \arrow[u] & 1
\end{tikzcd}
$$

$\mathscr{G}$ is an extension of finite point group F by lattice subgroup $\mathbb{Z}^d$.



Classification of $\mathscr{G}$-symmetric Hamiltonians
$\Leftrightarrow$ $\tau_{\mathscr{G}}$-*twisted* F-equivariant K-theory of $\mathbb{T}^d$
$\Leftrightarrow$ K-theory of $C_r^*(\mathscr{G})$
[Freed–Moore '13, T'16].

Crystallographic T-duality

[Gomi+T' 18] There is a zoo of "crystallographic T-dualities". Each d-dimensional space group $\mathscr{G}$ defines an isomorphism

$$T_{\mathscr{G}} : K_F^{d-\bullet+\sigma_{\mathscr{G}}}(T^d_{\mathrm{affine}}) \cong K_F^{-\bullet+\tau_{\mathscr{G}}}(\mathbb{T}^d_{\mathrm{dual}})$$

Technical subtlety: $\sigma_{\mathscr{G}}, \tau_{\mathscr{G}}$ are *graded, equivariant* twists (physics gave a clue). As a *set* (not as a group), these are classified by

$$H^3_F(\,\cdot\,;\mathbb{Z}) \times \underbrace{H^1_F(\,\cdot\,;\mathbb{Z}_2)}_{\text{graded}}.$$

Geometric formulation is a souped-up version of the Fourier–Mukai transform. Gives many previously unknown isomorphisms between twisted equivariant cohomology theories of tori.

Crystallographic T-duality and super Baum–Connes

Another formulation uses Baum–Connes for $\mathscr{G}$: Euclidean space R^d is a proper universal space $\underline{E}\mathscr{G}$. Quotient by $\mathbb{Z}^d$ gives orbifold $F\backslash T^d = \underline{B}\mathscr{G}$.

So there is an assembly isomorphism

$$K_\bullet^F(T^d) \cong K_\bullet(\underline{B}\mathscr{G}) \xrightarrow{\mu_{\mathscr{G}}} K_\bullet(C_r^*(\mathscr{G})).$$

LHS is $K_\bullet^F(T^d) \cong K_F^{d-\bullet+\sigma_{\mathscr{G}}}(T^d)$ by Poincaré duality, where the twist $\sigma_{\mathscr{G}}$ compensates for the failure of equivariant K-orientability.

RHS is $K_\bullet(C_r^*(\mathscr{G})) \cong K_F^{-\bullet+\tau_{\mathscr{G}}}(\mathbb{T}^d)$ by a Fourier transform, and twist $\tau_{\mathscr{G}}$ due to $\mathscr{G} \not\leftarrow F$.

Crystallographic T-duality and super Baum–Connes

Overall, $T_{\mathscr{G}} : K_F^{d-\bullet+\sigma_{\mathscr{G}}}(T^d) \xrightarrow{\sim} K_F^{-\bullet+\tau_{\mathscr{G}}}(\mathbb{T}^d)$.

Giving $\mathscr{G}$ a $\mathbb{Z}_2$-grading $\rightarrow$ extra H^1-type twist c on both sides.

$\Rightarrow$ a super Baum–Connes assembly map for the graded group $\mathscr{G}$ implements a morphism

$$\mu_{\mathscr{G},c} : K^{\mathscr{G}}_{\bullet+c}(T^d) \longrightarrow K^{\mathrm{gr}}_{\bullet}(C^*_r(\mathscr{G})).$$

"Ordinary Baum–Connes conjecture $\Rightarrow$ Super version" is not known or obvious (according to experts I've asked). Instead, let me give some examples/applications.

Crystallographic T-duality and super Baum–Connes

We've seen how $K_0^{\mathrm{gr}}(C_r^*(\mathrm{p11g})) \cong K_{\mathbb{Z}_2}^{0+\tau_{\mathrm{p11g}}+c}(\mathbb{T})$ on the RHS appears for crystallographic BBC for pg $\to$ p11g.

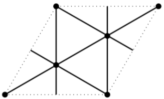
Gomi+physicists Shiozaki–Sato computed this to be $\mathbb{Z}/2$ (hard!).

LHS is much easier: $B\mathrm{p11g} = R/\mathrm{p11g} = \mathbb{Z}_2\backslash(R/\mathbb{Z}) = \mathbb{Z}_2\backslash(T_{\mathrm{free}})$ is a circle S^1, but remember that the up/down gets flipped when looping once. Twice a point is homologically trivial, so that

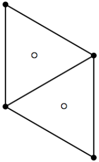
$$\mathbb{Z}/2 \cong H_{0+c}(S^1) \cong K_{0+c}(B\mathrm{p11g}) \equiv LHS \overset{s-BC}{\cong} RHS \equiv K_0^{\mathrm{gr}}(C_r^*(\mathrm{p11g})) \cong \mathbb{Z}/2$$

Crystallographic T-duality — orbifold exchange

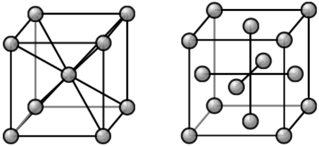
F-actions on T^d and $\mathbb{T}^d$ are generally inequivalent:



(a) p3m1



(b) p31m



Application: Topological phases for p3m1 are dual to those for p31m. Similar exchange of FCC $\leftrightarrow$ BCC (well-known in physics).

