# Unification of integrability in supersymmetric gauge theories

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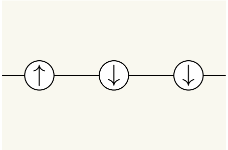
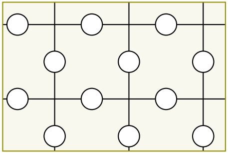
# Two of the most famous quantum integrable systems

XYZ spin chain



$\Longleftrightarrow$

8-vertex model



arise from various QFTs:

1. 2d/3d/4d gauge theories, 4 SUSY [NS '09, Maulik–Okounkov '12, . . . ]
2. 4d/5d/6d gauge theories, 8 SUSY [NS '09, Chen–Dorey–Hollowood–Lee '11]
3. 3d gauge theories, 4 SUSY [Bullimore–Dimofte–Gaiotto '15, Braverman–Finkelberg–Kamnitzer–Kodera–Nakajima–Webster–Weekes '16]
4. 4d "class-$\mathcal{S}_k$" theories [Gaiotto–Rastelli–Razamat '12, Gadde–Gukov '13, GR '15, Maruyoshi–Y '16, Y '17]
5. 4d "brane tiling" theories [Spiridonov '10, Yamazaki '13, Y '15, Maruyoshi–Y]
6. 4d Chern–Simons theory [Costello '13, C–Witten–Yamazaki '17, '18, CY '18]

Question:

Why does a single quantum integrable system appear in multiple QFT setups?

Answer:

They are different descriptions of the same physical system, related by dualities in string theory.

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Another motivation:

Understand 4d Chern–Simons theory.

4d CS has a direct relation to integrable lattice models.

But it is a very strange theory!

Reason:

4d CS arises from 6d maximally SUSY Yang–Mills.

In turn, this allows us to realize 4d CS in string theory.

## Plan

1. Motivations
2. 4d CS and integrable lattice models
3. 4d CS from 6d
4. String theory realization
5. Dualities
6. Summary & outlook

# 4d CS and integrable lattice models

4d CS is strange:

- It can only be defined on $\Sigma \times C$ with

$$C = \mathbb{C}\,, \quad \mathbb{C}^{\times} = \mathbb{C} \setminus \{0\} \quad \text{or} \quad E = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})\,.$$

- It has a complex gauge group $G_{\mathbb{C}}$.
- It has a complex action functional

$$S = \frac{\mathrm{i}}{\pi\hbar}\int_{\Sigma\times C} \mathrm{d}z \wedge \mathrm{CS}(\mathcal{A})\,,$$

$$\mathrm{CS}(\mathcal{A}) = \mathcal{A}\wedge \mathrm{d}\mathcal{A} + \frac{2}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}\,, \quad \hbar \in \mathbb{C}\,,$$

constructed from a partial connection

$$\mathcal{A} = \mathcal{A}_x \mathrm{d}x + \mathcal{A}_y \mathrm{d}y + A_{\bar{z}}\mathrm{d}\bar{z}\,, \quad \mathcal{A}_{x,y} = A_{x,y} + \mathrm{i}\phi_{x,y}\,.$$

How can we make sense of this theory?

We can still study 4d CS perturbatively. [C, CWY]

Basic observables are Wilson lines

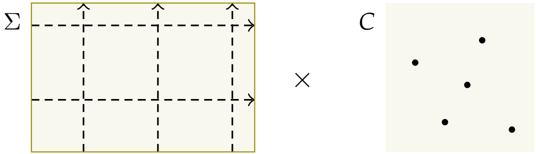
$$W_\alpha = \mathrm{Tr}_{V_\alpha} P \exp\left(\oint_{K_\alpha \times \{z_\alpha\}} \mathcal{A}\right)$$

in reps $G \to \mathrm{GL}(V_\alpha)$, around 1-cycles $K_\alpha \times \{z_\alpha\} \subset \Sigma \times C$.

