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Beyond uniform distribution

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Uniform distribution

Definition: A sequence of points $\{x_n: n = 1, 2, ...\} \subset \mathbb{R}/\mathbb{Z}$ in the unit interval/circle is asymptotically <u>uniformly distributed</u> if for any (fixed) subinterval $I \subset \mathbb{R}/\mathbb{Z}$,

$$\lim_{N \to \infty} \frac{1}{N} #\{n \le N \colon x_n \in I\} = |I|.$$

In particular, the <u>mean spacing</u> between the points is 1/N.

Example: If $\alpha \in \mathbb{R}$ then the sequence of fractional parts $\{\alpha \ 10^n\}$ is UD $\leftrightarrow \alpha$ is **normal** (in base 10) \leftrightarrow all (fixed) strings $b_1 \dots b_k$ of k digits occur with the same asymptotic frequency $(1/10^k)$ in the decimal expansion of α .

Weyl's criterion: $\{x_n: n = 1, 2, ...\}$ is UD \leftrightarrow for all fixed $0 \neq k \in \mathbb{Z}$, $\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i k x_n) = 0$ these sums are called ``Weyl sums''.

Examples

- If $\alpha \notin \mathbf{Q}$ is irrational, then the fractional parts $\{\alpha n\}$ are uniformly distributed
- Weyl (1916): for α irrational, the fractional parts αn^2 are UD.
- More generally, if $f(x) = a_d x^d + \cdots + a_1 x \in \mathbf{R}[x]$ is a polynomial with at least one irrational coefficient (not the constant term), then $\{\alpha f(n)\}$ is UD.
- The fractional parts \sqrt{n} are UD mod 1
- The fractional parts { $\log n$ } are <u>NOT</u> UD.
- <u>A metric theorem</u>: If $a(n) \in \mathbb{Z}$ are <u>distinct</u> integers, then for <u>almost all</u> α , the sequence { $\alpha a(n)$ } is UD.

Beyond Uniform Distribution

Once we know that a sequence is uniformly distributed (more generally, equidistributed w.r.t. some continuous measure), we should investigate finer statistics and compare with those of suitable random models.

We examine

- The distribution of normalized nearest neighbor gaps
- The pair correlation function
- The minimal gap statistic

Level spacing distribution

Given a sequence of points, let $\{E_n\}$ be the "order statistics": $E_1 \leq E_2 \leq \cdots \leq E_N \leq \cdots$

P(s) := limiting distribution of the **normalized gaps** δ_n between adjacent levels

Equivalently, for any test function $f \in C_c^{\infty}[0, \infty)$

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(\delta_n) = \int_0^\infty f(s) P(s) ds$$

Statistical models

a) "Picket fence": the levels are perfectly spaced: $x_n = n$. $\delta_n = x_{n+1} - x_n \equiv 1$, and $P(s) = \delta_0(s-1)$

b) <u>Uncorrelated levels</u>: E_n independent, uniform in [0,1] (homogeneous **Poisson** process on the line with intensity 1). level spacing distribution is P(s)=exp(-s)

c) E_n = eigenvalues of a random NxN symmetric matrix (Gaussian Orthogonal Ensemble) H=H^T, matrix elements=independent real Gaussians P(s) was computed by Gaudin and Mehta (1960's)



The Poisson model

Take N uniform IID's $x_1, ..., x_N \in [0,1) \simeq \mathbf{R}/\mathbf{Z}$. The order statistics are

$$x_{(1)} < x_{(2)} < \ldots < x_{(N)}$$

The normalized gaps are

$$\delta_n := N(x_{(n+1)} - x_{(n)}), \qquad x_{(N+1)} = x_{(1)}$$

Theorem: a) The normalized gaps are identically distributed.

b)
$$\delta_1$$
 has CDF $\operatorname{Prob}(\delta_1 < s) = 1 - \left(1 - \frac{s}{N}\right)^N \sim 1 - e^{-s} = \int_0^s e^{-t} dt$, $N \to \infty$

Random Matrix Theory (RMT)

<u>Wigner</u> modeled spectra of complex, many body, quantum systems (heavy nuclei, slow neutron resonances), by those of various ensembles of RANDOM matrices, e.g.:

