Notes for the IMS Summer School in Mathematical Logic-2019

W. Hugh Woodin

Version: July 19, 2019

Contents

1 Preliminaries

1.1 L and HOD

For each set X, $\mathcal{P}(X)$ denotes the set of all Y such that $Y \subseteq X$. $\mathcal{P}(X)$ is the *powerset* of X.

By the ZFC axioms, every set is generated by the transfinite iteration of the powerset operation:

Definition 1.1 (Cumulative Hierarchy). The sets V_α are defined by induction on the ordinal α as follows.

(1) $V_0 = \emptyset$.

(2) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.

(3) If α is a limit ordinal then $V_\alpha = \cup \left\{ V_\beta \mid \beta < \alpha \right\}$. □

For each set X, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the *definable powerset* of X. More precisely, $\mathcal{P}_{\mathrm{Def}}(X)$ is the set of all $Y \subseteq X$ such that Y is definable from parameters in the structure

$$(X, \in).$$

Definition 1.2 (Effective Cumulative Hierarchy). The sets L_α are defined by induction on the ordinal α as follows.

(1) $L_0 = \emptyset$.

(2) $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$.

(3) If α is a limit ordinal then $L_\alpha = \cup\left\{L_\beta \mid \beta < \alpha\right\}$. □

Definition 1.3 (Gödel). L is the class of all sets X such that

$$X \in L_\alpha$$

for some ordinal α. □

Definition 1.4 (Gödel). The axiom $V = L$ is the axiom which asserts that for all sets X, $X \in L$. □

The definition of L does not require the Axiom of Choice.

Theorem 1.5 (Gödel). *Assume $V = L$. Then the following hold.*

(1) *The Axiom of Choice.*

(2) CH.

(3) *There exists a wellordering $<_L$ of $\mathcal{P}(\omega)$ of length ω_1 such that*

$$<_L \in \mathcal{P}_{\mathrm{Def}}(V_{\omega+1} \times V_{\omega+1}).$$ □

Theorem 1.6 (Gödel). *Suppose $(M, E) \vDash$ ZF. Let L^M be the set of all $X \in M$ such that*

$$(M, E) \vDash \text{“}X \in L\text{”}.$$

Then the following hold.

(1) $(L^M, E|L^M) \vDash$ ZFC.

(2) $(L^M, E|L^M) \vDash$ “$V = L$”. □

The issue of whether $V = L$ is arguably settled by Scott's Theorem. Basic large cardinal notions are reviewed in Section 1.3.

Theorem 1.7 (Scott). *Assume there is a measurable cardinal. Then $V \neq L$.* □

We fix some standard notation. Suppose X is a set. Then $\mathrm{TC}(X)$ is the transitive closure of X, this defined to be the smallest transitve set Y such that $X \in Y$.

Remark 1.8. By the ZF axioms, for every set X, $\mathrm{TC}(X)$ exists. But this requires the Axiom of Replacement.

To see why consider the set

$$X = \{\{V_n\} \mid n < \omega\}.$$

Let M be the closure of $X \cup \omega$ under powerset. Then one can show that M is transitive and

$$M \models \mathrm{ZC}$$

where ZC denotes the ZFC axioms without the Axiom of Replacement. But

$$\mathrm{TC}(X) = V_\omega \cup \{X\}$$

and one can verify that $V_\omega \notin M$. □

Gödel defined another very important transitive class, HOD. This definition, like the definition of the class L, does not require the Axiom of Choice.

Definition 1.9 (Gödel). HOD is the class of all sets X such that for some ordinal α the following hold where $Y = \mathrm{TC}(X)$.

(1) $Y \in V_\alpha$.

(2) Suppose $A \in Y$. Then A is definable in the structure

$$(V_\alpha, \in)$$

from ordinal parameters.

Theorem 1.10. *Suppose* $(M, E) \models \mathrm{ZF}$. *Let* H *be the set of all* $X \in M$ *such that* (i) *holds where* $Y \in M$ *and*

$$(M, E) \models \text{“}Y = \mathrm{TC}(X)\text{”}.$$

(i) *Suppose* $Z \in M$ *and* $(Z, Y) \in E$. *Then* Z *is definable in* (M, E) *from parameters from the set*

$$\mathrm{Ord}^M = \{A \in M \mid (M, E) \models \text{“}A \text{ is an ordinal”}\}.$$

Then the following hold.

(1) H *is the set of all* $A \in M$ *such that* $(M, E) \models \text{“}A \in \mathrm{HOD}\text{”}$.

(2) $(H, E|H) \models \mathrm{ZFC}$. □

Remark 1.11. A key question is: Does $V = \mathrm{HOD}$? □

1.2 Vopěnka's Theorem

The following theorem of Vopěnka (which is a theorem of just ZF) plays an important role in modern Set Theory. This is particularly true in inner model theory and in the study of the Axiom of Determinacy.

Vopěnka's theorem has a remarkable metamathematical consequence. Every countable model

$$(M, E) \models \mathrm{ZF} + \neg\mathrm{AC} + \text{“There exists a set } X \text{ such that } V = L(X)\text{”}$$

is a symmetric forcing extension of a model of ZFC. Here $L(X)$ refers to L *relativized* to X, as in Definition 1.64 on page 25.

Theorem 1.12 (Vopěnka). *Assume* ZF *and suppose that* $A \subseteq \lambda$. *Then either* $A \in \mathrm{HOD}$ *or* $\mathrm{HOD}[A]$ *is a generic extension of* HOD.

Proof. Let $\mathbb{B}$ be the set of all

$$X \subset \mathcal{P}(\lambda)$$

such that X is definable from ordinal parameters. More precisely, $\mathbb{B}$ is the set of all

$$X \subset \mathcal{P}(\lambda)$$

such that there exists an ordinal $\alpha > \lambda$ such that X is definable in V_α from ordinal parameters.

By Axiom of Replacement, there exists an ordinal $\eta > \lambda$ such that for all $X \subseteq \mathcal{P}(\lambda)$ the following are equivalent.

(1.1) $X \in \mathbb{B}$.

(1.2) X is definable in V_η from ordinal parameters.

Fix η. There exists a Boolean algebra $\mathbb{B}_0 \in \mathrm{HOD}$ and an isomorphism

$$\pi_0 : \mathbb{B}_0 \to \mathbb{B}.$$

We prove:

(2.1) $\mathbb{B}_0$ is a complete Boolean algebra in HOD.

(2.2) Let $G_A = \{b \in \mathbb{B}_0 \mid A \in \pi_0(b)\}$. Then G_A is a HOD-generic ultrafilter and $A \in \mathrm{HOD}[G_A]$.

We first prove (2.1). Let $\mathcal{A}$ be an antichain in $\mathbb{B}_0$ with $\mathcal{A} \in \mathrm{HOD}$. Let

$$X = \cup\{\pi_0(b) \mid b \in \mathcal{A}\}.$$

Thus X is definable from ordinal parameters and so $X \in \mathbb{B}$. Let $b_0 \in \mathbb{B}_0$ be such that $\pi_0(b_0) = X$. Then b_0 is the least upper bound of $\mathcal{A}$ in $\mathbb{B}_0$. This proves (2.1).

We finish by proving (2.2). Let $\mathcal{A}$ be a maximal antichain in $\mathbb{B}_0$ with $\mathcal{A} \in \mathrm{HOD}$. Let

$$X = \cup \{\pi_0(b) \mid b \in \mathcal{A}\}.$$

Thus $X = \mathcal{P}(\lambda)$ (since $\mathcal{A}$ is a maximal antichain and since π_0 is an isomorphism). Therefore there exists $b \in \mathcal{A}$ such that $A \in \pi_0(b)$, But then $b \in G_A$ and so $G_A \cap \mathcal{A} \neq \emptyset$.

This proves that G_A is a HOD-generic ultrafilter. For each $\alpha < \lambda$, let

$$X_\alpha = \{E \subset \lambda \mid \alpha \in E\}$$

and let $b_\alpha \in \mathbb{B}_0$ be such that $\pi_0(b_\alpha) = X_\alpha$. Thus

$$\langle b_\alpha : \alpha < \lambda \rangle \in \mathrm{HOD}.$$

Finally

$$A = \{\alpha \mid b_\alpha \in G_A\}$$

and so $A \in \mathrm{HOD}[G_A]$. □

1.3 Measurable cardinals, supercompact cardinals, and extendible cardinals

This is a very quick review of some basic large cardinal notions.

Definition 1.13. Suppose U is an ultrafilter on a nonempty set X (so $U \subset \mathcal{P}(X)$).

(1) U is non-principal if for all $a \in X$, $\{a\} \notin U$.

(2) Suppose δ is a regular cardinal. Then U is δ-complete if for all $Z \subsetneq U$, if $|Z| < \delta$ then $\cap Z \in U$. □

Remark 1.14. Suppose U is an ultrafilter on a nonempty set X.

(1) U is ω-complete.

(2) U is non-principal if and only if every $Z \in U$ is infinite. □

Suppose U is an ultrafilter on a cardinal κ. Then U is *uniform* if for all $\alpha < \kappa$

$$\{\beta < \kappa \mid \alpha < \beta\} \in U.$$

If U is a κ-complete non-principal ultrafilter on κ then U must be a uniform ultrafilter on κ.

Definition 1.15 (Measurable cardinals). Suppose κ is an uncountable regular cardinal. Then κ is a *measurable* cardinal if there is an ultrafilter U on κ such that U is non-principal and κ-complete. □

Suppose κ is a measurable cardinal and U is a κ-complete non-principal ultrafilter on κ. Then we can form the ultrapower

$$\mathrm{Ult}_0(V, U) = V^\kappa / U.$$

Since the ultrafilter U is countable complete, the ultrapower $\mathrm{Ult}_0(V, U)$ is isomorphic to a transitive class, M_U.

This transitive class can also be defined as follows. For each transitive set X let X_U be the transitive set which isomorphic to the ultrapower, X^κ/U. Then M_U is simply the union of all the transitive sets X_U.

There is an associated (elementary) embedding

$$j_U : V \to M_U.$$

This is the ultrapower embedding given by U. For each transitive set X, let

$$j_U^X : X \to X_U$$

be the ultrapower embedding.

For all transitive sets X,

$$j_U(X) = X_U.$$

and

$$j_U|X = j_U^X.$$

Thus both M_U and j_U are Σ_2-definable classes from the parameter U.

Since U is a κ-complete ultrafilter,

$$j_U(\alpha) = \alpha$$

for all $\alpha < \kappa$, and since U is a uniform ultrafilter on κ,

$$j_U(\kappa) > \kappa.$$

Thus κ is the least ordinal γ such that $j_U(\gamma)$, this is the *critical point* of j_U and it is denoted by $\mathrm{CRT}(j_U)$.

Let

$$W = \{X \subset \kappa \mid \kappa \in j_U(X)\}.$$

Then W is a κ-complete uniform ultrafilter on κ. Suppose $f : \kappa \to \kappa$ and

$$\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in W.$$

Then $j_U(f)(\kappa) < \kappa$. Let $\alpha_0 = j_U(f)(\kappa)$. Then

$$\{\alpha < \kappa \mid f(\alpha) = \alpha_0\} \in W.$$

This motivates the following definition.

Definition 1.16. Suppose κ is an uncountable regular cardinal and that U is a uniform ultrafilter on κ. Then U is *normal* if for all functions

$$f : \kappa \to \kappa,$$

if $\{\alpha < \kappa \mid f(\alpha) < \alpha\} \in U$ then there exists $\alpha_0 < \kappa$ such that

$$\{\alpha < \kappa \mid f(\alpha) = \alpha_0\} \in U. \qquad \square$$

Lemma 1.17. *Suppose U is a normal ultrafilter on κ. Then U is κ-complete.* □

As above suppose U is a κ-complete uniform ultrafilter on κ,

$$j_U : V \to M_U$$

is the ultrapower embedding and

$$W = \{X \subseteq \kappa \mid \kappa \in j_U(X)\}.$$

Then W is a normal ultrafilter on κ and $W = U$ if and only if U is a normal ultrafilter on κ.

Supercompact cardinals are a generalization of measurable cardinals. Suppose $\kappa < \lambda$ and κ is an uncountable regular cardinal. Then

$$\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \mid |\sigma| < \kappa\}.$$

Definition 1.18. Suppose $\kappa < \lambda$ and κ is an uncountable regular cardinal. Suppose U is an ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Then:

(1) U is *fine* if for all $\alpha < \lambda$, $\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid \alpha \in \sigma\} \in U$.