Take

$$\Sigma = T^2$$

and make a lattice of Wilson lines:

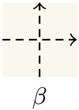


4d CS is topological on $\Sigma$ and holomorphic on $C$:

$$\int_{\Sigma \times C} dz \wedge \mathrm{CS}(\mathcal{A}) \,, \qquad \mathcal{A} = \mathcal{A}_x dx + \mathcal{A}_y dy + \mathcal{A}_{\bar{z}} d\bar{z} \,.$$

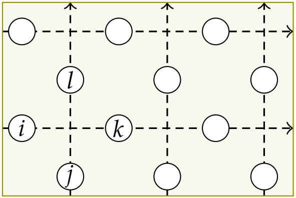
Topological invariance on $\Sigma \implies$ factorization into R-matrices

$$R_{\alpha\beta} = \alpha$$




$$\in \mathrm{End}(V_\alpha \otimes V_\beta) \,.$$

The correlator equals the partition function of a lattice model:

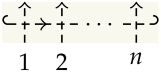



$$= \sum_{\text{configs}} \prod_{\text{vertices}} (R_{\alpha\beta})^{kl}_{ij} \,.$$

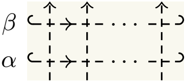
Spin sites = ◯, Boltzmann weights = $(R_{\alpha\beta})^{kl}_{ij}$.

"Extra dimensions" $C \implies$

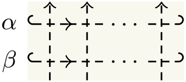
1. Wilson line $W_\alpha$ carries a spectral parameter $z_\alpha \in C$.
2. Transfer matrices

$$t_\alpha(z_\alpha) = \quad \alpha$$

$$\in \operatorname{End}\left(\bigotimes_{\beta=1}^{n} V_\beta\right)$$

commute:



$$=$$



$$\iff [t_\alpha(z_\alpha), t_\beta(z_\beta)] = 0\,.$$

$\implies$ series of commuting conserved charges:

$$t(z) = \sum_{n=-\infty}^{\infty} t_n z^n\,; \qquad [t_m, t_n] = 0\,.$$

The lattice model is integrable.

The R-matrix solves the Yang–Baxter equation.

For Wilson lines in the vector rep, we get the R-matrix for

- rational 6-vertex model for $C = \mathbb{C}$; [C]
- trigonometric 6-vertex model for $C = \mathbb{C}^\times$. [CWY]

Their conserved charges include Hamiltonians for

- XXX spin chain;
- XXZ spin chain.

For $C = E$, we get Felder's elliptic dynamical R-matrix, carrying an extra parameter $\lambda \in \mathfrak{h}^*_{\mathbb{C}}$ associated with d.o.f. on faces. [CY]

This is related by conjugation to Baxter's elliptic R-matrix for

$$\text{8-vertex model} \longleftrightarrow \text{XYZ spin chain}\,.$$

# 4d CS from 6d

6d MSYM is dimensional reduction of 10d SYM:

- 4 scalars $\phi_\mu = A_{6+\mu}$, $\mu = 1, \ldots, 4$
- R-symmetry $\mathrm{Spin}(4)_R$

Put 6d MSYM on

$$M \times C$$

and topologically twist along $M$:

- $\mathrm{Spin}(4)_M \to \mathrm{Spin}(4)'_M = \mathrm{diag}(\mathrm{Spin}(4)_M \times \mathrm{Spin}(4)_R)$
- Scalars become a 1-form $\phi = \phi_\mu \mathrm{d}x^\mu$ on $M$.
- $\exists$ supercharge $Q$ that is scalar on $M$ and

$$Q^2 = 0\,, \quad \partial_\mu = \{Q, \ldots\}\,, \quad \partial_{\bar{z}} = \{Q, \ldots\}\,.$$

$Q$-cohomology is topological on $M$ and holomorphic on $C$.

Take

$$M = \mathbb{R}^2 \times \Sigma$$

and describe the 6d theory as a 2d theory on $\mathbb{R}^2$.

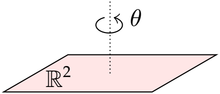
We get a B-twisted $\mathcal{N} = (2,2)$ SUSY gauge theory with

- gauge group $\mathcal{G} = \mathrm{Map}(\Sigma \times C, G)$;
- vector multiplet $A_1$, $A_2$, $\phi_1$, $\phi_2$;
- 3 adjoint chiral multiplets $\mathcal{A}_3 = A_3 + \mathrm{i}\phi_3$, $\mathcal{A}_4 = A_4 + \mathrm{i}\phi_4$, $A_{\bar{z}}$;
- superpotential

$$W = -\frac{\mathrm{i}}{e^2} \int_{\Sigma \times C} \mathrm{d}z \wedge \mathrm{CS}(\mathcal{A}) \, .$$