Gaussian Orthogonal Ensemble (GOE): NxN symmetric matrices H=H^T, matrix elements=independent real Gaussians

Gaussian Unitary Ensemble (GUE): NxN hermitian matrices H=H[†], matrix elements=independent complex Gaussians

Dyson's Circular Unitary Ensemble (CUE): NxN unitary matrices with Haar probability measure $N \rightarrow \infty$

Level repulsion in nuclear spectra

Landau & Smorodinsky, Wigner (1950's): "repulsion of energy levels" of the same symmetry type occurs in complex atomic spectra



FIG. 1. Segments of complex spectra, each containing 50 levels and rescaled to the same spectrum span. The first two show experimental results for neutron and proton resonances, while Fig. 1(c) shows the central region of a 1206-dimensional, $J^{\intercal}=2^{\star}$, T=0, shell-model spectrum; in these three cases all the states have the same exact symmetries. Figure 1(d) shows a Poisson sequence, while Figs. 1(e) and 1(f) show spectra with mixed exact symmetries, the first an experimental spectrum with $J=3^{\star}$, 4^{\star} and the second a shell model spectrum with $J=\frac{1}{2}^{-}$, $\frac{3}{2}^{-}$, \ldots , $\frac{19}{2}^{-}$. The "arrowheads" mark the occurrence of pairs of levels with spacings smaller than one quarter of the average.





FIG. 2. Nearest-neighbor spacing histograms for the six cases of Fig. 1, constructed by considering all the available levels instead of the 50 used in Fig. 1. Spacings S_0 are expressed in

Brody et al

Bohigas, Haq &Pandey, 1983 The nuclear data ensemble (experiment)

Pair correlation

Definition: Assume we have a sequence $\{x_i\}$ with <u>unit mean spacing</u>. The pair correlation function $R_2(s)$ (assuming it exists) is defined by

$$\lim_{N \to \infty} \frac{1}{N} \#\{i \neq j \le N : |x_i - x_j| \le t\} = \int_{-t}^{t} R_2(s) ds$$

Equivalently, for any even test function $f \in C_c^{\infty}(R)$

$$\lim_{N\to\infty}\frac{1}{N}\sum_{1\leq i\neq j\leq N}f(x_i-x_j)=\int_{-\infty}f(s)R_2(s)ds$$

Explain why we divide by N

measures spacings between <u>all</u> pairs of levels - avoids ordering the levels (not a *probability* distribution)

For the Poisson ensemble, $R_2(s)=1$

GUE:
$$R_2(x) = 1 - \left(\frac{\sin(\pi x)}{\pi x}\right)^2$$



The pair correlation function $R_2(s)$ for Poisson, GUE and GOE

Spacings of $\{\alpha n^2\}$

Theorem (Weyl 1916): If α irrational then fractional parts $\{\alpha \ n^2\}_{n=1}^{\infty}$ are uniformly distributed.

Normalization: $x_n := N\{\alpha \ n^2\}, \ n \le N$ has unit mean spacing $(x_n \text{ unordered}).$

<u>Conjecture (ZR-Sarnak-Zaharescu 2001)</u>: If α is <u>badly approximable</u> then spacings of $\{\alpha n^2\}$ are Poissonian. (i.e. all correlation functions are Poissonian).

Badly approximable:
$$\forall \epsilon > 0, \exists c(\alpha, \epsilon) > 0, |\alpha - \frac{p}{q}| > \frac{c}{q^{2+\epsilon}}, \forall \frac{p}{q}$$
 e.g. α algebraic (Roth), almost all α
Metric Theorems:
 $\sqrt{2} \cdot n^2 \mod 1, n \le 20$

Theorem (ZR and Sarnak 1998): For almost all α , the **pair correlation** of $\{\alpha n^2\}$ is Poissonian.

Same result for fractional parts { $\alpha f(n)$ }, f(x)=integer polynomial of degree ≥ 2 .



Spacings of $\{\alpha 2^n\}$

Thm (**ZR & A. Zaharescu, 2002**): For **almost all** α , the level spacing distribution of $\{\alpha 2^n\}$ is Poissonian

- showed that all correlation functions, in particular the pair correlation, is Poissonian for almost all α .

