Continued Fraction Normals and Subsequence selections - a combinatorial approach

Satyadev Nandakumar, Subin Pulari, Prateek Vishnoi, Gopal Viswanathan

IIT Kanpur

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Motivation

Every irrational real number in $r \in [0,1]$ has a unique continued fraction expansion of the form

$$r = 0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}},$$

where, for every $i \in \mathbb{N}^+$, we have $a_i \in \mathbb{N}^+$.

We denote this by $[0; a_1, a_2, a_3, \dots]$.

Normality for base-b expansions of reals

Let $b \geq 2$ be an integer. Denote the set $\{0, 1, \ldots, b-1\}$ by Σ_b, and the set of finite strings drawn from this alphabet by Σ_b^*. For a finite string w, let $|w|$ denote its length.

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Definition

A real r with base-b expansion $.r_1 r_2 \ldots$ is said to be *normal in base b*, if for every $w \in \Sigma_b^*$, we have

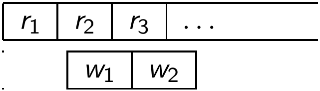
$$\lim_{n \to \infty} \frac{|\{1 \leq i < n - |w| + 1 \mid r_i \ldots r_{i+|w|-1} = w\}|}{n - |w| + 1} = \frac{1}{b^{|w|}}.$$

Sliding Block matching

r_1	r_2	r_3	$\dots$

w_1	w_2

Sliding Block matching



Normality for continued fractions

Denote the set of finite strings of positive integers by $\mathbb{N}_b^*$. For continued fractions, we consider the Gauss measure as the invariant measure.

Definition

For a Borel set $A \subseteq [0, 1]$, the *Gauss measure* of A is defined by

$$\gamma(A) = \frac{1}{\ln 2} \int_A \frac{1}{1+x} dx.$$

The left-shift transformation on continued fractions is ergodic wrt γ.

Continued fraction normality

Definition

An irrational r with continued fraction expansion $[0; a_1, a_2, \dots]$ is said to be *continued fraction normal*, if for every $w \in \mathbb{N}^*$,

$$\lim_{n\to\infty} \frac{|\{1 \leq i < n - |w| + 1 \mid a_i \dots a_{i+|w|-1} = w\}|}{n - |w| + 1} = \gamma(C_w),$$

where C_w is the cylinder set

$$\{r \in [0,1] - \mathbb{Q} \mid w \text{ is a prefix of the continued fraction expansion of } r\}.$$

Let $[0; a_1, a_2, \ldots]$ be a continued fraction normal, and let $(m, m+d, m+2d, \ldots)$, $m \geq 1$, $d \geq 2$ be an arithmetic progression of integers.

Question:

Is $[0; a_m, a_{m+d}, a_{m+2d}, \ldots]$ a continued fraction normal?

Base-b normality is preserved when we select a subsequence along an arithmetic progression. [Wall, 1949]

Subselections along Arithmetic Progression

Base-b normality is preserved when we select a subsequence along an arithmetic progression. [Wall, 1949]

Continued fraction normality is *not*!

Theorem ([Heersink and Vandehey, 2016])

For any $[0; a_1, a_2, \dots]$ *continued fraction normal, and any* $(m, m+d, m+2d, \dots)$*, the continued fraction* $[0; a_m, a_{m+d}, a_{m+2d}, \dots]$ *is not normal.*

The proof uses ergodic-theoretic techniques used in a result by Vandehey [Vandehey, 2017].

Proofs of Wall's Result

Wall's result on arithmetic progressions has different proofs using

1. Weyl's criterion
2. Automata Theoretic
3. Combinatorial (?)

A combinatorial approach

The key ingredients of the proof in [Heersink and Vandehey, 2016]:

1. $$\lim_{n\to\infty} \frac{|\{0 \leq i \leq N \mid T^{im}r \wedge T^{im+d}r \in C_{[0;1]}\}|}{N} = \sum_{a_1,a_2,\ldots,a_n\in\mathbb{N}^+} \gamma(C_{[0;1,a_1,\ldots,a_n,1]}).$$

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1. $$\lim_{n\to\infty} \frac{|\{0 \leq i \leq N \mid T^{im}r \wedge T^{im+d}r \in C_{[0;1]}\}|}{N} = \sum_{a_1,a_2,\ldots,a_n \in \mathbb{N}^+} \gamma(C_{[0;1,a_1,\ldots,a_n,1]}).$$

2. For any $n \geq 1$, $\sum_{a_1,a_2,\ldots,a_n \in \mathbb{N}^+} \gamma(C_{[0;1,a_1,\ldots,a_n,1]}) > \gamma(C_{[0;1,1]})$.

A combinatorial approach

Our approach:

1. Inequality by induction

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2. If a sequence is continued fraction normal, then the disjoint block frequencies also behave normally.

Key obstacle: loss of compactness, countably infinite alphabet!

Illustrative case for Step 1

Consider A.P.s with common difference 2.

Lemma

$$\sum_{a \in \mathbb{N}^+} \gamma(C_{[0;1,a,1]}) > \gamma(C_{[0;1,1]}).$$

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Proof Strategy: Show that the *Lebesgue measure* of $C_{[0;1,a,1]}$ is greater than that of $C_{[0;1,1,a]}$.

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Proof Strategy: Show that the *Lebesgue measure* of $C_{[0;1,a,1]}$ is greater than that of $C_{[0;1,1,a]}$.

Use the standard continued fraction recurrence for denominators of the extremities of the cylinders to show:

$$\text{denom}([0;1,1,a]) \times \text{denom}([0;1,1,(a+1)] > \text{denom}([0;1,a,1]) \times \text{denom}([0;1,a,2]).$$

Step 2: loss of compactness

Sliding block frequencies normal $\Rightarrow$ disjoint block frequencies normal.

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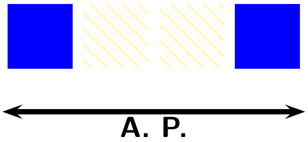
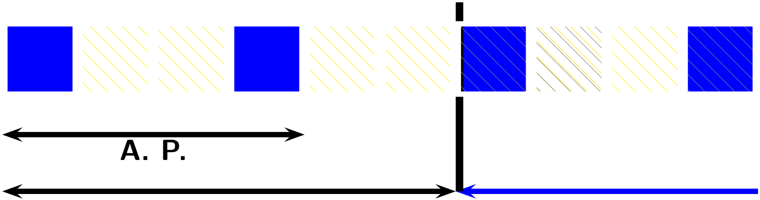
A consequence of the Piateskii-Shapiro Theorem. Follows the proof in Kuipers and Niederreiter, with a careful application of Helley Selection.

Tying things up



Tying things up



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Thank You!

Arbitrary length cylinders

Using the concavity of the cumulative distribution function of Gauss measure, we can reduce the general inequality to an algebraic inequality involving denominators.