For the CT group (charge-conjugation, time-reversal), get KO-KR-theory exchange and dual tenfold way [M+T, JPhysA '15].

String theory interpretation?

Crystallographic T-duality — computational trick

Application: For $d \geq 3$ odd, $T_{\mathscr{G}} : K_F^{0+\sigma_{\mathscr{G}}}(T^d) \xrightarrow{\sim} K_F^{-1+\tau_{\mathscr{G}}}(\mathbb{T}^d)$.

$$
\begin{array}{rl}
K^0_{D_2}(T^3_{\mathsf{P222}}) \cong \mathbb{Z}^{13}, & K^1_{D_2}(T^3_{\mathsf{P222}}) \cong \mathbb{Z} \text{ or } \mathbb{Z} \oplus \mathbb{Z}/2, \\
K^{0+\omega}_{D_2}(T^3_{\mathsf{P222}}) \cong \mathbb{Z}, & K^{1+\omega}_{D_2}(T^3_{\mathsf{P222}}) \cong \mathbb{Z}^{13}, \\
K^0_{D_2}(T^3_{\mathsf{C222}}) \cong \mathbb{Z}^{8}, & K^1_{D_2}(T^3_{\mathsf{C222}}) \cong \mathbb{Z}^2 \text{ or } \mathbb{Z}^2 \oplus \mathbb{Z}/2, \\
K^{0+\omega}_{D_2}(T^3_{\mathsf{C222}}) \cong \mathbb{Z}^2, & K^{1+\omega}_{D_2}(T^3_{\mathsf{C222}}) \cong \mathbb{Z}^{8}, \\
K^0_{D_2}(T^3_{\mathsf{I222}}) \cong \mathbb{Z}^{7} \oplus \mathbb{Z}/2, & K^1_{D_2}(T^3_{\mathsf{I222}}) \cong \mathbb{Z} \text{ or } \mathbb{Z} \oplus \mathbb{Z}/2, \\
K^{0+\omega}_{D_2}(T^3_{\mathsf{I222}}) \cong \mathbb{Z} \oplus \mathbb{Z}/2, & K^{1+\omega}_{D_2}(T^3_{\mathsf{I222}}) \cong \mathbb{Z}^{7}, \\
K^0_{D_2}(T^3_{\mathsf{F222}}) \cong \mathbb{Z}^{7}, & K^1_{D_2}(T^3_{\mathsf{F222}}) \cong \mathbb{Z} \oplus \mathbb{Z}_2^2 \text{ or } \mathbb{Z} \oplus \mathbb{Z}/2, \\
K^{0+\omega}_{D_2}(T^3_{\mathsf{F222}}) \cong \mathbb{Z}, & K^{1+\omega}_{D_2}(T^3_{\mathsf{F222}}) \cong \mathbb{Z}^{7} \text{ or } \mathbb{Z}^{7} \oplus \mathbb{Z}/2.
\end{array}
$$

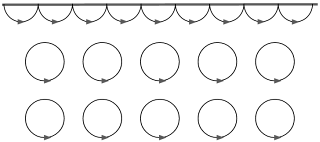
AHSS computes both sides, but RHS has extension problems.
Solved by simply inspecting K^0 on the LHS!

T-duality and BBC

In simplest models of BBC, the boundary is codim-1 hyperplane dividing Euclidean space into “bulk” and “vacuum”.

Machinery of Toeplitz extensions produces an index map ∂ from bulk to boundary “momentum space” $\sim$ integration-along-$k_\perp$

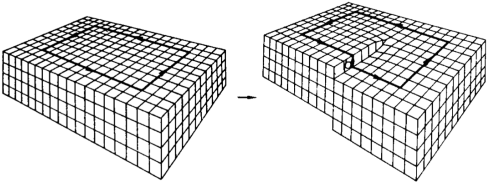
[Hannabuss+Mathai+T, CMP’16, LMP’18] ∂ is simply the T-dual of a geometric restriction-to-boundary map in position space.

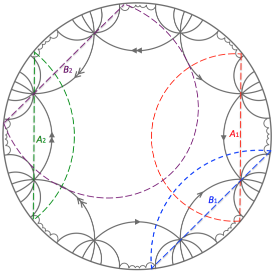


Intuitively “obvious” because Fourier transform converts integration along a circle into restriction to 0-th Fourier coefficient.

T-duality and BBC in non-Euclidean geometries

In Nil-geometry, lattice $\sim \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}, a, b, c \in \mathbb{Z} \right\}$ models screw dislocations [H+M+T, ATMP'16], and T-dual has H-flux. Get "screw modes" as argued in [Ran+Zhang+Vishwanath, Nature '19]





In hyperbolic plane, lattice $\sim$ surface/Fuchsian group, and there is fractional BBC [M+T '17]