Do not know any naturally occurring α where we can determine the spacings (same as for the question of uniform distribution).

"artificial" examples where normality (uniform distribution) is known, have been proved to NOT have a pair correlation function:

- Champernowne's base-2 number $\alpha = 0.01 \ 10 \ 11 \ 100 \ 101 \ 110 \ 111 \ 1000 \dots$ (Pirsic & Stockinger 2018)
- Stoneham numbers $\alpha_{2,3} = \sum_{m \ge 1} 1 / (3^m 2^{3^m})$ (Larcher & Stockinger 2018),....

A non-Poissonian example: *αn* mod 1

The spacings between <u>fractional parts</u> { α *n*}, *n* \leq *N*:

``3-gap Theorem'' (V. Sós 1957,....): Nearest neighbor spacing of fractional parts { αn }, $n \leq N$ takes at most 3 distinct values A_N, B_N, C_N .

<u>Consequence</u>: Level spacings distribution is not stationary; P(s) does not exist.



MV Berry & M Tabor 1977

A non-Poissonian example: Sqrt n mod 1

N. Elkis & C. McMullen (2004): The level spacing distribution of fractional parts of \sqrt{n} exists, is not Poissonian (e.g. is constant near 0, power law tails).

- Use homogeneous dynamics (Ratner's measure classification)

D. Elbaz, J. Marklof & I. Vinogradov (2015): The pair correlation is Poissonian



Fractional parts of $\alpha \sqrt{n}$

Conjecture

For any Diophantine irrational α , the level spacing distribution of the fractional parts of $\sqrt{\alpha}\sqrt{n}$ is Poissonian



Fractional part of $\sqrt[3]{n}$????



metric theory of pair correlation

In the past couple of years, there has been renewed interest in the metric theory of the pair correlation function. Earlier arguments were formalized, giving a formal connection between the property that the sequence $\alpha(n) \mod 1$ has Poissonian pair correlation for almost all α and the <u>additive energy</u> of the sequence a(n):

Suppose $\mathcal{A} = \{a(n)\}\$ is a sequence of increasing integers. The additive energy is

$$E(\mathcal{A}; N) = \#\{1 \le k, \ell, m, n \le N : a(k) + a(\ell) = a(m) + a(n)\}$$

Note that

$$N^2 \leq E(\mathcal{A}; N) \leq N^3$$

THM (Aistleitner, Larcher & Lewko 2017, Bloom, Chow, Gaffney & Walker 2018, Bloom & Walker 2019):

 $\exists C \gg 1$ such that if $E(\mathcal{A}; N) \ll N^3/(\log N)^C$ then $\alpha \alpha(n) \mod 1$ has Poissonian pair correlation for almost all α

<u>A. Walker (2018)</u>: $\mathcal{A} = \mathcal{P}$ = primes then $E(\mathcal{P}; N) \approx \frac{N^3}{\log N}$ (does not satisfy assumptions of thm) and for almost all α the pair correlation function is <u>NOT</u> Poissonian

The Riemann zeros

Riemann zeta function: For Re(s)>1

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} (1 - \frac{1}{p^s})^{-1}$$

Riemann (1858): Analytic continuation & functional equation

$$\zeta^*(s) \coloneqq \pi^{-s/2} \Gamma(\frac{s}{2}) \zeta(s) = \zeta^*(1-s)$$

is analytic in C, except for simple poles at s=0,1

The Riemann hypothesis: All zeros of $\zeta^*(s)$ are on the line Re(s)=1/2 ("nontrivial zeros")

Riemann von-Mangoldt formula

$$N(T) := \#\{0 < \gamma_j \le T\} = \frac{T}{2\pi} \log \frac{T}{2\pi e} + O(\log T)$$



Mean spacing between zeros is $\frac{2\pi}{\log T}$

Repulsion between zeros

A fundamental discovery was made in 1972 by Montgomery, who found that nearby zeros tend to "repel" each other :

