Low discrepancy sequences failing Poissonian pair correlations

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May 8, 2019

Discrepancy and Equidistribution

Given a sequence $\{x_n\}_{n\geq 1}$ of reals in $[0,1]$, its discrepancy is defined as

$$D_N = \sup_{0\leq\alpha\leq\beta\leq 1} \left| \frac{\#\{\, 0 \leq i \leq N \mid \alpha \leq x_i \leq \beta \,\}}{N} - (\beta - \alpha) \right|$$

The sequence is $\{x_n\}_{n\geq 1}$ is equidistributed if $\lim\limits_{N\to\infty} D_N = 0$.
That is, if for every $\alpha \leq \beta$:

$$\lim_{N\to\infty} \frac{\#\{\, 0 \leq i \leq N \mid \alpha \leq x_i \leq \beta \,\}}{N} = (\beta - \alpha).$$

Poissonian pair correlations

Given a sequence $(x_n)_{n\geq 1}$ in the unit interval, we define:

$$F_N(s) = \frac{1}{N} \# \left\{ (i,j) \,:\, 1 \leq i \neq j \leq N \text{ and } \|x_i - x_j\| < \frac{s}{N} \right\}.$$

A sequence $\{x_n\}_{n\geq 1}$ of numbers in $[0,1]$ has Poissonian pair correlations if it satisfies

$$\lim_{N\to\infty} F_N(s) = 2s.$$

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- The sequence of fractional parts of $(\sqrt{n})_{n\geq 1}$ has Poissonian pair correlations. (D.El Baz, J.Marklof, I.Vinogradov, 2015)

The question

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But what happens if α converges to normality really fast?

Pascal Matrices Modulo 2

We define a family of matrices using the Pascal triangle modulo 2:

$$
\begin{array}{cccccc}
\dots & 1 & 1 & 1 & 1 & 1 \\
\dots & 5 & 4 & 3 & 2 & 1 \\
\dots & 15 & 10 & 6 & 3 & 1 \\
\dots & 35 & 20 & 10 & 4 & 1 \\
\dots & 70 & 35 & 15 & 5 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
\longrightarrow
\begin{array}{cccccc}
\dots & 1 & 1 & 1 & 1 & 1 \\
\dots & 1 & 0 & 1 & 0 & 1 \\
\dots & 1 & 0 & 0 & 1 & 1 \\
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$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

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\dots & 0 & 1 & 1 & 1 & 1 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

$$
M_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}
$$

Alternative Characterization

The previous definition is equivalent to

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- $M_d \in \mathbb{F}_2^{2^d \times 2^d}$.
- M_d is inversible for all $d \geq 1$.
- The last column of M_d is always the vector of ones.

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- We take $d \geq 1$ and $e = 2^d$. We let

$$w_0, w_1, \ldots, w_{2^e-1}$$

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- We define the Big Block d as the concatenation of those products:

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- The Levin Constant is defined as the number whose fractional part is the concatenation of all Big Blocks:

$$\lambda = 0.\lambda_0 \lambda_1 \lambda_2 \ldots$$

The Levin Constant

$$\lambda = 0.\ \underbrace{0\,1}_{\lambda_0}\ \underbrace{00\,11\,10\,01}_{\lambda_1}\ \underbrace{\begin{matrix}\mathsf{0000\,1111\,1010\,0101\,1100\,0011\,0110\,1001}\\ 1000\,0111\,0010\,1101\,0100\,1011\,1110\,0001\end{matrix}}_{\lambda_2}\ \ldots$$

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Theorem (Levin, 1999)
The discrepancy of the first N terms of the sequence of fractional parts of $(\lambda 2^n)_{n\geq 1}$ is $O((\log N)^2/N)$.

Main Result

Theorem 1

The sequence of the fractional parts of $(\lambda 2^n)_{n\geq 1}$ does not have Poissonian pair correlations.

