Quantifying Equidistribution

A Survey of Finite-State Dimensions

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Equidistribution : Arithmetic, Computational, and Probabilistic Aspects
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Very Big Picture

Very Big Picture

Equidistribution
Weyl's criterion
1916

Very Big Picture

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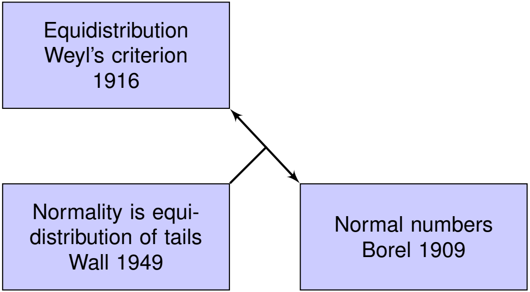
Weyl's criterion
1916

Normal numbers
Borel 1909

Very Big Picture



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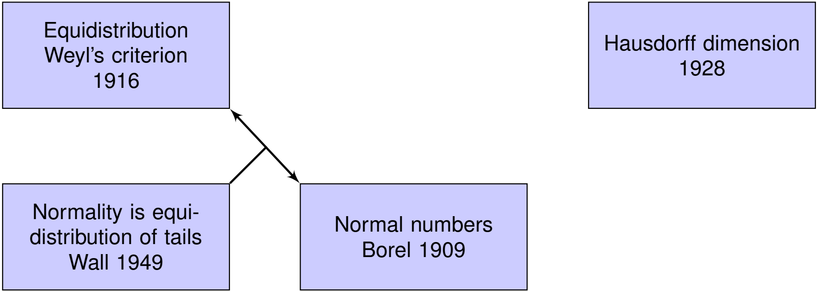

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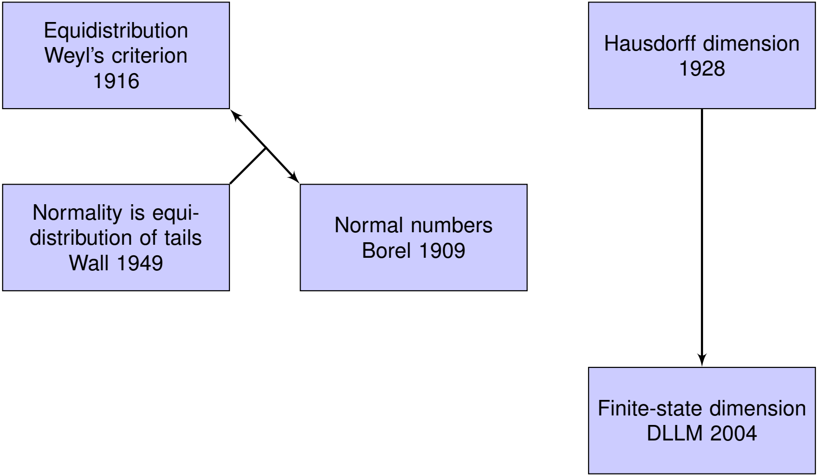

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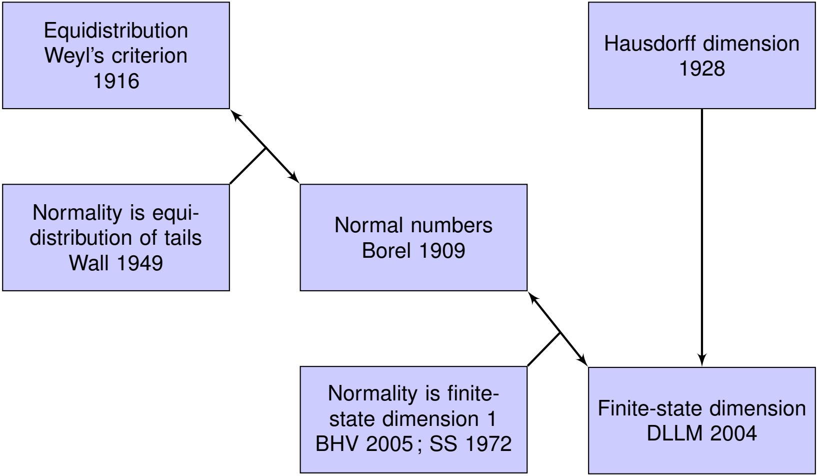

Equidistribution

A sequence $\alpha = (\alpha_n \mid n \in \mathbb{N})$ of real numbers is **uniformly distributed** (or **equidistributed**) **mod 1** if, for all $0 \leq a < b \leq 1$,

$$\lim_{m \to \infty} \frac{\left| \{ n < m \mid \{\alpha_n\} \in [a, b] \} \right|}{m} = b - a$$

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Weyl's criterion (1916)

This holds if and only if, for all $h \in \mathbb{Z}^+$,

$$\lim_{m \to \infty} \frac{1}{m} \sum_{n=0}^{m-1} e^{2\pi i h \alpha_n} = 0$$

Normality

$\Sigma = \{0, 1, \cdots, b-1\} \quad (2 \leq\in \mathbb{N})$
For $S \in \Sigma^\omega$, $w \in \Sigma^+$, and $n \in \mathbb{Z}^+$,

$$\begin{aligned} \mathrm{freq}_n(w, S) &= \frac{\big|\{i < n | S[i \mathinner{..} i + |w| - 1] = w\}\big|}{n} \\ &= n\text{-th frequency of } w \text{ in S} \end{aligned}$$

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A real number α is **normal in base** b if the base-b expansion of $\{\alpha\}$ is a normal sequence.

