Uniform distribution of generalized polynomials and applications

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Generalized polynomials

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obtained from the conventional polynomials by the use of

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  - uniform distribution of $q(n)$
  - applications to recurrence in dynamical systems

Plan

I. Sets of recurrence

Sárközy's theorem

Theorem

Let $E \subset \mathbb{N}$ be a set of positive upper density:

$$\overline{d}(E) := \limsup_{N\to\infty} \frac{|E \cap \{1, 2, \cdots, N\}|}{N} > 0.$$

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Then, $E - E := \{n : x, x + n \in E\}$ is combinatorially rich:
For any $k \in \mathbb{N}$,

$$(E - E) \bigcap \{n^k : n \in \mathbb{N}\} \neq \emptyset.$$

Dynamical Systems

- $(X, \mathcal{B}, \mu, T)$ a measure preserving system
  - $(X, \mathcal{B}, \mu)$: a probability space
  - $T : X \to X$: an invertible measure preserving map

$$\mu(A) = \mu(T^{-1}A)$$

Recurrence

- $(X, \mathcal{B}, \mu, T)$: a measure preserving system.
- $A \in \mathcal{B}$ with $\mu(A) > 0$.

Theorem (Poincaré)

$$\exists n \in \mathbb{N} : \ \mu(A \cap T^{-n}A) > 0.$$

Polynomial recurrence

- $(X, \mathcal{B}, \mu, T)$: a measure preserving system.
- $A \in \mathcal{B}$ with $\mu(A) > 0$.

Theorem (Furstenberg)

$q(n) \in \mathbb{Z}[n]$: a polynomial with $q(0) = 0$. Then

$$\exists n \in \mathbb{N} : \; \mu(A \cap T^{-q(n)}A) > 0.$$

Furstenberg Correspondence Principle

Theorem

For any $E \subset \mathbb{N}$ *with* $\overline{d}(E) > 0$,
$\exists (X, \mathcal{B}, \mu, T)$ *and* $A \in \mathcal{B}$ *with* $\mu(A) = \overline{d}(E)$ *such that*

$$\forall m \in \mathbb{Z} : \overline{d}(E \cap E - m) \geq \mu(A \cap T^{-m} A).$$

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- correspondence principle + polynomial recurrence
  $\Rightarrow$ for E with $\overline{d}(E) > 0$ and $k \in \mathbb{N}$,

$$\exists n : E \cap (E - n^k) \neq \emptyset \Leftrightarrow n^k \in (E - E)$$

Sets of recurrence

Definition

A set $R \subset \mathbb{Z}$ is a set of recurrence if for any measure preserving system $(X, \mathcal{B}, \mu, T)$ and for any $A \in \mathcal{B}$ with $\mu(A) > 0$

$$\exists n \in R (n \neq 0) : \ \mu(A \cap T^{-n} A) > 0.$$

Example: $\{q(n) : n \in \mathbb{N}\}$ for $q(n) \in \mathbb{Z}[n]$ with $q(0) = 0$.

Polynomial recurrence

Theorem

Let $q(n) \in R[n]$ be a polynomial with $q(\mathbb{Z}) \subset \mathbb{Z}$, $\deg(q) \geq 1$. Then the following conditions are equivalent:

1. *(intersective) For all $a \in \mathbb{N}$, $\{q(n) : n \in \mathbb{Z}\} \cap a\mathbb{N} \neq \emptyset$.*
2. *$\{q(n) : n \in \mathbb{Z}\}$ is a set of recurrence.*
3. *$\{q(n) : n \in \mathbb{Z}\}$ is a set of averaging recurrence: For any $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-q(n)}A) > 0.$$

Generalized polynomial

Question: Let $q(n)$ be an integer-valued generalized polynomial.

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(2) $\{[[\sqrt{2}n]\sqrt{2}]^2 : n \in \mathbb{Z}\}$ is a set of recurrence.