(2) U is *normal* if for all

$$f : \mathcal{P}_\kappa(\lambda) \to \lambda,$$

if $\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) \in \sigma\} \in U$ then there exists $\alpha_0 < \lambda$ such that

$$\{\sigma \in \mathcal{P}_\kappa(\lambda) \mid f(\sigma) \in \alpha_0\} \in U.$$ □

Definition 1.19. Suppose κ is an uncountable regular cardinal. Then:

(1) κ is a *strongly compact cardinal* if for all $\lambda > \kappa$ there is a κ-complete, fine, ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

(2) κ is a *supercompact cardinal* if for all $\lambda > \kappa$ there is a κ-complete, normel, fine, ultrafilter on $\mathcal{P}_\kappa(\lambda)$. □

Remark 1.20. By the theorem of Menas, if κ is a measurable cardinal and κ is a limit of strongly compact cardinals, then κ is a strongly compact cardinal. In contrast if κ is the least measurable cardinal which is a limit of *supercompact* cardinals then κ is *not* a supercompact cardinal.

Further by Magidor's theorem, if κ is a supercompact cardinal then there is a (class) forcing extension of V in which κ is a strongly compact cardinal and the *only* measurable cardinal.

A major open problem is Solovay's Conjecture which is the conjecture that supercompact cardinals and strongly compact cardinals are equiconsistent. □

Lemma 1.21. *Suppose κ is an uncountable regular cardinal. Then the following are equivalent.*

(1) *κ is a supercompact cardinal.*

(2) *For all $\lambda > \lambda$, there is a transitive class M and an elementary embedding*

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, *and* $\{j(\alpha) \mid \alpha < \lambda\} \in M$.

(3) *For all $\lambda > \lambda$, there is a transitive class M and an elementary embedding*

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, *and such that* $M^\lambda \subset M$. □

The notion that κ is a supercompact cardinal in naturally localized:

Definition 1.22. Suppose κ is an uncountable regular cardinal and $\lambda \geq \kappa$. Then κ is *λ-supercompact* if there is a κ-complete, fine, ultrafilter on $\mathcal{P}_\kappa(\lambda)$. □

Lemma 1.23. *Suppose κ is an uncountable regular cardinal and $\kappa \leq \lambda$. Then the following are equivalent.*

(1) *κ is λ-supercompact.*

(2) *There is a transitive class M and an elementary embedding*

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, *and* $\{j(\alpha) \mid \alpha < \lambda\} \in M$.

(3) *There is a transitive class M and an elementary embedding*

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, $j(\kappa) > \lambda$, *and such that* $M^\lambda \subset M$. □

Corollary 1.24. *Suppose κ is λ-supercompact and* $\mathrm{cof}(\lambda) < \kappa$.

$$\gamma = |\lambda^{\mathrm{cof}(\lambda)}|.$$

Then κ is γ-supercompact. □

Not all large cardinal notions are easily expressed in terms of ultrafilters.

Lemma 1.25. *Suppose*

$$\pi : V_\alpha \to V_\beta$$

is an elementary embedding. Then the following are equivalent.

(1) $\pi(x) = x$ *for all* $x \in V_\alpha$.

(2) $\pi(\eta) = \eta$ *for all* $\eta < \alpha$. □

If $\pi : V_\alpha \to V_\beta$ is a nontrivial elementary embedding then $\text{CRT}(\pi)$ is the least $\eta < \alpha$ such that $\pi(\eta) \neq \eta$.

Definition 1.26. Suppose κ is an uncountable regular cardinal. Then κ is an *extendible cardinal* if for all $\lambda > \kappa$, there exists an elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

such that $\kappa = \text{CRT}(\pi)$ and $\pi(\kappa) > \lambda$. □

Lemma 1.27. *Suppose that κ is an extendible cardinal. Then κ is a supercompact cardinal and κ is a limit ot supercompact cardinals.* □

1.4 Extenders

Any elementary embedding

$$j : V \to M$$

can be analyzed using *extenders*.

Definition 1.28. For each (infinite) cardinal γ, $H(\gamma)$ denotes the union of all transitive sets M such that $|M| < \gamma$. □

Lemma 1.29. *Suppose that γ is an infinite cardinal. Then:*

(1) $H(\gamma^+) \vDash \text{ZF}\backslash\text{Powerset}$.

(2) $H(\gamma^+) \vDash$ *"The Wellordering Principle".* □

Definition 1.30. A transitive class M is an *inner model of* ZFC if

(1) $\text{Ord} \subset M$,

(2) $M \vDash \text{ZFC}$.

This definition of an inner model of ZFC is *not* first order. The following lemma identifies the first order reformulation.

Lemma 1.31. *Suppose M is a transitive class containing* Ord*. Then the following are equivalent:*

(1) *M is an inner model of* ZFC*.*

(2) *For each infinite cardinal γ,*

$$M \cap H(\gamma^+) \vDash \text{ZFC}\backslash\text{Powerset}.$$ □

Remark 1.32. L and HOD are inner models of ZFC. □

Suppose N, M are inner models of ZFC and

$$j : N \to M$$

is an elementary embedding. Again this is not first order.

Lemma 1.33. *Suppose N, M are inner models of* ZFC *and*

$$j : N \to M$$

is a (class) function. Then the following are equivalent.

(1) *j is an elementary embedding.*

(2) *For all $X \in M$:*

$$j|X : X \to j(X)$$

is an elementary embedding from $(X, \in)$ *to* $(j(X), \in)$. □

Suppose N is an inner model of ZFC, $X \in N$, and U is an ultrafilter on $\mathcal{P}(X) \cap N$. Then the ultrapower of N by U is computed using only functions

$$F : X \to N$$

such that $F \in N$. This is denoted by $\mathrm{Ult}_0(N, U)$. If $\mathrm{Ult}_0(N, U)$ is wellfounded that M_U^N denotes the transitive collapse of $\mathrm{Ult}_0(N, U)$ and

$$j_U^N : N \to M_U^N$$

denotes the ultrapower embedding. If $N = V$ then M_U^V is also denoted by M_U and j_U^V is also denoted by j_U.

This ultrapower construction can be generalized as follows which involves the notion of an *extender*. Suppose N, M are inner models of ZFC and

$$j : N \to M$$

is an elementary embedding. If j is not the identity then there must exist an ordinal α such that $j(\alpha) \neq \alpha$.

The least such ordinal is the *critical point* of j and denoted $\mathrm{CRT}(j)$. This must be a regular cardinal of N and if $N = V$ then $\mathrm{CRT}(j)$ is a *measurable cardinal*.

Suppose that η is an ordinal and that $\mathrm{CRT}(j) < \eta$.

For each $s \in [\eta]^{<\omega}$ define an ultrafilter E_s as follows (and here for any set X, $[X]^{<\omega}$ denotes the set of all finite subsets of X).

Let $\bar{\eta}$ be the least ordinal α such that $\eta \leq j(\alpha)$. Then

$$E_s = \{Z \subseteq [\bar{\eta}]^{<\omega} \mid s \in j(Z)\},$$

and so E_s is on ultrafilter on $\mathcal{P}(X) \cap N$ where $X = [\bar{\eta}]^{<\omega}$. Note that

$$\{A \in [\bar{\eta}]^{<\omega} \mid |A| = |s|\} \in E_s.$$

Rephrased, $[\bar{\eta}]^k \in E_s$ where $k = |s|$ and where for any set X, and for any $k < \omega$, $[X]^k$ is the set of all $A \subset X$ such that $|A| = k$.

The sequence

$$E = \langle E_s : s \in [\eta]^{<\omega} \rangle$$

is the *N-extender* of length η defined by j. If $N = V$, then E is the extender of length η defined by j.

Let

$$X_E = \{j(f)(s) \mid f : [\bar{\eta}]^{<\omega} \to N, f \in N, s \in [\eta]^{<\omega}\}$$

Then $X_E \prec M$. Let M_E be the transitive collapse of X_E and let

$$\pi_E : M_E \to M$$

invert the transitve collapse. Thus

$$\pi_E : M_E \to M$$

is an elementary embedding and if $X_E \neq M$ then $\mathrm{CRT}(\pi_E) \geq \eta$ since $\eta \subset X_E$. Note $\mathrm{CRT}(\pi_E)$ is simply the least ordinal γ such that $\gamma \notin X_E$.

Let

$$j_E : N \to M_E$$

be the induced elementary embedding. Thus $j = \pi_E \circ j_E$ and this uniquely specifies j_E. Further, E is the extender of length η given by j_E and M_E is the *ultrapower* of N by E.

This ultrapower is defined as the direct limit of the ultrapowers of N by E_s where $s \in [\eta]^{<\omega}$.

The point here is that $[\eta]^{<\omega}$ is directed under the order $s \subseteq t$. Further if $s \subseteq t$ then there is a canonical map

$$\pi^t_s : [\bar{\eta}]^{|t|} \to [\bar{\eta}]^{|s|}$$

defined by $\pi(a) = b$ where $(a, b, <) \cong (t, s, <)$.

This induces and elementary embedding

$$e^s_t : \mathrm{Ult}_0(N, E_s) \to \mathrm{Ult}_0(N, E_t),$$

and so one has a directed system of ultrapowers and embeddings.

Thus for example in the case where $N = V$, one can now *define* an extender E of length η as a sequence

$$\langle E_s : s \in [\eta]^{<\omega} \rangle$$

of countably complete ultrafilters, such that the following hold.

(1) For some ordinal $\bar{\eta}$,

$$[\bar{\eta}]^{|s|} \in E_s$$

for all $s \in [\eta]^{<\omega}$,

(2) For all $s \subset t$ in $[\eta]^<$.

$$E_s = \left\{X \subseteq [\eta]^{|s|} \mid (\pi^t_s)^{-1}[X] \in E_t\right\}$$

where

$$\pi^t_s : [\bar{\eta}]^{|t|} \to [\bar{\eta}]^{|s|}$$

is defined by $\pi(a) = b$ where $(a, b, <) \cong (t, s, <)$.

(3) $\mathrm{Ult}_0(V, E)$ is wellfounded and if

$$j_E : V \to M_E \cong \mathrm{Ult}_0(V, E)$$

is the associated ultrapower embedding, then $\eta \leq j_E(\bar{\eta})$ and for all $s \in [\eta]^{<\omega}$,

$$E_s = \left\{X \subseteq [\bar{\eta}]^{|s|} \mid s \in j_E(X)\right\}.$$

Except for the requirement that $\mathrm{Ult}_0(V, E)$ be wellfounded, this definition can be localized to the structure

$$(\mathcal{P}([\bar{\eta}]^{<\omega}), E_s : s \in [\eta]^{<\omega}).$$

This is useful when dealing with N-extenders E when $E \notin N$, etc.

Lemma 1.34. *Suppose N, M are inner models of* ZFC *and*

$$j : N \to M$$

is an elementary embedding. Suppose $\mathrm{CRT}(j) < \eta$ *and E is the N-extender of length η given by j. Then E is uniquely determined by the function*

$$F : \mathcal{P}(\eta^{<\omega}) \to \mathcal{P}(\eta^{<\omega})$$

where $F(X) = j(X) \cap \eta^{<\omega}$. □

Using ultrapowers of V by extenders one can prove the following lemma.

Lemma 1.35. *Suppose κ is an uncountable regular cardinal. Then the following are equivalent.*

(1) κ is an extendible cardinal.

(2) For each $\lambda > \kappa$, there is a transitive class and an elementary embedding

$$j : V \to M$$

such that $\mathrm{CRT}(j) = \kappa$, *$j(\kappa) > \lambda$, and $V_{j(\lambda)+1} \subset M$.* □

One can use extenders to formulate most large cardinal axioms.

Definition 1.36. Suppose κ is a cardinal. Then κ is a *strong cardinal* if for all $\lambda > \kappa$ there exists an extender E such that the following hold where

$$j_E : V \to M_E \cong \mathrm{Ult}_0(V, E)$$

is the associated ultrapower embedding.

(1) $\mathrm{CRT}(j_E) = \kappa$ and $j_E(\kappa) > \lambda$.

(2) $V_\lambda \subset M_E$. □

1.5 Stationary sets and the Solovay Splitting Theorem

Definition 1.37. Suppose κ is an uncountable regular cardinal. Then

(1) Suppose $C \subset \kappa$ then C is closed if for all $\eta < \kappa$ such that $C \cap \eta$ is cofinal in η, $\eta \in C$.

(2) The filter $\mathcal{F}$ generated by the closed cofinal subsets of κ is the *club* filter at κ. □

The club filter at κ is κ-complete and uniform.