Lemmas

Lemma 1

Every Big Block λ_d is a concatenation of all words of length 2^d.

Lemmas

Lemma 2

Let $d \geq 0$. Take $n \geq 0$. Then $M_d w_n$ finishes with a zero if and only if n is even.

$$\lambda \quad = \quad 0. \quad \begin{array}{l} \color{red}{0}\,1 \\ 0\color{red}{0}\;11\;1\color{red}{0}\;01 \\ 000\color{red}{0}\;1111\;101\color{red}{0}\;0101\;110\color{red}{0}\;0011\;011\color{red}{0}\;1001 \\ 100\color{red}{0}\;0111\;001\color{red}{0}\;1101\;010\color{red}{0}\;1011\;111\color{red}{0}\;0001 \\ \ldots \end{array}$$

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Lemma 3

Let $d \geq 0$ and n an even number. Then $M_d w_n$ and $M_d w_{n+1}$ are complementary words.

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& \ldots
\end{aligned}
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Idea of the proof

- We take $d \geq 1$, $e = 2^d$ and $N_d = 2^{d+e+1}$.
- We will give a lower bound for

$$F_{N_d}(2) = \frac{1}{N_d} \# \left\{ (i,j) \,:\, 1 \leq i \neq j \leq N_d \text{ and } \|2^i\lambda - 2^j\lambda\| < \frac{2}{2^{d+e+1}} \right\}$$

  and prove that it diverges.
- Then $\lim_{d\to\infty} F_{N_d}(2) = \infty$
- We count occurrences of words of the type:

$$a = a_1 a_2 \dots a_{d+e} \in \{0,1\}^{d+e}$$

  such that $a_1 \dots a_d$ and $a_{e+1} \dots a_{e+d}$ are complementary words. We say that a is a witness.

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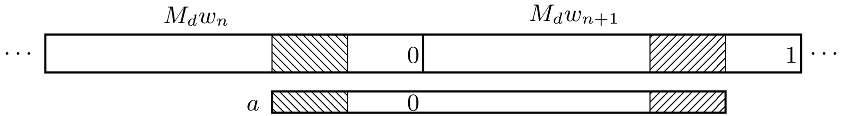
We count occurrences of witnesses in λ_d that start in an even word and that do not range over more than two words.



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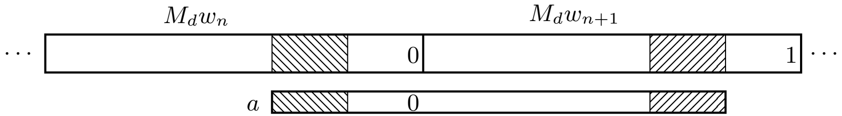
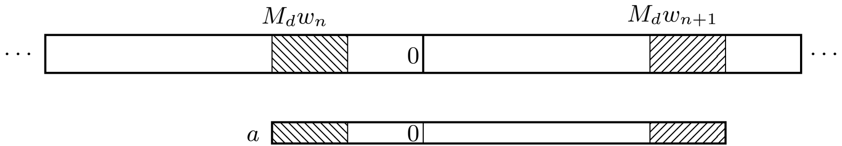


It turns out that there is as many occurrences of a witness a in λ_d as there are zeros in the middle part of a.



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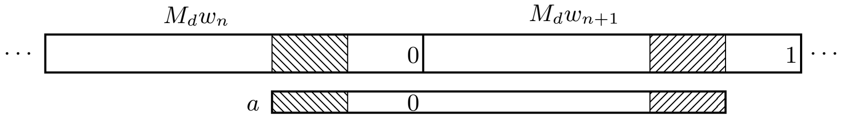

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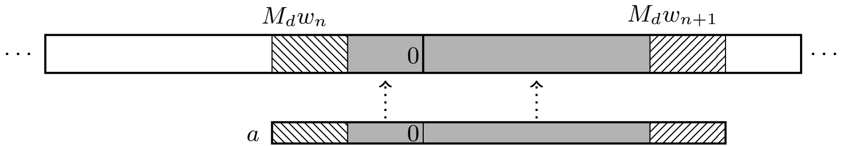

