Joint distribution of the base-q and Ostrowski digital sums

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Setup

Let q be an integer greater than or equal to 2, and let $S_q(n)$ denote the sum of digits of n in base q.

Gel′fond (1968)

If m_1 is coprime to $q-1$, then there exists $\delta > 0$ such that for all integers r,

$$|\{0 \leq n < N : S_q(n) \equiv r \pmod{m_1}\}| = \frac{N}{m_1} + O(N^{1-\delta}).$$

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History

- **Gel′fond's conjecture:** If $q_1, q_2, m_1, m_2 \geq 2$ are integers with

$$\gcd(q_1, q_2) = \gcd(m_1, q_1 - 1) = \gcd(m_2, q_2 - 1) = 1,$$

then there exists $\delta > 0$ such that for all integers r_1, r_2,

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Ostrowski α-numeration system

Let α be an irrational real number having continued fraction expansion $[a_0; a_1, \ldots]$. Let $(q_i)_{i\geq 0}$ be the sequence of the denominators of the convergents to the continued fraction expansion. For $i \geq 0$, $q_{i+1} = a_{i+1}q_i + q_{i-1}$.

Ostrowski (1922)

Every non-negative integer n can be expressed uniquely as

$$n = \sum_{0\leq i\leq \ell} b_i q_i,$$

where the b_i's are integers satisfying

(i) $0 \leq b_0 < a_1$.

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Example

If

$$\alpha = \frac{1+\sqrt{5}}{2} = [\overline{1}],$$

then

$$q_{-1} = 0, \; q_0 = 1,$$

and for $i \geq 0$,

$$q_{i+1} = a_{i+1}q_i + q_{i-1} = q_i + q_{i-1}.$$

The sequence (q_i) is the sequence of Fibonacci numbers. Every non-negative integer can be uniquely expressed as a sum of non-consecutive Fibonacci numbers. This representation is known as the *Zeckendorf representation* of integers.

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If the Ostrowski α-representation of a positive integer n is given by

$$n = \sum_{0 \le i \le \ell} b_i(n) q_i,$$

let

$$S_\alpha(n) = \sum_{0 \le i \le \ell} b_i(n)$$

be the sum of digits.

Comparison with S_q

	S_q	S_α
Submultiplicativity	✓ $S_q(n_1 n_2)$ $\leq S_q(n_1)S_q(n_2)$	✗ $\alpha = [0; \overline{1}]$ $(q_i)_{i\geq 0} : 1, 1, 2, 3, 5, \ldots$ $n_1 = 2, n_2 = 3$ $n_1 n_2 = q_4 + q_1$
Subadditivity	✓ $S_q(n_1 + n_2) \leq$ $S_q(n_1) + S_q(n_2)$	✗ $\alpha = [0; \overline{1, 3}]$, $(q_i)_{i\geq 0} : 1, 1, 4, 5, 19, 24, \ldots$ $n_1 = n_2 = 19$ $n_1 + n_2 = q_5 + 2q_3 + q_2$

History

Coquet, Rhin & Toffin (1983) gave three sufficient conditions for the set

$$\{n \in \mathbb{N} : S_q(n) \equiv r_1 \pmod{m_1},\ S_\alpha(n) \equiv r_2 \pmod{m_2}\}$$

to have asymptotic density equal to $1/(m_1 m_2)$.

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Coquet, Rhin & Toffin

If the sequence (q_i) is lacunary and $\gcd(a_i, m_2)$ is equal to one for infinitely many indices i, then as $N \to \infty$,

$$|\{0 \leq n < N : S_q(n) \equiv r_1 \pmod{m_1},\ S_\alpha(n) \equiv r_2 \pmod{m_2}\}| \sim \frac{N}{m_1 m_2}.$$

Note that this condition is satisfied for

$$\alpha = [0; \overline{1, m}] = \frac{-m + \sqrt{m^2 + 4m}}{2},\ m \geq 2$$

as

$$\frac{q_{i+1}}{q_i} \geq \begin{cases} 1 + \frac{q_{i-1}}{q_i} \geq 1 + \frac{1}{m+1} & \text{if } a_{i+1} = 1, \\ m + \frac{q_{i-1}}{q_i} \geq m + \frac{1}{2} & \text{if } a_{i+1} = m. \end{cases}$$

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Theorem (S., 2018)

Let $q \geq 2$ be an integer and let

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Let m_1, m_2 be positive integers with $\gcd(m, m_2) = 1$. There exists $\delta > 0$ such that for all integers r_1, r_2,

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- The case $\alpha = [\,\overline{1}\,]$ was proved by Spiegelhofer (2014).