Wall (1949)

A real number α is normal in base b if and only if the sequence $(b^n\alpha \mid n \in \mathbb{N})$ is uniformly distributed mod 1.

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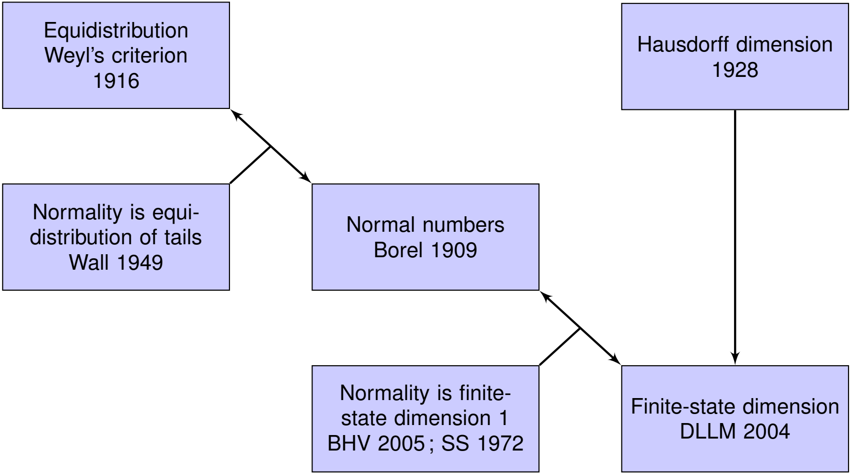

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Here we focus on the metric spaces Σ^ω equipped with the metric

$$d(S,T) = b^{-\min\{n \mid S[n] \neq T[n]\}}.$$

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The **Kolmogorov complexity** of a string $w \in \Sigma^*$ is

$$K(w) = K^{\emptyset}(w).$$

The **lower** and **upper algorithmic dimensions** of a sequence $S \in \Sigma^\omega$ are

$$\dim(S) = \liminf_{w \to S} \frac{K(w)}{|w| \log|\Sigma|} \quad \text{(Lutz 2003, Mayordomo 2002)}$$

and

$$\mathrm{Dim}(S) = \limsup_{w \to S} \frac{K(w)}{|w| \log|\Sigma|}, \quad \text{(Athreya, Hitchcock, Lutz, Mayordomo 2007)}$$

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In general, $0 \leq \dim(S) \leq \mathrm{Dim}(S) \leq 1$.

Define the relativized algorithmic dimensions $\dim^A(S)$ and $\mathrm{Dim}^A(S)$ analogously.

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Point-to-Set Principle (N. Lutz and J. Lutz 2018)

For every $E \subseteq \Sigma^\omega$

$$\dim_H(E) = \min_{A \in \mathbb{N}} \sup_{S \in E} \dim^A(S).$$

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N. Lutz and Stull (2017) and N. Lutz (2017) have used this method to solve open problems in classical geometric measure theory.

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Several equivalent characterizations are now known. Today we use the compression characterization.

Shannon (1948)

*A **finite-state compressor (FSC)** is a 5-tuple*

$$C = (Q, \Sigma, \delta, s, v),$$

where

- *Q is a finite set of **states**;*
- *$\Sigma = \{0, 1, \cdots, b-1\}$ is the **input alphabet**;*
- *$\delta : Q \times \Sigma \to Q$ is the **transition function**;*
- *$s \in Q$ is the **start state**; and*
- *$v : Q \times \Sigma \to \{0, 1\}^*$ is the **output function**.*

*Write $C(w)$ for the **cumulative output** of C on an input $w \in \Sigma^*$.*

An **information-lossless finite-state compressor (ILFSC)** is an FSC $C = (Q, \Sigma, \delta, s, v)$ for which the function

$$\begin{aligned} \Sigma^* &\to \{0,1\}^* \times Q \\ w &\mapsto \big(C(w), \delta(s,w)\big) \end{aligned}$$

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Today's definitions :

$$dim_{\mathrm{FS}}(S) = \inf_C \liminf_{w \to S} \frac{C(w)}{|w| \log|\Sigma|}. \quad \text{(Dai, Lathrop, Lutz, Mayordomo 2004)}$$

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In general, $0 \le \mathrm{dim}_{\mathrm{FS}}(S) \le \mathrm{Dim}_{\mathrm{FS}}(S) \le 1$.

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Elvira will tell you about very recent work extending Schnorr and Stimm's remarkable theorem.