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(2) $\{[[\sqrt{2}n]\sqrt{2}]^2 : n \in \mathbb{Z}\}$ is a set of recurrence.
(3) Both $\{[[\sqrt{3}n]\sqrt{3}] : n \in \mathbb{Z}\}$ and $\{[[\sqrt{3}n]\sqrt{3}]^2 : n \in \mathbb{Z}\}$ are not sets of recurrence.

II. Equidistribution

Uniform distribution mod 1

Definition

$(x_n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1 (u.d. mod 1)
if for any continuous function $f : [0,1] \to \mathbb{R}$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\{x_n\}) = \int_0^1 f(x)\, dx.$$

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Theorem (Weyl Criterion)

$(x_n)_{n\in\mathbb{N}}$ is u.d. mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i h x_n} = 0 \quad \textit{for all integers } h \neq 0.$$

Spectral Theorem

- For any $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$a_m = \mu(A \cap T^{-m}A) \text{ is positive definite.}$$

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- spectral theorem: $\exists$ a measure ν on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$

$$\mu(A \cap T^{-m}A) = \int_{\mathbb{T}} e^{2\pi imt}\, d\nu(t) \quad \text{for all } m \in \mathbb{Z}.$$

Recurrence and equidsitribution

- the limiting behavior of

$$\frac{1}{N}\sum_{n=1}^{N}\mu(A\cap T^{-q(n)}A)=\int\frac{1}{N}\sum_{n=1}^{N}e^{2\pi iq(n)t}\,d\nu(t).$$

- need to understand the behavior of

$$q(n)t \mod 1, \quad n=1,2,\ldots$$

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2. $q(n) \notin \mathbb{Q}[n] + \mathbb{R}$:

- Weyl: $(q(n))$ is uniformly distributed $\mod 1$
- Vinogradov, Rhin: $q(p_n)$ is uniformly distributed $\mod 1$.

(Here (p_n) is the sequence of primes in increasing order.)

Some examples of generalized polynomials

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(1) $q_1(n) = \{\sqrt{2}n\}^2$ is equidistributed with respect to $\frac{dx}{2\sqrt{x}}$:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} F(\{\sqrt{2}n\}^2) = \int_0^1 F(x) \frac{1}{2\sqrt{x}}\, dx.$$

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(2) $q_2(n) = \{\sqrt{2}n\}\{\sqrt{3}n\}$ is equidistributed w.r.t. $-\log x\, dx$.

(3) $q_3(n) = [\sqrt{2}(n+1)] - [\sqrt{2}n] - [\sqrt{2}]$
is equidistributed with respect to $(1 - \{\sqrt{2}\})\delta_0 + \{\sqrt{2}\}\delta_1$.

Dynamical representation of generalized polynomials

Theorem (Bergelson and Leibman, 2007)

Let $q(n)$ be a bounded generalized polynomial. Then there exist

- *a nilmanifold $X = G/\Gamma$*
- *an ergodic nilrotation on X by $a \in G$*
- *$x_0 \in X$*
- *$f : X \to \mathbb{R}$ is a nice function*

such that

$$q(n) = f(a^n x_0)$$

Piecewise polynomials

A piecewise polynomial $f : Q \to \mathbb{R}^m$ is a function

- $Q = \cup_{i=1}^{k} Q_i$ is a finite partition, where Q_i is determined by polynomial inequalities
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Example

$$f(x,y) = \begin{cases} xy + 1 & y \le x^2 \\ x^2 + 3y + 5 & y > x^2, y^2 < x \\ y - 1 & y^2 \ge x \end{cases}$$

is a pp-function on $[0,1]^2$.

Canonical form of GP

Theorem (Leibman, 2012)

Let $q(n)$ be a bounded generalized polynomial. Then there exist

- *generalized polynomials $v_1, \ldots, v_k$*
- *a subgroup Λ with $[\mathbb{Z} : \Lambda] < \infty$*

such that

1. *$(\{v_1(n)\}, \ldots, \{v_k(n)\})$ is equidistributed on $[0,1]^k$*
2. *on any translate $\Lambda' = n_0 + \Lambda$, there is a pp-function f on $[0,1]^k$ such that*

$$q|_{\Lambda'}(n) = f(\{v_1(n)\}, \cdots, \{v_k(n)\}).$$

III. Main Results

cc-polynomials

- Every generalized polynomial can be written as

$$q(n) = b_k(n)n^k + b_{k-1}(n)n^{k-1} + \cdots + b_1(n)n + b_0(n),$$

where $b_i(n)$ is a bounded generalized polynomial.