Relation to exponential sums

Let $e(x)$ denote $\exp(2\pi ix)$. Since

$$\frac{1}{b}\sum_{0\le \ell<b} e\left(\frac{a}{b}\ell\right) = 1 \text{ or } 0$$

according to whether b divides a or not, we have

$$\begin{aligned}
&|\{0 \le n < N : S_q(n) \equiv r_1 \pmod{m_1},\ S_\alpha(n) \equiv r_2 \pmod{m_2}\}| \\
&= \sum_{n<N} \frac{1}{m_1} \sum_{0\le k_1<m_1} e\left(k_1 \frac{S_q(n)-r_1}{m_1}\right) \frac{1}{m_2} \sum_{0\le k_2<m_2} e\left(k_2 \frac{S_\alpha(n)-r_2}{m_2}\right) \\
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Let $e(x)$ denote $\exp(2\pi ix)$. Since

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Theorem

Let $q \geq 2$ be an integer and let

$$\alpha = [0; \overline{1, m}], \ \ m \geq 2.$$

Let $\theta, \gamma \in \mathbb{R}$ with $\|m\gamma\| = \min_{j \in \mathbb{Z}} |m\gamma - j| \neq 0$. Then there exists $\delta > 0$ such that

$$\sum_{n<N} e(\theta S_q(n) + \gamma S_\alpha(n)) = O(N^{1-\delta}).$$

Method of proof

- Using the Weyl-van der Corput inequality, the problem is reduced to estimation of sums involving

$$\sum_{n<N} e(\theta(S_q(n+r)-S_q(n)))e(\gamma(S_\alpha(n+r)-S_\alpha(n))).$$

- It is *enough* to consider truncated sum-of-digits functions. Given positive integers k and t, let $t_\alpha(n;k)$ denote the truncation of the Ostrowski α-representation of n after k digits, i.e.,

$$t_\alpha(n;k) = \sum_{0\le i\le k-1} b_i(n)q_i,$$

$$S_{\alpha,k}(n) = \sum_{0\le i\le k-1} b_i(n)$$

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Discrete Fourier Transform

Let $G_t(\ell) = G_t(\ell, \theta)$ denote the discrete Fourier coefficients of the function $e(\theta S_{q,t}(n))$, i.e.,

$$G_t(\ell, \theta) = \frac{1}{q^t} \sum_{u < q^t} e(\theta S_{q,t}(u) - \ell u q^{-t}).$$

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An analytic way of capturing $t_\alpha(n;k)$ (k even)

- Let

$$\varphi = m + 1 + \alpha = \frac{m + 2 + \sqrt{m^2 + 4m}}{2}.$$

Let $k \geq 2$ and $0 \leq u < q_k$. There is a set $I_k(u)$ such that given $n \geq 0$, we have

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We have

$$n\varphi - t_\alpha(n;k)\varphi = \varphi \sum_{\ell \geq k} b_\ell(n) q_\ell$$

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The sets

$$I_k(u) + \mathbb{Z},\ 0 \le u < q_k,$$

form a partition of $\mathbb{R}$.

Proof. For each integer u with $0 \le u < q_k$, let

$$J_k(u) = I_k(u) \bmod 1.$$

- The sum of the measures of the sets $J_k(u)$ is 1.
- The sets $J_k(u)$ cover the interval $[0, 1)$. If not, pick

  $$x \in [0,1) \setminus \bigcup_{0 \le u < q_k} J_k(u).$$

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  Then, there exists $\epsilon > 0$ such that the sets $[x, x + \epsilon]$ and $\bigcup J_k(u)$ are disjoint. Since the sequence $(\{n\varphi\})$ is dense in $[0, 1)$, there is an integer n_0 such that $\{n_0\varphi\} \in [x, x + \epsilon]$. Therefore $\{n_0\varphi\} \notin \bigcup_{0 \le u < q_k} J_k(u)$, which is a contradiction.
- The sets $J_k(u)$ are disjoint.

The sets

$$I_k(u) + \mathbb{Z},\ 0 \le u < q_k,$$

form a partition of $\mathbb{R}$.

Proof. For each integer u with $0 \le u < q_k$, let

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An analytic way of capturing $t_\alpha(n; k)$ (k even)

- Let
$$\varphi = m + 1 + \alpha = \frac{m + 2 + \sqrt{m^2 + 4m}}{2}.$$
Let $k \geq 2$ and $0 \leq u < q_k$. There is a set $I_k(u)$ such that given $n \geq 0$, we have
$$t_\alpha(n; k) = u$$
if and only if
$$n\varphi \in I_k(t_\alpha(n; k)) + \mathbb{Z}.$$
- This characterization, together with trigonometric approximation of the characteristic function of the sets $I_k(u) + \mathbb{Z}$, yields a representation for $e(\gamma S_{\alpha,k}(n))$.

A representation for $e(\gamma S_{\alpha,k}(n))$

- Let H be a positive integer. Then

$$e(\gamma S_{\alpha,k}(n)) = \sum_{|h|\leq H} e((-1)^k \varphi hn) M_k(h,\gamma),$$

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Distribution of the sum of digits function in residue classes

Let $\alpha = [0; \overline{1, m}]$, $m \geq 2$.

Theorem

Let m_1, m_2 be positive integers with $\gcd(m, m_2) = 1$. There exists $\lambda < 1$ such that for all integers r_1, r_2,

$$|\{n < N : n \equiv r_1 \pmod{m_1},\ S_\alpha(n) \equiv r_2 \pmod{m_2}\}| = \frac{N}{m_1 m_2} + O(N^\lambda).$$

Corollary

Let z, m_2 be positive integers > 1 with $\gcd(m, m_2) = 1$. Then there exists $\lambda' < 1$ such that for all integers r,

$$|\{n < N : n \text{ is } z\text{-power free},\ S_\alpha(n) \equiv r \pmod{m_2}\}| = \frac{N}{m_2 \zeta(z)} + O(N^{\lambda'}).$$

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Thank You!