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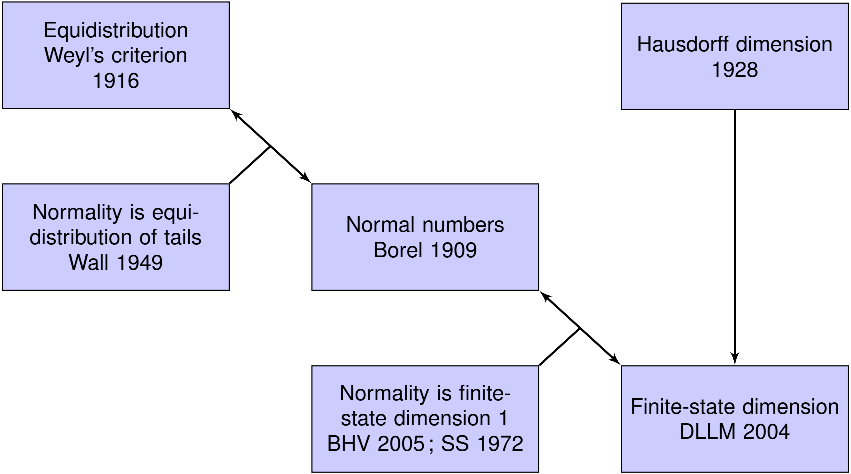

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QUESTION : Which theorems about normality can be better understood as the dimension-1 special case of quantitative theorems about finite-state dimension ?

EXAMPLE 1 : Copeland-Erdös sequences

$$2 \leq b \in \mathbb{N}, \Sigma = \{0, 1, \cdots, b-1\}.$$
For $n \in \mathbb{Z}^+$, $\sigma_b(n) =$ the base-b expansion of n.

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The **base-*b* Copeland-Erdös sequence** of an infinite set

$$A = \{a_1 < a_2 < \cdots\} \subseteq \mathbb{Z}^+$$

is the sequence $\mathrm{CE}_b(A) = \sigma_b(a_1)\sigma_b(a_2)\cdots \in \Sigma^\omega$.

Champernowne (1933)

*The **decimal Champernowne** sequence*

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Champernowne conjectured that

$$\mathrm{CE}_{10}(\mathit{PRIMES}) = 2357111317\cdots$$

is also normal.

Copeland-Erdös (1946)

For all b,
$\mathrm{CE}_b(\mathit{PRIMES})$ *is normal.*

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Outline of proof

1. For all sufficiently dense $A \subseteq \mathbb{Z}^+$, $\mathrm{CE}_b(A)$ is normal.
2. PRIMES is sufficiently dense. $\square$

Zeta dimensions (invented **many** times over the past 200 years)

*For $A \subseteq \mathbb{Z}^+$, define the A-**zeta function***

$$\zeta_A : [0, \infty) \to [0, \infty]$$
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Clearly, $0 \leq \mathrm{dim}_\zeta(A) \leq \mathrm{Dim}_\zeta(A) \leq 1$.

Gu, Lutz, and Moser (2007)

For every infinite $A \subseteq \mathbb{Z}^+$,

$$\dim_{FS}(\mathrm{CE}_b(A)) \geq \dim_\zeta(A)$$

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These inequalities are tight : For any real numbers

$$\begin{array}{ccccccc} & & \gamma & \leq & \delta & \leq & 1 \\ & & \vee\!\mid & & \vee\!\mid & & \\ 0 & \leq & \alpha & \leq & \beta & & \end{array}$$

there exists an infinite $A \subseteq \mathbb{Z}^+$ with $\dim_\zeta(A) = \alpha$, $\mathrm{Dim}_\zeta(A) = \beta$, $\dim_{FS}(\mathrm{CE}_b(A)) = \gamma$, and $\mathrm{Dim}_{FS}(\mathrm{CE}_b(A)) = \delta$.

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Remarks :

- The Copeland-Erdös density criterion is equivalent to $\dim_\zeta(A) = 1$.
- The dimension theorem required a different method.

EXAMPLE 2 : Finite-state dimension and real arithmetic.

Wall (1949)

If q is a non-zero rational, then,
for every real number α,
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The dimension result required a **very** different method.

I'm asking again, and this time I'm asking you!

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Further Directions

(1) It is routine to generalize normality to α-normality, where α is a probability measure on Σ. One can also define finite-state dimensions with respect to such α.

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If α and β are positive probability measures on Σ and $R \in \Sigma^\omega$ is α-normal, then

$$\dim_{FS}^{\beta}(R) = \mathrm{Dim}_{FS}^{\beta}(R) = \frac{\mathcal{H}(\alpha)}{\mathcal{H}(\alpha) + \mathcal{D}(\alpha \| \beta)},$$

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Can this be usefully generalized?

(2) In many contexts, it is natural to ask ‘’how much randomness‘’ a probability theorem requires. Classical examples include the law of large numbers and ergodic theorems. Current examples include stochastic chemical reaction networks and Banach-Tarski type constructions. The answers range from normality to algorithmic randomness and beyond. More general notions of normality, along lines that we’ve seen this week, may be needed to address such questions.

Thank You !