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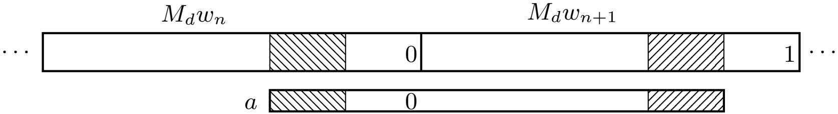

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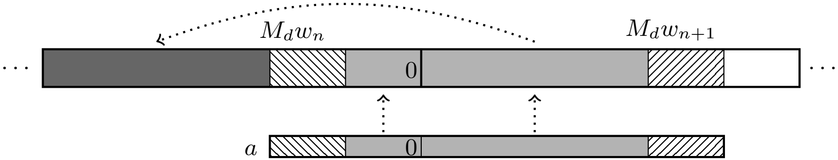

Then the number of pairs of occurrences ofthese type of words in λ_d gives us a lower bound for $F_{N_d}(2)$:

$$\begin{aligned}
F_N(2) &\geq \frac{1}{N} \sum_{k=0}^{e-d+1} 2\left(2^{d-1}\binom{e-d+1}{k}\binom{k}{2}\right) \\
&= \frac{1}{2^{e+1}} \sum_{k=0}^{e-d+1} \binom{e-d+1}{k}\binom{k}{2} \\
&= \frac{1}{2^{e+1}} \binom{e-d+1}{2} 2^{e-d+1-2} \\
&= \frac{1}{8e}(e-d+1)(e-d) \to_{d\to\infty} \infty
\end{aligned}$$

So the sequence of the fractional parts of the Levin constant does not have Poisson pair correlations.

Levin-like Constants

- Let $d \geq 1$ and $e = 2^d$. A tuple $\nu = (n_1, \ldots, n_e)$ of non-negative integers is suitable if $n_e = 0$ and $n_{i+1} \leq n_i \leq n_{i+1} + 1$ for all i.
  - $(1, 1, 1, 0)$ is suitable
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- If $\nu = (n_1, \ldots, n_e)$ is suitable and $C_1, \ldots, C_e$ are the columns of M_d, define

$$M_d^\nu = (\sigma^{n_1}(C_1), \ldots, \sigma^{n_e}(C_e))$$

  where σ rotates a vector one position.

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- Given a sequence $\nu_1, \nu_2, \nu_3, \dots$ of suitable tuples, define
$$\overline{\lambda_d} = (M_d^{\nu_d} w_0) \dots (M_d^{\nu_d} w_{2^e - 1})$$

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- Define
$$\overline{\lambda} = \overline{\lambda_0}\, \overline{\lambda_1}\, \overline{\lambda_2} \ldots$$
- **Theorem** (V.Becher, O.Carton, 2018)
The discrepancy $D_N((\{\lambda 2^n\})_{n \geq 1})$ is $O((\log N)/N^2)$.

Question

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How do you prove it?

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Are **any** of the Levin-like constants Poissonian pair correlated? No.

How do you prove it? It's the same.

Lemmas

Lemma 4

Take ν a suitable tuple and $d \geq 1$. Then M_d^ν is invertible.

Lemma 5

Take ν a suitable tuple and $d \geq 1$.Then there are two possibilities, depending of ν:

- *For all w: w finishes with a zero if and only if $M_d^\nu w$ finishes with a zero;*
- *For all w: w finishes with a zero if and only if $M_d^\nu w$ starts with a zero.*

Lemma 6

Take ν suitable, $d \geq 1$ and n an even number. Then $M_d^\nu w_n$ and $M_d^\nu w_{n+1}$ are complementary words.

Thank you!