Definition 1.38. Suppose κ is an uncountable regular cardinal and $S \subseteq \kappa$. Then S is *stationary* if for all closed cofinal $C \subseteq \kappa$,

$$C \cap S \neq \emptyset.$$ □

Remark 1.39. Suppose κ is an uncountable regular cardinal and that $\gamma < \kappa$ is an infinite regular cardinal. Let

$$S = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \gamma\}.$$

Then S is stationary subset of κ □

Theorem 1.40 (Solovay Splitting Theorem). *Suppose κ is an uncountable regular cardinal and that $S \subseteq \kappa$ is stationary. Then there is a partition*

$$\langle S_\alpha : \alpha < \kappa \rangle$$

of S into κ many stationary subsets. □

Solovay proved the following remarkable theorem about normal fine ultrafilters, and there is a version of the theorem for just fine δ-complete (even just countably complete) ultrafilters on $\mathcal{P}_\delta(\lambda)$.

Theorem 1.41 (Solovay). *Suppose $\kappa < \lambda$, λ is a regular cardinal, and U is a κ-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Then there exists a set $X \in U$ such that the function*

$$F : X \to \lambda$$

where $F(\sigma) = \sup(\sigma)$ for all $\sigma \in X$, is 1-to-1 on X. □

Solovay's original proof used Jonsson algebras but there is alternative proof using the Solovay Splitting Theorem and this proof only requires the Solovay Splitting Theorem in the case where

$$S = \{\alpha < \lambda \mid \mathrm{cof}(\alpha) = \omega\}.$$

The proof of Theorem 1.42 is in essence a simple version the proof of Theorem 6.33 which we do give, see page 53.

Theorem 1.42 (Solovay). *Suppose $\kappa < \lambda$, λ is a regular cardinal, and U is a κ-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$. Let*

$$S = \{\eta < \lambda \mid \mathrm{cof}(\eta) = \omega\}$$

and let

$$\langle S_\alpha : \alpha < \lambda\rangle$$

be a partition of S into λ-many stationary sets. Define X to be the set of all $\sigma \in \mathcal{P}_\kappa(\lambda)$ such that σ is the set of all $\beta < \sup(\sigma)$ such that

$$S_\beta \cap C \neq \emptyset$$

for all closed cofinal subsets C of $\sup(\sigma)$. Then

(1) *For all $\sigma, \tau \in X$, if $\sup(\sigma) = \sup(\tau)$ then $\sigma = \tau$.*

(2) *$X \in U$.* □

1.6 $^\infty$Borel sets

Definition 1.43 ($^\infty$Borel-codes). The class of $^\infty$Borel-codes for $^\infty$Borel subsets of ω^ω is the smallest class I such that the following hold.

(1) $(0, s) \in I$ for all $s \in \omega^{<\omega}$.

(2) Suppose $\langle w_\alpha : \alpha < \eta\rangle$ is a sequence such that $0 < \eta$ and such that $w_\alpha \in I$ for all $\alpha < \eta$. Then

$$(1, \langle w_\alpha : \alpha < \eta\rangle) \in I.$$

(3) Suppose $w \in I$. Then $(2, w) \in I$. □

Definition 1.44. The interpretation of an $^\infty$Borel-code w is the set $A_w \subseteq \omega^\omega$ defined by induction on w as follows.

(1) Suppose $w = (0, s)$. Then $A_w = \{x \in \omega^\omega \mid s \subset x\}$.

(2) Suppose $w = (1, \langle w_\alpha : \alpha < \eta\rangle)$. Then

$$A_w = \cup \{A_{w_\alpha} \mid \alpha < \eta\}.$$

(3) Suppose $w = (2, w_0)$. Then $A_w = \omega^\omega \backslash A_{w_0}$. □

Definition 1.45. Suppose $A \subseteq \omega^\omega$. Then A is $^\infty$Borel if there exists an $^\infty$Borel-code w such that

$$A = A_w.$$ □

Remark 1.46. Assuming the Axiom of Choice, or just that ω^ω can be wellordered, then every set $A \subseteq \omega^\omega$ is ${}^\infty$Borel.

Thus in the context of the Axiom of Choice, there may exist $A \subseteq \omega^\omega$ such that A is OD but such that there is no ${}^\infty$Borel-code, $w \in \mathrm{HOD}$, such that

$$A = A_w.$$

Similarly, if the Axiom of Choice does not hold then there can exist $A \subseteq \omega^\omega$ such that A is not ${}^\infty$Borel. □

We fix some notation.

Definition 1.47. (1) $\mathfrak{B}^\infty$ is the class of all ${}^\infty$Borel-codes.

(2) $\sim^\infty$ denotes the equivalence relation on $\mathfrak{B}^\infty$ where

$$w_1 \sim^\infty w_2$$

if $A_{w_1} = A_{w_2}$.

(3) $\mathbb{B}^\infty$ is the complete Boolean algebra given by $\mathfrak{B}^\infty/\sim^\infty$.

(4) For each $w \in \mathfrak{B}^\infty$, $[w]^\infty$ is the class of all $u \in \mathfrak{B}^\infty$ such that $A_w = A_u$. □

Remark 1.48. The elements of $\mathbb{B}^\infty$ are really just the equivalence classes $[w]^\infty$.

Definition 1.49 (ZF). Suppose that N is an inner model of ZFC. Then N is $(\sim^\infty)$-*closed* if for all ordinals α,

$$\{(w_1, w_2) \in \mathfrak{B}^\infty \times \mathfrak{B}^\infty \mid w_1 \sim^\infty w_2\} \cap V_\alpha \cap N \in N.$$ □

Lemma 1.50. HOD *is* $\sim^\infty$*-closed.* □

We note the following lemma.

Lemma 1.51. *Suppose that N is an inner model of* ZFC*. Then the following are equivalent.*

(1) *N is $(\sim^\infty)$-closed.*

(2) *For all ordinals α, $\{w \in \mathfrak{B}^\infty \mid A_w \neq \emptyset\} \cap V_\alpha \cap N \in N$.* □

Lemma 1.52 (${}^\infty$Borel Genericity Lemma). (ZF) *Suppose N is an inner model of* ZFC *and N is $\sim^\infty$-closed. Let*

$$\mathbb{B}^\infty_N = (\mathfrak{B}^\infty \cap N)/(\sim^\infty \cap N).$$

Then the following hold.

(1) *$\mathbb{B}^\infty_N$ is a complete Boolean algebra in N.*

(2) *Suppose* $x \in \omega^\omega$. *Let*

$$G_x \subset \mathbb{B}^\infty_N$$

be the set of all $[w] \in \mathbb{B}^\infty_N$ *such that* $x \in A_w$. *Then* G_x *is an* N*-generic ultrafilter on* $\mathbb{B}^\infty_N$ *and*

$$N[x] = N[G_x].$$

Proof. The proof is essentially the same as the proof of Vopěnka's theorem, Theorem 1.12. We first prove (1). Let $\mathcal{A} \in N$ be an antichain in $\mathbb{B}^\infty_N$. Choose a sequence $\langle w_\alpha : \alpha < \eta \rangle \in N$ of elements of $\mathfrak{B}^\infty \cap N$ such that that

$$\mathcal{A} = \{[w_\alpha]^\infty \cap N \mid \alpha < \eta\}$$

Let

$$w = (1, \langle w_\alpha : \alpha < \eta \rangle).$$

Thus $w \in \mathfrak{B}^\infty \cap N$ and

$$A_w = \cup \{A_{w_\alpha} \mid \alpha < \eta\}.$$

Thus $[w]^\infty$ is the least upper bound of $\mathcal{A}$ in $\mathbb{B}^\infty$ and so $[w]^\infty \cap N$ is the is the least upper bound of $\mathcal{A}$ in $\mathbb{B}^\infty_N$.

Thus $w \in \mathfrak{B}^\infty \cap N$ and $[w]^\infty \cap N$ is a least upper bound of $\mathcal{A}$ in $\mathbb{B}^\infty_N$.

We finish by proving (2). Suppose Let $\mathcal{A} \in N$ be a maximal antichain in $\mathbb{B}^\infty_N$. Choose a sequence $\langle w_\alpha : \alpha < \eta \rangle \in N$ of elements of $\mathfrak{B}^\infty \cap N$ such that that

$$\mathcal{A} = \{[w_\alpha]^\infty \cap N \mid \alpha < \eta\}$$

Let

$$w = (1, \langle w_\alpha : \alpha < \eta \rangle).$$

Thus $w \in \mathfrak{B}^\infty \cap N$ and since $\mathcal{A}$ is a maximal antichain in $\mathbb{B}^\infty_N$, necessarily

$$A_w = \cup \{A_{w_\alpha} \mid \alpha < \eta\} = \omega^\omega.$$

Therefore there exists $\alpha < \eta$ such that $x \in A_{w_\alpha}$ and so

$$G_x \cap \mathcal{A} \neq \emptyset.$$

This proves that G_x is an N-generic ultrafilter on $\mathbb{B}^\infty_N$. For each $s \in \omega^{<\omega}$, let

$$w_s = (0, s).$$

Thus $\langle w_s : s \in \omega^{<\omega} \rangle \in N$ and so

$$x = \cup \{s \in \omega^{<\omega} \mid [w_s]^\infty \cap N \in G_x\}.$$

Therefore $x \in N[G_x]$, and this proves (2). □

We fix some more notation

Definition 1.53 (ZF). Suppose $0 < k < \omega$.

(1) $\mathfrak{B}^{(k)}_\infty$ is the class of all $^\infty$Borel-codes for subsets of the product space

$$(\omega^\omega)^k = \omega^\omega \times \cdots \times \omega^\omega.$$

(2) $\sim^{(k)}_\infty$ is the equivalence relation on $\mathfrak{B}^{(k)}_\infty$ where

$$w_1 \sim^{(k)}_\infty w_2$$

if $A_{w_1} = A_{w_2}$.

(3) Suppose that N is an inner model of ZFC. Then N is $(\sim^{(k)}_\infty)$-*closed* if for all ordinals α,

$$\left\{(w_1, w_2) \in \mathfrak{B}^{(k)}_\infty \times \mathfrak{B}^{(k)}_\infty \mid w_1 \sim^{(k)}_\infty w_2\right\} \cap V_\alpha \cap N \in N. \qquad \square$$

Lemma 1.54 (ZF). *Suppose N is an inner model of* ZFC. *Then the following are equivalent.*

(1) *N is $(\sim^\infty)$-closed.*

(2) *For all $0 < k < \omega$, N is $(\sim^{(k)}_\infty)$-closed.*

Definition 1.55. Suppose N is an inner model of ZFC and N is $(\sim^\infty)$-closed. Then N is *strongly* $(\sim^\infty)$-*closed* if for all

$$w \in \mathfrak{B}^{(2)}_\infty \cap N$$

there exists $u \in \mathfrak{B}^{(1)}_\infty \cap N$ such that

$$A_u = \{x \in \omega^\omega \mid (x, y) \in A_w \text{ for some } y \in \omega^\omega\}. \qquad \square$$

Remark 1.56. Suppose N is an inner model of ZFC and $\omega^\omega \subset N$. Then N is strongly $(\sim^\infty)$-closed. $\square$

We now come to a key theorem.

Theorem 1.57 (ZF). *Suppose that N is an inner model of* ZFC *and that N is strongly $(\sim^\infty)$-closed. Then for all $x \in \omega^\omega$, $N[x]$ is strongly $(\sim^\infty)$-closed.*

Proof. We first show that $N[x]$ is $(\sim^\infty)$-closed. It will follow easily from this that $N[x]$ is necessarily strongly $(\sim^\infty)$-closed and we will give that argument after showing that $N[x]$ is $(\sim^\infty)$-closed.

Let $\mathcal{T}_N$ be the class of all

$$\tau \in N^{\mathbb{B}^\infty_N}$$

such that τ is a term for an element of $\mathfrak{B}^\infty$ (with Boolean value 1).

Fix $\tau \in \mathcal{T}_N$ and let A be the set of all pairs $(x_0, y_0) \in \omega^\omega \times \omega^\omega$ such that

$$y_0 \in A_{w_0}$$

where G_{x_0} is the N-generic ultrafilter on $\mathbb{B}^\infty_N$ given by x_0 and w_0 is the interpretation of τ by G_{x_0}.

There exists $u_0 \in \mathfrak{B}^{(2)}_\infty \cap N$ such that $A_{u_0} = A$. Since N is strongly $(\sim^\infty)$-closed, there exists $u_1 \in \mathfrak{B}^\infty \cap N$ such that

$$B = A_{u_1}$$

where $B = \{t_0 \in \omega^\omega \mid (t_0, t_1) \in A \text{ for some } t_1 \in \omega^\omega\}$.