Quite generally, a B-twisted gauge theory on $\mathbb{R}^2$ can be subjected to Ω-deformation: [Nekrasov '02, Y '14, Luo–Tan–Y–Zhao '14]

$$Q^2 = \epsilon \partial_\theta \,, \quad \epsilon \in \mathbb{C} \,.$$




This reduces the path integral to

$$\int_{\gamma/\mathcal{G}_\mathbb{C}} \mathrm{d}\varphi_0 \exp\left(\frac{2\pi}{\epsilon} W(\varphi_0)\right) ,$$

where

- $\varphi_0$ is the constant mode of the chiral multiplet $\varphi$;
- $\gamma$ is constructed from gradient flows ("Lefschetz thimble").

This is a 0d gauged sigma model on $\gamma$, with gauge group $\mathcal{G}_\mathbb{C}$.

Apply this mechanism to the 6d theory on $\mathbb{R}^2 \times \Sigma \times C$.

Viewing it as B-twisted gauge theory on $\mathbb{R}^2$, we get

$$\int_{\gamma/\mathcal{G}_{\mathbb{C}}} \mathcal{D}\mathcal{A} \exp\left(\frac{\mathrm{i}}{\pi\hbar}\int_{\Sigma\times C} \mathrm{d}z \wedge \mathrm{CS}(\mathcal{A})\right), \quad \hbar = -\frac{\epsilon e^2}{2\pi^2}\,.$$

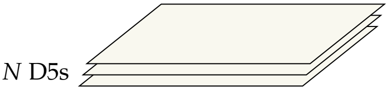
Therefore,

4d CS = 6d MSYM, twisted and $\Omega$-deformed on $\mathbb{R}^2$ .

This provides a nonperturbative definition of 4d CS.

[cf. Ashwinkumadr–Tan–Qin]

# String theory realization



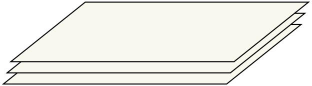

Consider a stack of $N$ D5-branes.

Put them on

$$M \times C \subset T^*M \times C\,.$$

On D5s lives 6d MSYM with $G = \mathrm{SU}(N)$.

Normal directions to D5s are parametrized by "scalars" $\phi_\mu$.

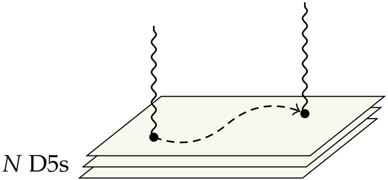



These "scalars" are really a 1-form $\phi = \phi_\mu \mathrm{d}x^\mu$

$\implies$ 6d MSYM is twisted.

Take $M = \mathbb{R}^2 \times \Sigma$. $\Omega$-deformation is given by a nontrivial RR 2-form $C_2$. ("RR fluxtrap" [Hellerman–Orlando–Reffert])

This realizes 4d CS.




Wilson lines are open strings ending on D5s.

$N$ choices of D5s to end on, hence the vector rep of $\mathrm{SU}(N)$.

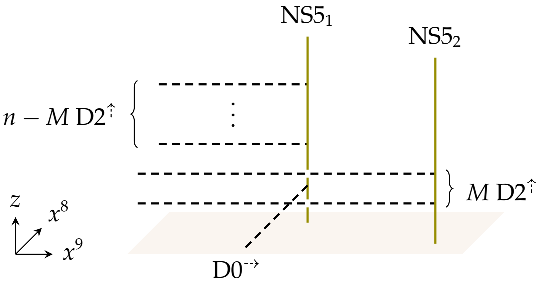
Other reps have more elaborate construction involving additional branes. [Yamaguchi ’06, Gomis–Passerini ’06]