Montgomery studied the "pair correlation function", which measures the repulsion between pairs of levels $\frac{1}{2}+i\gamma$, $\frac{1}{2}+i\gamma$ ' as measured in the scale of their mean spacing:

$$\frac{1}{N(t)} \#\{i \neq j, \gamma_i, \gamma_j \sim T: \quad \frac{\log T}{2\pi} |\gamma_i - \gamma_j| < a\} \sim \int_{-a}^{a} 1 - (\frac{\sin \pi x}{\pi x})^2 dx$$
repulsion

Dyson: this is the same as that for GUE in Random Matrix Theory !! – originally used to model spectra of complex atoms

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1.0

FIGURE 1

Pair correlation of veros of the zeta function. Solid line: GUE prediction. Scatter lot: empirical data based on zeros γ_{n} ,

20

2.5

3.0

9

0.8

density 0.6

4

 $1 \leq n \leq 10^5$.

Pair correlation fuction, N = 0

P(s) for the zeros of Riemann's zeta function

<u>Conjecture</u>: P(s) for the <u>zeros</u> of $\zeta(s)$ coincides with that of CUE/GUE.

Extensive numerical work by Odlyzko (1980-present)

Theory:

- Montgomery(1973), pair correlation
- Rudnick & Sarnak (1994) higher correlations

- agreement with GUE in restricted range



FIGURE 4

Probability density of the normalized spacings δ_n . Solid line: GUE prediction. Scatter plot: empirical data based on zeros γ_n , $10^{12} + 1 \le n \le 10^{12} + 10^5$.

Odlyzko: 10⁵ zeros near 10¹²-th zero

Minimal gaps

The minimal gap between the first N levels of a sequence $x_1, \dots, x_N, \dots \in [0,1)$

$$\delta_{\min}(N) = \min_{i \neq j \le N} |x_i - x_j|$$

For uncorrelated levels - N uniform IID's x_1, \dots, x_N on (0,1), (Poisson sequence), the minimal gap is of size

$$\delta_{\min}^{Poisson}(N) \approx \frac{1}{N^2}$$
 - exercise (the birthday problem)

Paul Lévy (1939) $N^2 \delta_{\min}(N)$ has a limiting exponential distribution.

For GOE, it is much larger, $\delta_{\min}^{GOE}(N) \approx \frac{1}{N^{3/2}}$

For CUE/GUE, the minimal gap is of size $\delta_{\min}^{GUE}(N) \approx \frac{1}{N^{4/3}}$ Vinson 2001, Ben Arous & Bourgade 2013

(Normalize **mean** gap to be 1/N)

A metric theory of minimal gaps

 $\delta_{\min}^{\alpha}(N) \gg \frac{1}{N(\log N)^{2+o(1)}}$

Suppose $\mathcal{A} = \{a(n)\}\$ is a sequence of increasing integers. Define $\delta_{\min}^{\alpha}(N)$:=minimal gap between $\alpha a(n) \mod 1$ Recall that for a random sequence, $\delta_{\min}^{Poisson}(N) \approx 1/N^2$

<u>THM (ZR, 2018)</u> If $\mathcal{A} = \{a(n) \in \mathbb{Z}\}$ are <u>distinct</u> integers, and the additive energy satisfies $E(\mathcal{A}; N) \ll N^{2+o(1)}$ then for <u>almost all</u> α , $\delta_{\min}^{\alpha}(N) < \frac{1}{N^{2-\varepsilon}}$ for all $\varepsilon > 0$

-- consistent with Poisson statistics for the minimal gap

Examples: $a(n) = n^d$, $d \ge 2$; $a(n) = 2^n$

"Bad" example:
$$\mathcal{A} = \mathcal{P}$$
 = primes then $E(\mathcal{P}; N) \approx \frac{N^3}{\log N}$ (does not satisfy assumptions of thm) and

not Poissonian!

for almost all α ,

Open problems

- If α is **badly approximable** then spacings of $\{\alpha n^2\}$ are Poissonian.
- For any Diophantine irrational α , the level spacing distribution of the fractional parts of $\sqrt{\alpha}\sqrt{n}$ is Poissonian
- Metric theory: ditto for almost every α
- Minimal spacings?

Thank you for your attention!