Thus for all $t_0 \in \omega^\omega$, the following are equivalent where $G_{t_0} \subset \mathbb{B}^\infty_N$ is the N-generic filter given by t_0,

(1.1) $b \in G_{t_0}$ where $b \in \mathbb{B}^\infty_N$ is that element given by u_1.

(1.2) Let w be the interpretation of τ by G_{t_0}. Then $A_w \neq \emptyset$.

Suppose $G_0 \subset \mathbb{B}^\infty_N$ is N-generic and let $x_0 \in (\omega^\omega)^{N[G_0]}$ be the element given by G_0. We claim the following are equivalent and this claim is interpreted as a first order statement in N (and not about generic objects actually in V). This claim follows immediately from the choice of u_1 and the metamathematics of forcing.

(2.1) $b \in G_0$ where $b \in \mathbb{B}^\infty_N$ is that element given by u_1.

(2.2) There exists $G \subset \mathbb{B}^{(2)}_\infty \cap N$ such that G is N-generic and such that if

$$(x^G_0, y^G_0) \in (\omega^\omega \times \omega^\omega)^{N[G]}$$

is the element given by G then:

a) $x_0 = x^G_0$.

b) $y^G_0 \in (A_{w_0})^{N[G]}$ where w_0 is the interpretation of τ by G_0.

The equivalence of (2.1) and (2.2), together with the equivalence of (1.1) and (1.2), implies that

$$\{w \in \mathfrak{B}^\infty \cap N[x] \mid A_w \neq \emptyset\}$$

is definable in the structure $(N[x], N, \in)$ from the parameter $\mathbb{B}^{(2)}_\infty \cap N$. By Lemma 1.51, this implies that $N[x]$ is $(\sim^\infty)$-closed.

It now follows easily that $N[x]$ is strongly $(\sim^\infty)$-closed. To see this, let $\mathcal{T}^{(2)}_N$ be the class of all

$$\tau \in N^{\mathbb{B}^\infty_N}$$

such that τ is a term for an element of $\mathfrak{B}^{(2)}_\infty$ (with Boolean value 1).

Fix $\tau \in \mathcal{T}^{(2)}_N$ and let A be the set of all $(x_0, y_0, z_0) \in \omega^\omega \times \omega^\omega \times \omega^\omega$ such that

$$(y_0, z_0) \in A_{w_0}$$

where G_{x_0} is the N-generic ultrafilter on $\mathbb{B}^\infty_N$ given by x_0 and w_0 is the interpretation of τ by G_{x_0}.

There exists $u_0 \in \mathfrak{B}^{(3)}_\infty \cap N$ such that $A_{u_0} = A$. Since N is strongly $(\sim^\infty)$-closed, there exists $u_1 \in \mathfrak{B}^{(2)}_\infty \cap N$ such that

$$B = A_{u_1}$$

where $B = \{(x_0, y_0) \in \omega^\omega \times \omega^\omega \mid (x_0, y_0, z_0) \in A \text{ for some } z_0 \in \omega^\omega\}$.

The key point is the u_1 naturally defines $\sigma \in \mathcal{T}_N$. If $G_0 \subset \mathbb{N}^\infty_N$ is N-generic and if $x_0 \in (\omega^\omega)^{N[G_0]}$ is the element given by G_0, then σ is the term for the $^\infty$Borel-code w obtained from u_1 and x_0 for the set of all $y \in (\omega^\omega)^{N[G_0]}$ such that

$$(x_0, y) \in (A_{u_1})^{N[G_0]}.$$

Now let $G_x \subset \mathbb{B}^\infty_N$ be the N-generic filter given by x. Let w be the interpretation of τ by G_x and let u be the interpretation of σ by G_x. Then $w \in \mathfrak{B}^{(2)}_\infty$, $u \in \mathfrak{B}^{(1)}_\infty$, and

$$A_u = \{t_0 \in \omega^\omega \mid (t_0, t_1) \in A_w \text{ for some} t_1 \in \omega^\omega\}.$$

Thus since $N[x]$ is $(\sim^\infty)$-closed, $N[x]$ is strongly $(\sim^\infty)$-closed. □

Theorem 1.58 (ZF). *Suppose N is an inner model of* ZFC *and that N is strongly* $(\sim^\infty)$-*closed. Then*

$$N(\omega^\omega)$$

is a symmetric forcing extension of N.

Proof. Let

$$\mathfrak{B}^{<\omega}_\infty = \cup_{k<\omega} \mathfrak{B}^{(k)}_\infty$$

For each $w \in \mathfrak{B}^{<\omega}_\infty$ let $k(w)$ denote k where

$$w \in \mathfrak{B}^{(k)}_\infty.$$

Thus $A_w \subseteq (\omega^\omega)^{k(w)}$.

Define a partial order on $\mathfrak{B}^{<\omega}_\infty$ by $w_1 \leq w_2$ if:

(1.1) $k(w_2) \leq k(w_1)$.

(1.2) $\{x|k(w_2) \mid x \in A_{w_1}\} \subseteq A_{w_2}$.

Let $\mathbb{P}^\infty$ be the induced partial order on the equivalence classes $[w]$ given by $\sim^{(k(w))}_\infty$ and let

$$\mathbb{P}^\infty_N = \mathbb{P}^\infty | N.$$

Since N is strongly $(\sim^\infty)$-closed, $\mathbb{P}^\infty_N \in N$.

We now come to the key claim. Suppose

$$G \subseteq \text{Coll}(\omega, \omega^\omega)$$

be V-generic. Let G_N be the maximal filter on $\mathbb{P}^\infty_N$ given by the class of $w \in \mathfrak{B}^{<\omega}_\infty \cap N$ such that

$$G|k(w) \in A_w.$$

Then:

(2.1) G_N is N-generic.

Suppose $D \subset \mathbb{P}^{\infty}_N$ is open dense and $D \in N$. Assume toward a contradiction that
$$D \cap G_N = \emptyset.$$
Then there exist $k < \omega$ such that $G|k$ forces this. Let $G^{(k)}_N$ be the set of all $w \in \mathfrak{B}^{(k)}_{\infty} \cap N$ such that
$$G|k \in A_w.$$

Thus $G^{(k)}_N$ defines an N-generic filter on $\mathbb{B}^{(k)}_{\infty}|N$. Therefore there must exist $w \in G^{(k)}_N$ such that for all $u \in D$ with $k(u) > k$,
$$\{f|k \mid f \in A_u \cap N\} \cap A_w = \emptyset.$$
But D is open dense and so there must exist $u_0 \in D$ such that $k(u_0) > k$ and such that
$$\{f|k \mid f \in A_{u_0} \cap N\} \subseteq A_w.$$
But this contradicts the choice of w and this proves (2.1).

Fix $k < \omega$. Again let $G^{(k)}_N$ be the set of all $w \in \mathfrak{B}^{(k)}_{\infty} \cap N$ such that
$$G|k \in A_w.$$
Thus
$$N[G|k] = N[G^{(k)}_N]$$
and by Theorem 1.57, $N[G|k]$ is strongly $(\sim^{\infty})$-closed. Let $G^{(k)}$ be the tail of G so that
$$G^{(k)}(i) = G(k+i).$$
Thus $G^{(k)}$ is V-generic for $\mathrm{Coll}(\omega, \omega^{\omega})$.

Let $G^{(k)}_{N[G|k]}$ be the maximal filter on $\mathbb{P}^{\infty}_{N[G|k]}$ given by the class of $w \in \mathfrak{B}^{<\omega}_{\infty} \cap N[G|k]$ such that
$$G^{(k)}|k(w) \in A_w.$$
Thus $G^{(k)}_{N[G|k]}$ is $N[G|k]$-generic.

By the definability of forcing we claim the following. Suppose $H \subseteq \mathbb{P}^{\infty}_N$ is N-generic and let
$$g_H : \omega \to (\omega^{\omega})^{N[H]}$$
be the function naturally given by H. Let $(\omega^{\omega})_H$ be the range of g_H. Then the following hold.

(3.1) $(\omega^{\omega})_H = (\omega^{\omega})^{N[H]} \cap N((\omega^{\omega})_H)$.

(3.2) g_H is $N((\omega^{\omega})_H)$-generic for $\mathrm{Coll}(\omega, (\omega^{\omega})_H)$.

(3.3) N is strongly $(\sim^{|infty})$-closed in $N((\omega^{\omega})_H)$ and
$$\mathbb{P}^{\infty}_N = (\mathbb{P}^{\infty}_N)^{N((\omega^{\omega})_H)}.$$

(3.4) For each $k < \omega$, let $g^{(k)}_H(i) = g_H(k+i)$ for $i < \omega$. Then

a) $g_H^{(k)}$ is $N[g_H|k]((\omega^\omega)_H)$-generic for $\mathrm{Coll}(\omega, (\omega^\omega)_H)$.

b) Let

$$H^{(k)} \subset \left(\mathbb{P}^\infty_{N[g_H|k]}\right)^{N((\omega^\omega)_H)}$$

be the filter given by $g_H^{(k)}$. Then $H^{(k)}$ is an $N[g_H|k]$-generic filter.

Suppose not. Then some $w \in \mathbb{P}^\infty_N$ must force that either (3.1), (3.2), or (3.3) fails. Let $k = k(w)$ and choose $G \subset \mathrm{Coll}(\omega, \omega^\omega)$ such that G is V-generic with $G|k \in A_w$.

Let $H = G_N$ as above. Then $H \subseteq \mathbb{P}^\infty_N$, H is N-generic, and $w \in H$. Further $g_H = G$ and $(\omega^\omega)_H = (\omega^\omega)^V$. But then (3.1), (3.2), and (3.3) all hold for H and this contradicts the choice of w. Replacing N by $N[g_H|k]$ yields (3.4).

The theorem now follows by standard arguments using finite permutations of ω to generate automorphisms of $\mathrm{Coll}(\omega, \omega^\omega)$ and of the partial order $\mathbb{P}^\infty_N$. □

As a corollary of the (proof of) Theorem 1.58, we obtain the following.

Theorem 1.59 (ZF). *Suppose N is an inner model of* ZFC *and that N is strongly* $(\sim^\infty)$-*closed.*

(1) *Suppose $A \in \mathcal{P}(\omega^\omega) \cap N(\omega^\omega)$ and A is definable in*

$$(N(\omega^\omega), N, \in)$$

from ordinal parameters. Then

$$A = A_w$$

for some $w \in \mathfrak{B}^\infty \cap N$.

(2) *Suppose $A \in \mathcal{P}(\omega^\omega) \cap N(\omega^\omega)$. Then*

$$A = A_w$$

for some $w \in \mathfrak{B}^\infty \cap N(\omega^\omega)$.

Definition 1.60 (ZF + DC). Suppose U is an ultrafilter on $\mathcal{P}_{\omega_1}(\omega^\omega)$.

(1) U is *fine* if for all $x \in \omega^\omega$,

$$\{\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega) \mid x \in \sigma\} \in U.$$

(2) U is *normal* if for all functions

$$F : \mathcal{P}_{\omega_1}(\omega^\omega) \to \mathcal{P}_{\omega_1}(\omega^\omega),$$

if $\{\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega) \mid F(\sigma) \subseteq \sigma\} \in U$ then for some $x \in \omega^\omega$,

$$\{\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega) \mid x \in F(\sigma)\} \in U.$$ □

Definition 1.61 (HOD relativized to a set Z). (ZF) Suppose Z is a set. Then HOD_Z is the class of all sets X such that for some ordinal α the following hold where $Y = \mathrm{TC}(X)$.

(1) $Z \in V_\alpha$.

(2) $Y \in V_\alpha$.

(3) Suppose $A \in Y$. Then A is definable in the structure
$$(V_\alpha, \in)$$
from ordinal parameters and parameters from Z. □

Remark 1.62. (1) HOD_Z is a transitive class and $\mathrm{HOD}_Z \vDash \mathrm{ZF}\backslash\mathrm{Powerset}$.

(2) If there exists an ordinal α such that $Z \in V_\alpha$ and such that Z is definable in V_α with parameters from $\alpha \cup Z$, then
$$\mathrm{HOD}_Z \vDash \mathrm{ZF}.$$
In this case, $\mathrm{HOD}_Z = \mathrm{HOD}_X$ where $X = Z \cup \{Z\}$.

(3) For any *finite* set Z, HOD_Z is an inner model of ZFC. □

We shall be mostly interested in the case of HOD_Z in the case where Z is finite.