# Dualities

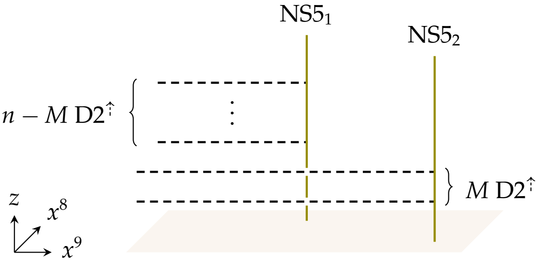
Apply S-duality and T-duality in the horizontal direction of $T^2$:

$$\begin{array}{ccccc} \text{D5} & \xrightarrow{S} & \text{NS5} & \xrightarrow{T} & \text{NS5} \\ \text{F1}^{\dashrightarrow} & \xrightarrow{S} & \text{D1}^{\dashrightarrow} & \xrightarrow{T} & \text{D0}^{\dashrightarrow} \\ \text{F1}^{\uparrow} & \xrightarrow{S} & \text{D1}^{\uparrow} & \xrightarrow{T} & \text{D2}^{\uparrow} \\ C_2 & \xrightarrow{S} & B_2 & \xrightarrow{T} & B_2 \end{array}$$

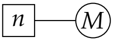
For $N = 2$, the brane configuration looks like




The magnon number $M$ counts "up" spins in the spin chain.

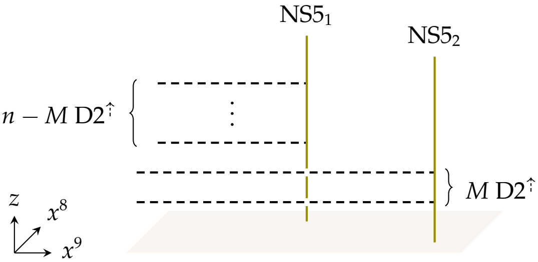



For $C = \mathbb{C}$ and $B_2 = 0$, NS5–D2 realizes 2d $\mathcal{N} = (4,4)$ gauge theory with $G = \mathrm{U}(M)$ and $n$ hypermultiplets:




$z_\alpha$ determine twisted masses associated with $\mathrm{U}(n)$ flavor group.

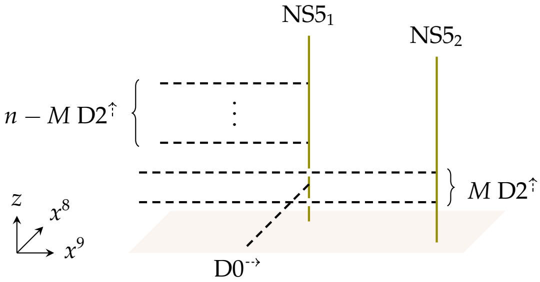
$B_2$ gives twisted masses breaking $\mathcal{N} = (4,4)$ SUSY to $(2,2)$.




Generically, the theory is in the Higgs phase.

For $z_\alpha$ all equal and $B_2 = 0$, the theory flows to A-model on $T^*\mathrm{Gr}(M, n)$.

For $N \geq 2$, the target space is $T^*$(partial flag manifold).




D0 represents a local operator and generates the chiral ring.

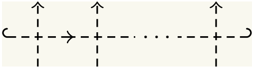
The chiral ring is the quantum cohomology

$$QH(T^*\mathrm{Gr}(M,n)) \,.$$

For generic $z_\alpha$ and $B_2$, it is equivariant quantum cohomology

$$QH_{(\mathbb{C}^\times)^n\times\mathbb{C}^\times}(T^*\mathrm{Gr}(M,n)) \,.$$

In the lattice model, D0 represents a transfer matrix:



Therefore,

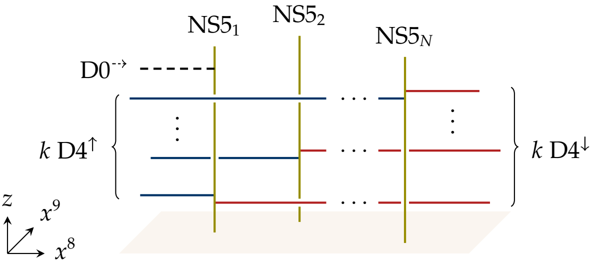
$QH_{(\mathbb{C}^\times)^n \times \mathbb{C}^\times}(T^*\mathrm{Gr}(M,n)) =$ algebra of conserved charges in $M$-magnon sector of XXX spin chain of length $n$.

This is 2d Nekrasov–Shatashvili correspondence.

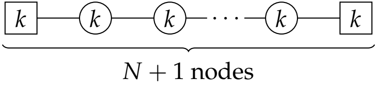
[NS, Maulik–Okounkov, ...]