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- Definition: $q(n)$ is called a cc-polynomial at least one of $b_i(n)$, where $i \neq 0$, is constant.

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Theorem

If $q(n)$ is a cc-polynomial, then there is a set $J \subset \mathbb{N}$ with $d(J) = 0$ such that

$$\lim_{n \notin J, n \to \infty} |q(n)| = \infty.$$

Main results I - equidistirbution of cc-polynomials

Theorem

Let $q(n)$ be a cc-polynomial.
Then

1. $q(n)\lambda$ *is uniformly distributed* $\bmod 1$ *for all but countably many* λ.
2. $q(p_n)\lambda$ *is uniformly distributed* $\bmod 1$ *for all but countalby many* λ.

the W-tricked von Mangoldt functions

- $\Lambda'(n) = 1_{\mathcal{P}}(n) \log n$
- For $W, r \in \mathbb{N}$ with $(r, W) = 1$,

$$\Lambda'_{W,r}(n) = \frac{\phi(W)}{W} \Lambda'(Wn + r).$$

Theorem (Green and Tao, 2010)

• $\eta(n) = f(a^n x_0)$: *a basic nilsequence*

$$\lim_{\substack{W \in \mathcal{W} \\ W \to \infty}} \limsup_{N \to \infty} \sup_{\substack{\eta \in \mathcal{L}_{D,L,M} \\ r \in R(W)}} \left| \frac{1}{N} \sum_{n=1}^{N} (\Lambda'_{W,r}(n) - 1)\eta(n) \right| = 0.$$

Main results II - recurrecne for cc-polynomials

Theorem

Let $q(n)$ be a cc-polynomial. Then the following are equivalent:
(1) For any $k \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$

$$\{n : \|q(n)\alpha_i\| < \epsilon\}$$

has positive upper density for any $\epsilon > 0$
(2) For any $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-q(n)}A) > 0.$$

Main results III - recurrecne for cc-polynomials

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Examples

- Let $q(n) = [\sqrt{11}n + 2]$.
  - $\{q(n) : n \in \mathbb{Z}\} \cap a\mathbb{N} \neq \emptyset$.
  - $\{q(n) : n \in \mathbb{Z}\}$ is not a set of recurrence.

Examples

- Let $q(n) = [\sqrt{11}n + 2]$.
  - $\{q(n) : n \in \mathbb{Z}\} \cap a\mathbb{N} \neq \emptyset$.
  - $\{q(n) : n \in \mathbb{Z}\}$ is not a set of recurrence.
- Let $q(n) = 2n^2 - 1 + [1 - \{[\{\alpha n\}n]\alpha\}]$, where $\alpha = \sum_{j=1}^{\infty} \frac{1}{10^{j!}}$.
  - $\{q(n) : n \in \mathbb{Z}\}$ is a set of recurrence.
  - $\{q(n) : n \in \mathbb{Z}\}$ is not a set of averaging recurrence.

Recurrence for $[q(n)]$

Theorem

Let $q(n) = a_k n^k + \cdots + a_1 n + a_0 \notin \mathbb{Q}[n] + \mathbb{R}$.
Then the following are equivalent.

1. $\{[q(n)] : n \in \mathbb{Z}\}$ *is a set of recurrence*
2. $\{[q(n)] : n \in \mathbb{Z}\}$ *is an averaging set of recurrence*
3. $q(n)$ *is one of the following form:*
   - $\exists i, j$ *such that* $a_i / a_j \notin \mathbb{Q}$ $(i, j \neq 0)$
   - $q(n) = \alpha q_0(n) + \beta$, *where* $\alpha \notin \mathbb{Q}$, $q_0(n) \in \mathbb{Z}[n]$ *intersective and* $0 \leq \beta \leq 1$.

Thank you for your attention!