Theorem 1.63 (ZF + DC). *Suppose that U is a fine ultrafilter on $\mathcal{P}_{\omega_1}(\omega^\omega)$. Then for all sets X,*
$$\mathrm{HOD}_{(U,X)}$$
is strongly $(\sim^\infty)$-closed.

Proof. Trivially, $\mathrm{HOD}_{(U,X)}$ is $(\sim^\infty)$-closed. Therefore we only have to verify that for $w \in \mathfrak{B}^{(2)}_\infty \cap \mathrm{HOD}_{(U,X)}$, there exists $u \in \mathfrak{B}^{(1)}_\infty \cap \mathrm{HOD}_{(U,X)}$ such that
$$A_u = \{f \in \omega^\omega \mid (f, h) \in A_w \text{ for some } h \in \omega^\omega\}.$$

Fix $w \in \mathfrak{B}^{(2)}_\infty \cap \mathrm{HOD}_{(U,X)}$. Suppose $\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega)$. Suppose w_1 and w_2 are ${}^\infty$Borel-codes. Define
$$w_1 \sim_\sigma w_2$$
if $A_{w_1} \cap \sigma = A_{w_2} \cap \sigma$. Thus $\mathrm{HOD}_{(U,X,\sigma),\sigma}$ is $(\sim_\sigma)$-closed in the natural sense that for all ordinals α,
$$\{(w_1, w_2) \in \mathfrak{B}^\infty \times \mathfrak{B}^\infty \mid w_1 \sim_\sigma w_2\} \cap V_\alpha \cap \mathrm{HOD}_{(U,X,\sigma)} \in \mathrm{HOD}_{(U,X),\sigma}.$$
Let $\mathbb{B}_\sigma$ be the complete Boolean algebra of $\mathrm{HOD}_{(U,X,\sigma)}$ given by
$$\mathrm{HOD}_{(U,X,\sigma)} \cap \mathfrak{B}^\infty/\sim_\sigma.$$
For each $f \in \sigma$, let G^σ_f be the ultrafilter on $\mathbb{B}_\sigma$ given by the class of all
$$U \in \mathfrak{B}^\infty \cap \mathrm{HOD}_{(U,X,\sigma)}$$
such that $f \in A_u$. Thus G^σ_f is a $\mathrm{HOD}_{(U,X,\sigma)}$-generic filter and
$$\mathrm{HOD}_{(U,X,f)}[f] = \mathrm{HOD}_{(U,X,f)}[G^\sigma_f].$$

Let,

(1.1) $H_\infty = \prod_\sigma \text{HOD}_{(U,X,\sigma)}/U$,

(1.2) $\mathbb{B}_\infty = \prod_\sigma \mathbb{B}_\sigma/U$,

(1.3) $w_\infty = \prod_\sigma w/U$,

(1.4) and for each $f \in \omega^\omega$, let $G^\infty_f = \prod_\sigma G^\sigma_f/U$;

where the ultraproducts are computed using all functions, $F : \omega^\omega \to V$.

Thus for all $f \in \omega^\omega$,

(2.1) $w_\infty \in \mathfrak{B}^{(2)}_\infty$.

(2.2) G^∞_f is an H_∞-generic ultrafilter in $\mathbb{B}_\infty$.

(2.3) $H_\infty[f] = H_\infty[G^\infty_f]$.

Suppose $f_0, f_1 \in \omega^\omega$ and $(f_0, f_1) \in A_w$. Let $f \in \omega^\omega$ be the function where $f(2k) = f_0(k)$ and $f(2k+1) = f_1(k)$ for all $k < \omega$. Then

(3.1) $(f_0, f_1) \in A_{w_\infty}$.

(3.2) $H_\infty[G^\infty_f] = H_\infty[f] = H_\infty[f_0][f_1] = H_\infty[G^\infty_{f_0}][f_1]$.

So by factoring, f_1 is $H_\infty[G^\infty_{f_0}]$-generic for a Boolean algebra,

$$\mathbb{B}^{f_0}_{f_1} \in H_\infty[f_0].$$

such that

$$|\,\mathbb{B}^{f_0}_{f_1}\,|^{H_\infty[f_0]} \leq |\mathbb{B}_\infty|^{H_\infty[f_0]}.$$

Let

$$\kappa_\infty = |\mathbb{B}_\infty|^{H_\infty}.$$

We come to the key point.

(4.1) $(H_\infty[f_0])^{\text{Coll}(\omega,\kappa_\infty)} \vDash$ "There exists $f^* \in \omega^\omega$ such that $(f_0, f^*) \in A_{w_\infty}$".

For each $\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega)$, let

$$\kappa_\sigma = |\mathbb{B}_\sigma|^{\text{HOD}_{(U,X,\sigma)}}.$$

Thus

$$\kappa_\infty = \left(\prod_\sigma \kappa_\sigma\right)/U$$

We prove the following. Suppose that $f_0 \in \omega^\omega$. Then the following are equivalent.

(5.1) There exists $f_1 \in \omega^\omega$ such that $(f_0, f_1) \in A_w$.

(5.2) $(H_\infty[f_0])^{\text{Coll}(\omega,\kappa_\infty)} \vDash$ "There exists $f^* \in \omega^\omega$ such that $(f_0, f^*) \in A_{w_\infty}$".

(5.3) There exists $Z \in U$ such that for all $\sigma \in U$, $f_0 \in \sigma$ and

$$(\mathrm{HOD}_{(U,X,\sigma)}[f_0])^{\mathrm{Coll}(\omega,\kappa_\sigma)} \models \text{"There exists } f^* \in \omega^\omega \text{ such that } (f_0, f^*) \in A_w\text{"}.$$

(5.4) There exists $Z \in U$ such that for all $\sigma \in U$, $f_0 \in \sigma$ and if

$$G \subset \mathrm{Coll}(\omega, \kappa_\sigma)$$

is $\mathrm{HOD}_{(U,X,\sigma)}[f_0]$-generic with $G \in V$, then there exists

$$f_1 \in \omega^\omega \cap \mathrm{HOD}_{(U,X,\sigma)}[f_0][G]$$

such that $(f_0, f_1) \in A_w$.

We have already proved that (5.1) implies (5.2). By Łos, (5.2) is equivalent to (5.3) and trivially (5.3) implies (5.4). Thus we just have to show that (5.4) implies (5.1). The issue is that (5.4) might be vacuously true since there may be no such G *in* V.

Let

$$\pi : \mathcal{P}_{\omega_1}(\omega^{\omega_1}) \to \omega_1$$

be the function $\pi(\sigma) = \sup\{\eta_x \mid x \in \sigma\}$ where for each $x \in \omega^\omega$, η_x is the least ordinal admissible relative to x.

Let U_π be the ultrafilter on ω_1 defined by $I \in U_\pi$ if

$$\{\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega) \mid \pi(\sigma) \in I\}.$$

Since U is a fine ultrafilter, it follows that U_π is a countable complete uniform ultrafilter on ω_1. Clearly U_π is definable from U and so for all $\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega)$,

$$U_\pi \cap \mathrm{HOD}_{(U,X,\sigma)} \in \mathrm{HOD}_{(U,X,\sigma)}.$$

Thus for all $\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega)$, ω_1^V is a measurable cardinal in $\mathrm{HOD}_{(U,X,\sigma)}$. But this implies:

(6.1) For all $\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega)$, $\kappa_\sigma < \omega_1$,

(6.2) For all $\sigma \in \mathcal{P}_{\omega_1}(\omega^\omega)$, for all $f \in \sigma$, there exists a filter

$$G \subset \mathrm{Coll}(\omega, \kappa_\sigma)$$

such that $G \in V$ and such that G is $\mathrm{HOD}_{(U,X,\sigma)}[f]$-generic.

Now assume (5.4). Choose $\sigma \in Z$ and choose a filter

$$G \subset \mathrm{Coll}(\omega, \kappa_\sigma)$$

such that G is $\mathrm{HOD}_{(U,X,\sigma)}[f_0]$-generic. Then there exists $f_1 \in \omega^\omega \cap \mathrm{HOD}_{(U,X,\sigma)}[f_0][G]$ such that $(f_0, f_1) \in A_w$. This shows (5.4) implies (5.1).

There must exist $b \in \mathbb{B}_\infty$ such that (and using the informal language of forcing), the following are equivalent for all G such that G is H_∞-generic for $\mathbb{B}_\infty$.

(7.1) $b \in G$.

(7.2) Let $f_G \in (\omega^\omega)^{H_\infty[G]}$ be the element given by G. Suppose g is $H_\infty[G]$-generic for $\mathrm{Coll}(\omega, \kappa_\infty)$. Then there exists $f^* \in (\omega^\omega)^{H_\infty[G][g]}$ such that

$$(f_G, f^*) \in (A_{w_\infty})^{H_\infty[G][g]}.$$

Let

$$\sim_\infty = \left(\prod \sim_\sigma \cap \mathrm{HOD}_{(U,X,\sigma)}\right)/U$$

noting that this in general is not the equivalence relation $\sim^\infty$. Thus $\sim_\infty \subset H_\infty$, $\sim_\infty$ is an equivalence relation on $\mathfrak{B}^\infty \cap H_\infty$, and

$$\mathbb{B}_\infty = \mathfrak{B}^\infty \cap H_\infty/\sim_\infty.$$

Thus there exists $u \in \mathfrak{B}^\infty \cap H_\infty$ such that

$$b = [u]_{\sim_\infty}.$$

where $[u]_{\sim_\infty}$ denotes the equivalence class of u relative to $\sim_\infty$. Thus

$$A_u = \{f \in \omega^\omega \mid (f, h) \in A_w \text{ for some } h \in \omega^\omega\}.$$

Thus proves the theorem. □

Definition 1.64 (L relativized to (S, ω^ω) where $S \subset \mathrm{Ord}$). (ZF). Suppose $S \subset \mathrm{Ord}$.

(1) The sets $L_\alpha(S, \omega^\omega)$ are defined by induction on the ordinal α as follows.

 a) $L_0(S, \omega^\omega) = \omega^\omega$.
 b) $L_{\alpha+1}(S, \omega^\omega) = \mathcal{P}_{\mathrm{Def}}(L_\alpha(S, \omega^\omega) \cup \{S \cap \alpha\})$.
 c) If α is a limit ordinal then $L_\alpha(S, \omega^\omega) = \cup\left\{L_\beta(S, \omega^\omega) \mid \beta < \alpha\right\}$.

(2) $L(S, \omega^\omega)$ is the class of all sets X such that $X \in L_\alpha(S, \omega^\omega)$ for some ordinal α. □

Theorem 1.65 (ZF + DC). *Suppose that there is a fine ultrafilter on $\mathcal{P}_{\omega_1}(\omega^\omega)$. Then for all sets $S \subset \mathrm{Ord}$, every set*

$$A \in \mathcal{P}(\omega^\omega) \cap L(S, \omega^\omega)$$

is ${}^\infty$Borel.

Proof. Let U a fine ultrafilter on $\mathcal{P}_{\omega_1}(\omega^\omega)$ and let Let

$$N = \mathrm{HOD}_{(U,S)}.$$

By Theorem 1.63, N is strongly $(\sim^\infty)$-closed and so by Theorem 1.59, every set

$$A \in \mathcal{P}(\omega^\omega) \cap N(\omega^\omega)$$

is ${}^\infty$Borel. Finally

$$L(S, \omega^\omega) \subseteq N(\omega^\omega)$$

and this proves the theorem □

Suppose $w \in \mathfrak{B}^\infty$ and ω_1^V is strongly inaccessible in $L[w]$. Then A_w is Lebesgue measurable and has the property of Baire. In fact A_w has all the regularity properties connected to forcing.

Suppose that U a fine countably complete ultrafilter on $\mathcal{P}_{\omega_1}(\omega^\omega)$. Then one can show (in fact without using countable choice) that ω_1 must be a regular cardinal and further for all sets X, ω_1^V is a measurable cardinal in $\mathrm{HOD}_{(U,X)}$. Thus for all $w \in \mathfrak{B}^\infty$, ω_1^V is strongly inaccessible in $L[w]$.

Thus the following theorem is an immediate corollary of Theorem 1.63.

Theorem 1.66 (ZF + DC). *Suppose that there is a fine countably complete ultrafilter on $\mathcal{P}_{\omega_1}(\omega^\omega)$. Then for all sets $S \subset \mathrm{Ord}$, every set*

$$A \in \mathcal{P}(\omega^\omega) \cap L(S, \omega^\omega)$$

is Lebesgue measurable and has the property of Baire. □

1.7 Determinacy axioms and AD^+

Let $(\mathcal{D}, \leq_T)$ denote the partial order of the Turing degrees.