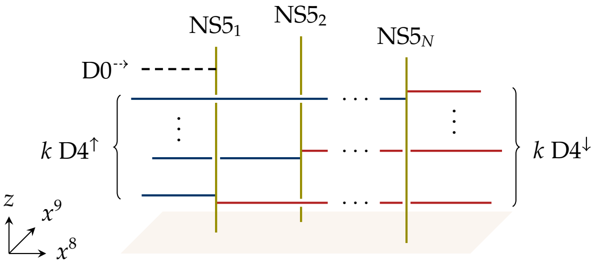
Brane construction provides a concrete realization of the transfer matrix, which has not been understood geometrically.

Replace D2s with D4s:



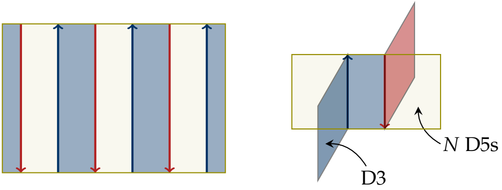

D4–NS5 realizes 4d $\mathcal{N} = 2$ SUSY gauge theory described by a linear quiver:

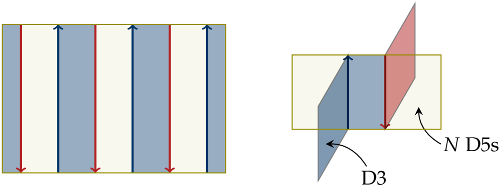





In the original duality frame, D4s are D3s.

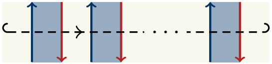
D3s combine with D5s and produce strips of surface operators:






A strip of surface operator is a "thick" line operator, carrying a Verma module of $\mathfrak{sl}_N$. [CY, C–Gaiotto–Y]

The corresponding transfer matrix



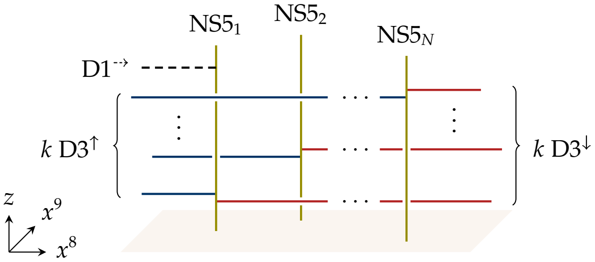
produces a "noncompact" XXX spin chain.

This explains 4d Nekrasov–Shatashvili correspondence.

[NS, Chen–Dorey–Hollowood–Lee, . . . ]

What about open spin chains?

If the horizontal direction of the lattice is not $S^1$, we cannot apply the T-duality. But we can still apply S-duality:




D3–NS5 realizes 3d $\mathcal{N} = 4$ linear quiver theory.

Open XXX spin chain has a large symmetry, Yangian.

This explains why Yangian appears in this 3d theory.

[Bullimore–Dimofte–Gaiotto, BFKKNWW]

Other chains of dualities produce other QFT setups:

- Apply T-duality twice:

$$\text{D1–D3–NS5} \longrightarrow \text{D3–D5–NS5}$$

This produces 4d $\mathcal{N} = 1$ "brane tiling" theories + surface operators. [Maruyoshi–Y]

- Further apply T-duality and lift to M-theory:

$$\text{D3–D5–NS5} \nearrow \text{D2–D4–NS5} \longrightarrow \text{M2–M5}$$
$$\text{D3–D5–NS5} \searrow \text{D4–D4–NS5} \longrightarrow \text{M5–M5}$$

We get 4d $\mathcal{N} = 1$ theories of "class $\mathcal{S}_k$" + surface operators

[Gaiotto–Rastelli–Razamat, Gadde–Gukov, Gaiotto–Razamat, ...]

Summary:

4d CS arises from Ω-deformed twisted 6d MSYM. The latter can be embedded into string theory, and string dualities allow us to connect it to various other QFTs in which integrable systems have been found to arise.

Outlook:

- More chains of dualities.
- T-operators from Wilson lines, Q-operators from ’t Hooft lines. [Work in progress with Costello and Gaiotto]
- Relation to spin chains in AdS/CFT integrability?