Diophantine Approximation and Recursion Theory

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Introduction

I don't know you.
You've been lately on my mind.

New Riders of the Purple Sage, 1971

Randomness

From Recursion Theory

Definition

A real number ξ is *Martin-Löf random* if it does not belong to any effectively-null G_δ set. Precisely, if $(O_n : n \in \mathbb{N})$ is a uniformly computably enumerable sequence of open sets such that for all n, O_n has measure less than $1/2^n$, then $\xi \notin \cap_{n \in \mathbb{N}} O_n$.

This is not mysterious: Identify a family of sets of measure 0, and say that ξ is random if it does not belong to any set in the family.

Randomness

From Algorithmic Information Theory

Definition

A real number ξ is *algorithmically incompressible* iff there is a C such that for all ℓ, $K(\xi \upharpoonright \ell) > \ell - C$, where K denotes prefix-free Kolmogorov complexity and $\xi \upharpoonright \ell$ denotes the first ℓ bits in the base 2 representation of ξ.

This is also not mysterious: Say that ξ is incompressible when for all ℓ, it takes ℓ bits of information to describe $\xi \upharpoonright \ell$.

Schnorr's Theorem

Theorem (Schnorr 1973)

ξ is Martin-Löf random iff it is algorithmically incompressible.

An analogy

For each b, the set of numbers in $[0, 1]$ that are simply normal to base b has measure 1.

Normal and absolutely normal numbers play the role of random reals for avoiding the null sets of simple normality for powers of b and simple normality for all b, respectively.

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Disanalogous Unlike in the algorithmic case, the integer bases provide a family of criteria of similar type for randomness.

An Extension of the Schmidt-Cassels Theorem

Theorem (Becher, Bugeaud and Slaman 2013)

Let M be a set of natural numbers greater than or equal to 2 such that the following necessary conditions hold.

- *For any b and positive integer m, if $b^m \in M$ then then $b \in M$.*
- *For any b, if there are infinitely many positive integers m such that $b^m \in M$, then all powers of b belong to M.*

There is a real number ξ such that for every base b, ξ is simply normal to base b iff $b \in M$.

Irrationality Exponents

Definition

For a real number ξ, the *irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$$

is satisfied by an infinite number of integer pairs (p, q) with $q > 0$.

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When z is large, instances of $0 < \left|\xi - \frac{p}{q}\right| < \frac{1}{q^z}$ are instances of algorithmic compression.

Normality and Irrationality Exponents

Disanalogous

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A few years ago, we extended work of Amou and Bugeaud.

Theorem (Becher and Slaman)

Suppose $a \in [2, \infty]$ and M is a subset of the integers greater than or equal to 2 which satisfies the conditions for a set of bases of simple normality. Then there is a real number ξ such that ξ is simply normal to exactly the bases in M and ξ has exponent of irrationality a.

Irrationality Exponents Relative to Independent Bases

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Definition (following Amou and Bugeaud 2010)

For a real number ξ, the *base-b irrationality exponent of* ξ is the least upper bound of the set of real numbers z such that

$$0 < \left| \xi - \frac{p}{b^k} \right| < \frac{1}{(b^k)^z}$$

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Theorem (Amou and Bugeaud 2010)

Suppose that a_2 and a_3 are greater than $1 + \frac{1+\sqrt{5}}{2}$. There is a real number whose base-2 and base-3 exponents of irrationality are a_2 and a_3, respectively.

Speculation

Question

Is there a real number which is normal to base-3 and has base-2 exponent of irrationality equal to ∞?

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Thank you for listening. Even more, thank you for any help you might offer on this question.