Definition 1.67 (Turing Determinacy). *Turing Determinacy* is the assertion that for all $X \subset \mathcal{D}$, there exists $d_0 \in \mathcal{D}$ such that either

$$\{d \in \mathcal{D} \mid d_0 \leq_T d\} \subset X$$

or

$$\{d \in \mathcal{D} \mid d_0 \leq_T d\} \subset \mathcal{D} \backslash X.$$ □

Note that (assuming countable choice) if Turing Determinacy holds then there is a definable fine countably complete ultrafilter on $\mathcal{P}_{\omega_1}(\omega^\omega)$.

Theorem 1.68 ($\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}}$). *Assume Turing Determinacy. Suppose $S \subset \mathrm{Ord}$. Then the following hold.*

(1) *Suppose $A \in \mathcal{P}(\omega^\omega)$ and A is definable in $L(S, \omega^\omega)$ from S and ordinal parameters. Then there exists*

$$w \in \mathfrak{B}^\infty \cap \mathrm{HOD}_{\{S\}}^{L(S,\omega^\omega)}$$

such that $A = A_w$.

(2) *Every*

$$A \in \mathcal{P}(\omega^\omega) \cap L(S, \omega^\omega)$$

is ${}^\infty$Borel *with an* ${}^\infty$Borel-*code in $L(S, \omega^\omega)$.* □

Suppose $A \subset X^\omega$. Then A defines a game on X where the players choose elements of X, defining $f \in X^\omega$ after ω-many moves. Player I wins if $f \in A$ and Player II wins if $f \notin A$.

The game G_A is *determined* if there is a function

$$\tau : X^{<\omega} \to X$$

which is a *winning strategy* for either Player I or Player II.

Here, τ is a winning strategy for Player I if for all $f \in X^\omega$, if

$$f(2k) = \tau(f|2k)$$

for all $k < \omega$, then $f \in A$. Similarly, τ is a winning strategy for Player II if for all $f \in X^\omega$, if

$$f(2k+1) = \tau(f|(2k+1))$$

for all $k < \omega$, then $f \notin A$.

If $X = \omega$ then winning strategies are in essence elements of ω^ω.

The Axiom of Determinacy, AD, is the axiom that for every set $A \subseteq \omega^\omega$ the corresponding game G_A on ω is determined. This axiom, like Turing Determinacy, contradicts the Axiom of Choice and the context is ZF + $\mathrm{DC}_{\mathbb{R}}$.

Lemma 1.69 (Martin). *Assume* AD. *Then Turing Determinacy holds.* □

$\mathrm{AD}_{\mathbb{R}}$ is the axiom which asserts that for every $A \subset (\omega^\omega)^\omega$, the game on ω^ω given by A is determined–so here strategies are in essence elements of $\mathcal{P}(\omega^\omega)$. What about other generalizations of AD?

Lemma 1.70 (ZF). *There exists a set $A \subseteq \omega_1^\omega$ such that A is not determined.* □

Definition 1.71. Θ denotes the supremum of the ordinals α such that there is a surjection

$$\pi : \omega^\omega \to \alpha.$$ □

Assuming the Axiom of Choice, $\Theta = c^+$. But assuming AD, Θ is always a limit cardinal. This is an immediate corollary of the Moschovakis Coding Lemma which yields the following lemma as an immediate corollary.

Lemma 1.72 (Moschovakis). *Assume* AD *and that $\alpha < \Theta$. Then there is a surjection*

$$\pi : \omega^\omega \to \mathcal{P}(\alpha).$$ □

The axiom AD^+ is the following technical variation of AD. The axiom AD^+ is in essence simply a "structural" enhancement of AD.

Definition 1.73 (AD^+). Assume ZF + $\mathrm{DC}_{\mathbb{R}}$. Then AD^+ is the conjunction of the following.

(1) Suppose $A \subseteq \omega^\omega$, $\lambda < \Theta$, and that

$$\pi : \lambda^\omega \to \omega^\omega$$

is a continuous function. Then the game on λ given by $\pi^{-1}[A]$ is determined.

(2) Suppose $A \subseteq \omega^\omega$. Then A is ${}^\infty$Borel. □

Remark 1.74. It is conjectured that assuming ZF + $\mathrm{DC}_\mathbb{R}$, AD and AD^+ are actually equivalent. The following summarizes roughly what is known at present.

(1) Assume ZF + DC + $\mathrm{AD}_\mathbb{R}$. Then AD^+ holds.

(2) Assume ZF + $\mathrm{DC}_\mathbb{R}$ + AD + $\neg\mathrm{AD}^+$. Then ZF + DC + $\mathrm{AD}_\mathbb{R}$ is consistent.

Thus if ZF + $\mathrm{DC}_\mathbb{R}$ + AD is consistent with ($\neg\mathrm{AD}^+$), that theory is much stronger that just ZF + $\mathrm{DC}_\mathbb{R}$ + AD. □

2 Universally Baire sets and the axiom V = Ultimate-L

Definition 2.1. Suppose $A \subseteq \omega^\omega$. Then A is *universally Baire* if for all topological spaces Ω and for all continuous functions

$$\pi : \Omega \to \omega^\omega,$$

the set $\pi^{-1}[A]$ has the property of Baire in the topological space Ω. □

The notion of being universally Baire is more usefully formulated in terms of Suslin representations and forcing. We fix some standard notation.

Suppose X is a set. A tree T on X is a set $T \subseteq X^{<\omega}$ which is closed under initial segments, more precisely for all $s \in T$,

$$s|k \in T$$

for all $k \in \mathrm{dom}(s)$.

If T is a tree on X then $[T]$ is the set of all $f \in X^\omega$ such that

$$f|k \in X$$

for all $k < \omega$. The set $[T]$ is the set of infinite branches of the tree T.

Suppose λ is an ordinal and that T is a tree on $\omega \times \lambda$. Then:

(1) We view $[T]$ as the set of all pairs (x, f) such that

$$(x, f) \in \omega^\omega \times \lambda^\omega$$

and such that $\langle (x(i), f(i)) : i \leq k \rangle \in T$ for all $k < \omega$.

(2) $p[T] = \{x \in \omega^\omega \mid (x, f) \in [T] \text{ for some } f \in \lambda^\omega\}$.

Definition 2.2. Suppose $A \subseteq \omega^\omega$ and $\lambda \in \mathrm{Ord}$. Then A is *λ-Suslin* if there is a tree T on $\omega \times \lambda$ such that $A = p[T]$. □

Lemma 2.3. *Suppose that $A \subseteq \omega^\omega$. Then the following are equivalent.*

(1) *A is universally Baire.*

(2) *For all partial orders $\mathbb{P}$ there exist λ and trees S, T on $\omega \times \lambda$ such that the following hold.*

  (a) *$p[S] = A$ and $p[T] = \omega^\omega \backslash A$.*

  (b) *Suppose $G \subset \mathbb{P}$ is V-generic. Then in $V[G]$:*

$$p[S] = (\omega^\omega)^{V[G]} \backslash p[T].$$ □

Theorem 2.4. *Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \omega^\omega$ is universally Baire. Then*

(1) *Every set $B \in \mathcal{P}(\omega^\omega) \cap L(A, \omega^\omega)$ is universally Baire.*

(2) $L(A, \omega^\omega) \vDash \mathrm{AD}^+$. □

Definition 2.5 (V = Ultimate-L). (1) There is a proper class of Woodin cardinals.

(2) For each Σ_2-sentence ϕ, if ϕ holds in V then there exists a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\omega^\omega)} \vDash \phi.$$ □

The axiom V = Ultimate-L implies a number of L-like consequences, and these are proved by using the connections with the theory of AD^+.

Theorem 2.6. *Assume V = Ultimate-L. Then the following hold.*

(1) CH.

(2) $V = \mathrm{HOD}$.

(3) *V is not a generic extension of any inner model.* □

Assuming V = Ultimate-L one also has what is arguably the simplest possible wellordering of the reals, in the context of a proper class of Woodin cardinals.

Theorem 2.7. *Assume V = Ultimate-L. Then the following hold.*

(1) *Suppose $x \in \mathbb{R}$. Then $x \in \mathrm{HOD}^{L(A,\mathbb{R})}$ for some universally Baire set $A \subset \mathbb{R}$.*

(2) *There is a wellordering of $\mathbb{R}$ which is Σ_1-definable from $\mathbb{R}$ is the structure*

$$\langle Hc^+), \Gamma^\infty, \in\rangle$$

where Γ^∞ is the set of all the universally Baire sets $A \subseteq \mathbb{R}$. □

3 The HOD Hypothesis and the HOD Conjecture

Suppose that κ is an uncountable regular cardinal and $S \subseteq \kappa$ is a stationary set. Then by the Solovay Splitting Theorem, there is a partition

$$\langle S_\alpha : \alpha < \kappa \rangle$$

of S into stationary subsets of S. But the proof is not effective.

Question 3.1. *Suppose that κ is an uncountable regular cardinal and let*

$$S = \{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}.$$

Is there a partition of S into two stationary sets each of which is in HOD*?* □

This question is really the simplest case of the *effective splitting problem.*

Lemma 3.2. *Suppose that κ is an uncountable regular cardinal and let*

$$S = \{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}.$$

Suppose there is no splitting of S into two stationary sets each of which is in HOD*. Then κ is a measurable cardinal in* HOD.

Proof. Let $\mathcal{F}$ be the club filter on κ. Then

$$\mathcal{F} \cap \mathrm{HOD} \in \mathrm{HOD}.$$

Clearly $S \in \mathrm{HOD}$. Thus

$$\mathcal{F} \cap \mathcal{P}(S) \cap \mathrm{HOD}$$

is an ultrafilter if and only if there is no partition of S into two stationary sets each of which is in HOD.

But $\mathcal{F} \cap \mathrm{HOD}$ is a κ-complete filter in HOD since $\mathcal{F}$ is a κ-complete filter in V. □

We recall some definitions from [7].

Definition 3.3. Suppose that κ is an uncountable regular cardinal. Then κ is *ω-strongly measurable in* HOD if there exists $\lambda < \kappa$ such that the following hold where $S = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \omega\}$.

(1) $(2^\lambda)^{\mathrm{HOD}} < \kappa$.

(2) There is no partition

$$\langle T_\alpha : \alpha < \lambda \rangle \in \mathrm{HOD}$$

of S into stationary sets. □

Remark 3.4. If κ is ω-strongly measurable in HOD then the Boolean algebra,

$$(\mathcal{P}(S) \cap \text{HOD}) / \mathcal{I},$$

is atomic (with fewer than λ many atoms for each λ with $(2^\lambda)^{\text{HOD}} < \kappa$) where

$$S = \{\alpha < \kappa \mid \text{cof}(\alpha) = \omega\}$$

and where $\mathcal{I}$ is the nonstationary ideal at κ. Since $S \in \text{HOD}$ and since $\mathcal{I} \cap \text{HOD} \in \text{HOD}$, necessarily κ is a measurable cardinal in HOD. □

Remark 3.5. Key questions arise:

(1) How many regular cardinals can there be which are ω-strongly measurable in HOD?

(2) Suppose γ is a singular strong limit cardinal and $\text{cof}(\gamma) > \omega$. Can γ^+ *ever* be ω-strongly measurable in HOD?

(3) Suppose δ is supercompact. Can there exist *any* regular cardinal $\kappa > \delta$ such that κ is ω-strongly measurable in HOD? What if δ is an extendible cardinal? □

Definition 3.6. (1) (HOD Hypothesis) There is a proper class of regular cardinals which are not ω-strongly measurable in HOD.

(2) (Weak HOD Conjecture) The HOD Hypothesis is provable from

ZFC + "There is an extendible cardinal".

(3) (HOD Conjecture) The HOD Hypothesis is provable from

ZFC + "There is a supercompact cardinal".

(4) (Strong HOD Conjecture) The HOD Hypothesis is provable from ZFC. □

Remark 3.7. The HOD Hypothesis is from [7] but there it is referred to as the HOD Conjecture. In [10], we altered the definitions to make clear the potential ambiguities. Of course similar issues arise with both the Ultimate-L Conjecture and the Ω Conjecture and moreover these conjectures are really only interesting if their conclusions are provable from (some) large cardinal hypotheses. This is particularly true for the Ultimate-L Conjecture.

However the HOD Hypothesis is easily verified to be *consistent* with the existence of an extendible cardinal (by forcing $V = \text{HOD}$) whereas for both the Ultimate-L Conjecture and the Ω Conjecture even the consistency of the statements with the existence of extendible cardinal is open. Therefore the situation for the HOD Conjecture is really quite different. □

We note the following weak version of the HOD Dichotomy Theorem of [10]. This theorem shows that if there is an extendible cardinal then either V is very "close" to HOD, or V is very "far" from HOD. We will sketch a proof of this theorem on page 54.

Theorem 3.8 (HOD Dichotomy Theorem). *Suppose that δ is an extendible cardinal. Then one of the following hold.*

(1) **Every** *regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD.

(2) **No** *regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD. □

If any of the HOD Conjectures are true then the "far" option in the HOD Dichotomy Theorem is vacuous and so one obtains that large cardinals imply that HOD is close to V. This would suggest that there is *some* generalization of the axiom $V = L$ for which there is no generalization of Scott's Theorem.

4 The sealing theorems

Remark 4.1. Suppose $A \subseteq \omega^\omega$ is universally Baire and $V[G]$ is a generic extension of V. Then A has a canonical interpretation A_G as a universally Baire set in $V[G]$. This is defined as follows. Choose trees $S, T \in V$ on $\omega \times \lambda$ for some λ such that

(1) $A = p[S]$ and $A \backslash \omega^\omega = p[T]$.

(2) $(p[S])^{V[G]} = (\omega^\omega)^{V[G]} \backslash (p[S])^{V[G]}$.

Then $A_G = (p[S])^{V[G]}$. This is well-defined. □

It is convenient to fix the following notation and from now on $\mathbb{R}$ denotes ω^ω.

Definition 4.2. Suppose that there is a proper class of Woodin cardinals.

(1) Γ^∞ denotes the set of all $A \subset \mathbb{R}$ such that A is universally Baire.

(2) Suppose $V[g]$ is a set-generic extension of V. Then

$$(\Gamma^\infty_g, \mathbb{R}_g) = ((\Gamma^\infty)^{V[g]}, \mathbb{R}^{V[g]}).$$ □

The following theorem shows that $L(\Gamma^\infty, \mathbb{R})$ can be *sealed* in a very strong sense relative to set-generic extensions.

Theorem 4.3 (Sealing Theorem). *Suppose that δ is supercompact and that there is a proper class of Woodin cardinals. Suppose $V[G] \subset V[H]$ are set-generic extensions of V and $V_{\delta+1}$ is countable in $V[G]$. Then the following hold.*

(1) $\Gamma^\infty_G = \mathcal{P}(\mathbb{R}_G) \cap L(\Gamma^\infty_G, \mathbb{R}_G)$.

(2) *Suppose that* γ *is a limit of Woodin cardinals in* V *and that* G *is* V*-generic for some partial* $\mathbb{P} \in V_\gamma$*. Then*

$$(\Gamma^\infty)^{V_\gamma[G]} = \Gamma^\infty_G.$$

(3) $\Gamma^\infty_H = \mathcal{P}(\mathbb{R}_H) \cap L(\Gamma^\infty_H, \mathbb{R}_H)$.

(4) *There is an elementary embedding*

$$j : L(\Gamma^\infty_G, \mathbb{R}_G) \to L(\Gamma^\infty_H, \mathbb{R}_H)$$

such that for all $A \in \Gamma^\infty_G$*,* $j(A) = (A)^{V[H]}$*, where* $(A)^{V[H]}$ *is the interpretation of* A *in* $V[H]$. □

Remark 4.4. Theorem 4.3 raises the key question of whether the conclusion is *provable* (more precisely that there is no requirement that $V[G]$ be different that V) from some large cardinal hypothesis such as the existence of a proper class of Vopěnka cardinals or something stronger such as the existence of a proper class of λ such that the Axiom I_0 holds at λ. Any such theorem would be a very strong *anti* inner model theorem. In particular, such a theorem would give a generalization of Scott's Theorem for the axiom $V =$ Ultimate-L. This is because if $V =$ Ultimate-L then necessarily

$$L(\Gamma^\infty, \mathbb{R}) \vDash \mathrm{AC}$$

and so $\mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R}) \neq \Gamma^\infty$, see Theorem 2.7.

Before dismissing this possibility it is prudent to note that there is a version of Theorem 4.3 for the projective sets, this is the Projective Sealing Theorem, and the hypothesis only requires that there are infinitely many strong cardinals below δ. Here *projective sealing* is the assertion that if $V[G] \subset V[H]$ are generic extensions of V then

$$V[G]_{\omega+1} \prec V[H]_{\omega+1}.$$

The Projective Sealing Theorem shows that if δ is a limit of strong cardinals and $V[G]$ is a generic extension of V in which δ is countable, then projective sealing holds in $V[G]$.

The point here is that the existence of even a proper class of strong cardinals is consistent with the existence of a projective wellordering of the reals, and so *cannot* imply projective sealing.

But by increasing this large cardinal hypothesis through requiring in addition that δ be a limit of Woodin cardinals, or even just assuming there is a proper class of Woodin cardinals, one obtains the conclusion that projective sealing holds in V (and so without having to pass to a generic extension of V). □

Theorem 4.3 shows that if there is a proper class of Woodin cardinals and there is a supercompact cardinal, then associated to V there is a canonical model of AD^+.

Theorem 4.5. *Suppose that δ is supercompact and that there is a proper class of Woodin cardinals. Suppose $V[G]$ is a generic extension of V in which $V_{\delta+1}$ is countable. Then*

$$L(\Gamma^\infty_G, \mathbb{R}_G) \vDash \text{ZF} + \text{AD}_{\mathbb{R}} + \text{“}\Theta \textit{ is regular”}. \qquad \square$$

There is a weak version of the Sealing Theorem which is important formulating stronger versions of the Ultimate-L Conjecture. We fix some additional notation.

Definition 4.6. Suppose that there is a proper class of Woodin cardinals and that

$$V[G] \subseteq V[H]$$

are generic extensions of V. Then

$$\pi^G_H : L_{\Theta_G}(\Gamma^\infty_G, \mathbb{R}_G) \to L_{\Theta_H}(\Gamma^\infty_H, \mathbb{R}_H)$$

is the Σ_0-embedding defined by

$$\pi^G_H(A) = A_H$$

for all $A \in \Gamma^\infty_G$, where:

(1) $\Theta_G = \sup\left\{\Theta^{L(A,\mathbb{R}_G)} \mid A \in \Gamma^\infty_G\right\}$.

(2) $\Theta_H = \sup\left\{\Theta^{L(A,\mathbb{R}_H)} \mid A \in \Gamma^\infty_H\right\}$.

(3) $L_{\Theta_G}(\Gamma^\infty_G, \mathbb{R}_G) = \cup\left\{L_{\Theta_G}(A, \mathbb{R}_G) \mid A \in \Gamma^\infty_G\right\}$

(4) $L_{\Theta_H}(\Gamma^\infty_H, \mathbb{R}_H) = \cup\left\{L_{\Theta_H}(A, \mathbb{R}_H) \mid A \in \Gamma^\infty_H\right\}$. $\square$

Remark 4.7. Suppose that there is a proper class of Woodin cardinals and that

$$V[G] \subseteq V[H]$$

are generic extensions of V. Then by AD^+-theory, π^G_H is necessarily a Σ_1-embedding.

If $V[G]$ seals Γ^∞ then of course π^G_H is fully elementary and much more. $\square$

Definition 4.8. Suppose that there is a proper class of Woodin cardinals and that $V[G]$ is a set generic extension of V. Then $V[G]$ *weakly seals* Γ^∞ if for all generic extensions $V[H]$ of $V[G]$, π^G_H is a Σ_2-embedding. $\square$

Suppose there is a proper class of Woodin cardinals and that δ is supercompact. The following theorem shows that if

$$G \subset \text{Coll}(\omega, {<}\delta)$$

is V-generic, than $V[G]$ very nearly seals Γ^∞.

Theorem 4.9 (Weak Sealing Theorem). *Suppose that there is a proper class of Woodin cardinals, δ is supercompact, and that $G \subset \text{Coll}(\omega, {<}\delta)$ is V-generic. Then:*

(1) *$V[G]$ weakly seals Γ^∞.*

(2) $\Gamma^\infty_G = \mathcal{P}(\mathbb{R}_G) \cap L(\Gamma^\infty_G, \mathbb{R}_G)$.

(3) *Suppose $V[H]$ is a generic extension of V, $G \in V[H]$, and that $V_{\delta+1}$ is countable in $V[H]$. Then*

$$(L(\Gamma^\infty_G, \mathbb{R}_G), A : A \in \Gamma^\infty_G) \equiv (L(\Gamma^\infty_H, \mathbb{R}_H), A_H : A \in \Gamma^\infty_G)$$

where for each $A \in \Gamma^\infty_G$, $A_H \in \Gamma^\infty_H$ is the interpretation of A in $V[H]$. □

The following version of sealing involves the generic elementary embeddings associated to the stationary towers, $\mathbb{P}_{<\delta}$ and $\mathbb{Q}_{<\delta}$, where δ is a Woodin cardinal, [4]. The partial orders have a remarkable property which is summarized in the following theorems, [4].

Theorem 4.10. *Suppose δ is a Woodin cardinal and that G is V-generic for $\mathbb{P}_{<\delta}$. Then in $V[G]$ there is a V-extender E of length δ such that if*

$$j_E : V \to M_E \cong \mathrm{Ult}_0(V, E)$$

is the associated ultrapower embedding then the following hold:

(1) $j_E(\delta) = \delta$.

(2) $M_E^{<\delta} \subset N$ *in* $V[G]$.

(3) δ *is a Woodin cardinal in* $V[G]$.

Further for each regular uncountable cardinal $\gamma < \delta$, there is a condition $p \in \mathbb{P}_{<\delta}$ such that if $p \in G$ then $\mathrm{CRT}(j_E) = \gamma$. □

Theorem 4.11. *Suppose δ is a Woodin cardinal and that G is V-generic for $\mathbb{Q}_{<\delta}$. Then in $V[G]$ there is a V-extender E of length δ such that if*

$$j_E : V \to M_E \cong \mathrm{Ult}_0(V, E)$$

is the associated ultrapower embedding then the following hold:

(1) $\mathrm{CRT}(j_E) = \omega_1^V$ *and* $j_E(\omega_1^V) = \delta$.

(2) $M_E^\omega \subset N$ *in* $V[G]$. □

Definition 4.12 (Tower Sealing). Suppose that there is a proper class of Woodin cardinals and that δ is a Woodin cardinal. Then **Tower Sealing** holds at δ if whenever G is V-generic for either the $\mathbb{P}_{<\delta}$-stationary tower at δ or the $\mathbb{Q}_{<\delta}$-stationary tower at δ, then

$$j(\Gamma^\infty) = \Gamma^\infty_G$$

where

$$j : V \to M \subset V[G]$$

is the generic elementary embedding given by G. □

Lemma 4.13. *Suppose that δ is a Woodin cardinal which is a limit of Woodin cardinals, and that Tower Sealing holds at δ. Then $\Gamma^\infty = \mathcal{P}(\mathbb{R}) \cap L(\Gamma^\infty, \mathbb{R})$.*

The following theorem shows that after collapsing $V_{\delta+1}$ where δ is an extendible cardinal, one obtains a much stronger version of the Sealing Theorem, Theorem 4.3.

Theorem 4.14 (Strong Sealing Theorem). *Suppose that δ is an extendible cardinal and that $\lambda > \delta$. Then there is a proper class of κ such that:*

(1) *κ is a measurable Woodin cardinal and $\kappa > \lambda$.*

(2) *Suppose G is V-generic for some $\mathbb{P} \in V_\lambda$ and $V_{\delta+1}$ is countable in $V[G]$. Then Tower Sealing holds at κ in $V[G]$.*

The sealing theorems motivate the following sealing hypotheses, formulated using the notion of grounds, due to Hamkins. A inner model N is a ground of V if V is a generic extension of N.

By the Definability Theorem, Theorem 6.15 on page 46, grounds are uniformly Σ_2-definable classes from parameters. Thus these sealing hypotheses are first order.

Definition 4.15 (Sealing Hypothesis). There exists a ground N of V and there exists δ such that

(1) δ is an supercompact cardinal in N,

(2) $N \cap V_{\delta+1}$ is countable in V. □

Definition 4.16 (Strong Sealing Hypothesis). There exists a ground N of V and there exists δ such that

(1) δ is an extendible cardinal in N,

(2) $N \cap V_{\delta+1}$ is countable in V. □

5 The Ultimate-L Conjecture

The key question for the axiom V = Ultimate-L is whether there is a generalization Scott's Theorem for this axiom. More precisely whether some large cardinal hypothesis implies the axiom V = Ultimate-L does not hold.

It turns out that there is an explicit conjecture, this is the Ultimate-L Conjecture, which implies that there is no generalization of Scott's Theorem for the axiom V = Ultimate-L. The statment of this conjecture requires the following definition.

Definition 5.1. Suppose $\delta > \omega$ is a supercompact cardinal and that N is an inner model of ZFC such that $\text{Ord} \subset N$. Then N is a *weak extender* model of δ is supercompact if for all $\lambda > \delta$ there exists a δ-complete, normal, fine, ultrafiter U on $\mathcal{P}_\delta(\lambda)$ such that:

(1) $\mathcal{P}_\delta(\lambda) \cap N \in U$,

(2) $U \cap N \in N$. □

We will see in Section 6 that if N is a weak extender model of δ is supercompact then necessarily N is *very close* to V. This motivates the Ultimate-L Conjecture which if true then by virtue of the necessarily closeness of weak extender models of δ is supercompact to V, implies (as claimed above) that there can be no Scott Theorem for the axiom, $V = \text{Ultimate-}L$.

Definition 5.2 (Ultimate-L Conjecture). Suppose that there is an extendible cardinal. Then (provably) there exists δ and there exists a weak extender model N of δ is supercompact such that the following hold.

(1) $N \vDash$ "$V = \text{Ultimate-}L$".

(2) N has the δ-genericity property. □

An inner model N has the δ-genericity property if for every bounded set $\sigma \subset \delta$, σ is N-generic for some partial order $\mathbb{P} \in N$ with $|\mathbb{P}|^N < \delta$.

The δ-genericity property and the related δ-approximation and δ-cover properties of Hamkins, are the subject of Section 6, as are weak extender models of δ is supercompact.

The following theorem shows that the Ultimate-L Conjecture implies the Weak HOD Conjecture.

Theorem 5.3 (Ultimate-L Conjecture). *Suppose that δ is an extendible cardinal. Then the* HOD *Hypothesis holds.* □

6 δ-cover, δ-approximation, and δ-genericity

Hamkins [3] , motivated by the relationship between V and $V[G]$, abstracted out two general properties that one can naturally impose on an inner model $N \subseteq V$ such that $\mathrm{Ord} \subset N$ and $N \vDash \mathrm{ZFC}$. Coincidentally, this turns out to be closely related to (and more general than) the notion that N be a weak extender model of δ is supercompact for some δ, which while implicit in [7], first appears in [9], but primarily as an expository device.

Suitable extender models, the version of weak extender models defined and analyzed in [7], add a key additional condition. But this additional condition can also be naturally simplified to a more abstract condition.

Putting everything together, one obtains what seems to be a very useful criterion for the closeness of an inner model N to V, for which one can both obtain all the results Hamkins proved and all those results proved for suitable extender models, in a single unified approach.

We begin with Hamkins' conditions.

Definition 6.1 (Hamkins, [3]). Suppose that $\delta > \omega$ is a regular cardinal and that N is a transitive inner model of ZFC containing the ordinals. Then:

(1) N has the *δ-cover property* if for all $\sigma \subset N$ with $|\sigma| < \delta$, there exists $\tau \in N$ such that $\sigma \subset \tau$ and such that $|\tau| < \delta$.

(2) N has the *δ-approximation property* if for all $X \subset N$ the following are equivalent.

a) $X \in N$.

b) $X \cap \tau \in N$ for all $\tau \in N$ with $|\tau| < \delta$. □

Remark 6.2. Defining when N has the ω-cover property and the ω-approximation property in the natural way and assuming N is a transitive inner model of ZFC such that $\mathrm{Ord} \subset N$:

(1) N has the ω-cover property.

(2) N has the ω-approximation property if and only if $N = V$. □

Suppose that for some regular uncountable cardinal δ, $V = N[G]$ where $G \subset \mathbb{P}$ is N-generic and $\mathbb{P}$ is δ-cc in V. Then N has the δ-cover property and the δ-approximation property in V. The following remarkable theorem of Bukovský [1] (see also [2]) reinforces the motivation for the definitions of δ-cover and δ-approximation, from the perspective of the relationship between V and N. The theorem requires the following definition.

Definition 6.3. Suppose that $\delta > \omega$ is a regular cardinal and that N is a transitive inner model of ZFC containing the ordinals. Then N has the *uniform δ-cover property* if for all ordinals θ and all functions

$$F : \theta \to \mathcal{P}_\delta(N)$$

there exists a function

$$H : \theta \to \mathcal{P}_\delta(N)$$

such that

(1) $H \in N$,

(2) $F(\alpha) \subseteq H(\alpha)$ for all $\alpha < \theta$. □

Remark 6.4. Suppose $V[G]$ is a δ-cc extension of V. Then V fails to have the δ-approximation property in $V[G]$ if and only if there is a Suslin tree T in V at δ such that $V[G]$ adds a cofinal branch of T. Thus V has the κ-approximation property in $V[G]$ for all regular cardinals $\kappa > \delta$. □

Theorem 6.5 (Bukovský). *Suppose that $\delta > \omega$ is a regular cardinal and that N is a transitive inner model of* ZFC *containing the ordinals. Then the following are equivalent.*

(1) *N has the uniform δ-cover property.*

(2) *There exist a partial order $\mathbb{P} \in N$ and an N-generic filter $G \subseteq \mathbb{P}$ such that $V = N[G]$ and such that $\mathbb{P}$ is δ-cc in N.*

Proof. We first prove the following.

(1.1) Suppose $A \subset \mathrm{Ord}$. Then $N[A]$ is a δ-cc generic extension of N.

Fix A and fix γ such that $A \subset \gamma$. Let $\mathcal{L}_N$ be infinitary language $\mathcal{L}_{\gamma^+,\omega}$ as defined in N with constants c_α for the ordinals $\alpha < \gamma$, a unary predicate P, and a binary predicate E. Thus in V, the set of sentences of $\mathcal{L}_N$ such that

$$(\gamma, A, <) \vDash \phi$$

defines a complete $\mathcal{L}_N$-theory (interpreting each c_α by α, P by A, and E by $<$).

Let $\mathcal{S}_N$ be the set of all sentences of $\mathcal{L}_N$. By absoluteness for each set $T \subseteq \mathcal{S}_N$ with $T \in N$, T is consistent in V if and only if T is consistent in N. Here we can simply define T to be consistent if T has a model after collapsing $T \times \gamma$ to be countable. The absoluteness then is just the absoluteness of $\mathbf{\Sigma}^1_1$.

Working in N, let I be the set of all consistent theories $T \in N$ of $\mathcal{L}_N$ such that for all $\phi \in \mathcal{S}_N$ if $T \cup \{\phi\}$ is inconsistent then $(\neg\phi) \in T$.

For each $T \in I$, $(\mathcal{S}_N, T)$ defines in N, a γ^+-complete Boolean algebra $\mathbb{B}_T$. This is just the quotient of the (infinitary) Lindenbaum algebra associated to the infinitary language $\mathcal{L}_N$, using the filter defined by T.

To prove (1.1) it suffices to prove that for some $T \in I$:

(2.1) $\mathbb{B}_T$ is δ-cc in N.

(2.2) $(\gamma, A, <) \models \phi$ for each $\phi \in T$.

To see this suppose $T \in I$ satisfies (2.1) and (2.2). Let

$$G = \{\phi \in \mathcal{S}_N \mid (\gamma, A, <) \models \phi\}$$

Then by (2.1), $\mathbb{B}_T$ is a complete Boolean algebra in N (since $\mathbb{B}_T$ is a γ^+-complete Boolean algebra in N) and by (2.2), G defines an ultrafilter on $\mathbb{B}_T$ which is necessarily N-generic. Finally $N[A] = N[G]$.

Assume toward a contradiction no such $T \in I$ exists. Let

$$e : I \to N$$

be a partial function such that

(3.1) $\mathrm{dom}(e)$ is the set of $T \in I$ such that $\mathbb{B}_T$ is not δ-cc in N.

(3.2) $e(T) \subset \mathcal{S}_N$ and defines a maximal δ-antichain in $\mathbb{B}_T$, more precisely:

a) $|e(T)|^N = \delta$, $e(T) \cap T = \emptyset$, and $\bigvee \{\phi \mid \phi \in e(T)\} \in T$,

b) for all $\phi_1, \phi_2 \in e(T)$, if $\phi_1 \neq \phi_2$ then $(\neg(\phi_1 \wedge \phi_2)) \in T$.

Choose (in V) a function

$$F : I \to N$$

such that for all $T \in I$, either

(4.1) $F(T) \in T$ and $(\gamma, A, <) \models (\neg F(T))$, or

(4.2) $(\gamma, A, <) \models T$, $T \in \mathrm{dom}(e)$, $F(T) \in e(T)$, and $(\gamma, A, <) \models F(T)$.

Since N has the uniform δ-cover property, there exists a function

$$H : I \to \mathcal{P}_\delta(\mathcal{S}_N)$$

such that $H \in N$ and such that $F(T) \in H(T)$ for all $T \in I$.

For each $T \in I \backslash \mathrm{dom}(e)$, let ϕ_T be the sentence

$$\phi_T = \bigvee \left\{ (\neg\phi) \mid \phi \in H(T) \cap T \right\}$$

and for each $T \in \mathrm{dom}(e)$, let ϕ_T be the sentence

$$\phi_T = \left(\bigvee \left\{ (\neg\phi) \mid \phi \in H(T) \cap T \right\} \right) \vee \left(\bigvee (H(T) \cap e(T)) \right).$$

The key point is that:

(5.1) For each $T \in I$, $(\gamma, A, <) \models \phi_T$.

Let $T^* = \{\phi_T \mid T \in I\}$. Thus $T^* \in N$ and by (5.1), T^* is a consistent $\mathcal{L}_N$-theory. Let T_0 be the set of all $\phi \in \mathcal{S}_N$ such that $T^* \cup \{(\neg\phi)\}$ is not consistent. Thus by (5.1)

(6.1) $(\gamma, A, <) \models T_0$,

and so $T_0 \in I$. But then since $T^* \subseteq T_0$,

$$\phi_{T_0} \in T_0.$$

By (6.1) and the definition of F, $T_0 \in \text{dom}(e)$ and $F(T_0)$ is defined according to (4.2). But then by the definition of ϕ_{T_0} and since $(\gamma, A, <) \models T_0$, necessarily

$$\bigvee (H(T_0) \cap e(T_0)) \in T_0$$

which is a contradiction, since $|H(T_0)|^N < \delta$.

Thus as claimed, there exists $T \in I$ such that (2.1) and (2.2) both hold. This proves (1.1).

Choose $A \subset \text{Ord}$ such that $\mathcal{P}(\delta) \subset L[A]$. We prove

(7.1) $V = N[A]$.

Assume toward a contradiction that $V \neq N[A]$. Choose $A^* \subset \text{Ord}$ such that $A \in L[A^*]$ and $A^* \notin N[A]$. By (1.1), $N[A^*]$ is a δ-cc extension of N. But $A \in L[A^*]$ and so by factoring, $N[A^*]$ is a δ-cc extension of $N[A]$. Thus by the choice of A,

$$\mathcal{P}(\delta) \cap N[A] = \mathcal{P}(\delta) \cap N[A^*]$$

and this implies $N[A] = N[A^*]$, which is a contradiction.

The key point here is the general fact that a nontrivial δ-cc forcing extension must add a new subset of δ. To see this let $\mathbb{P}$ be δ-cc and suppose $G \subset \mathbb{P}$ is V-generic. Let λ be least such that there exists $Z \subset \lambda$ such that $Z \in V[G]$, $Z \notin V$, and $Z \cap \alpha \in V$ for all $\alpha < \lambda$. We must show $\lambda \leq \delta$. Assume toward a contradiction that $\lambda > \delta$.

Fix a term τ for Z and we can (by replacing $\mathbb{P}$ with $\mathbb{P}|p$ for some $p \in G$ if necessary) reduce to the case that (τ, λ) does not depend on G.

Fix $X \prec V_\gamma$ for large enough γ with $(\delta, \tau, \lambda, \mathbb{P}) \in X$, $\delta \subset X$, and $|X|^V = \delta$. Clearly we can reduce to the case that $G \cap X \in V$ for otherwise $\lambda \leq \delta$.

Note that $G \cap X$ is X-generic for $\mathbb{P}$. If $\text{cof}(\lambda)^V \leq \delta$ then $Z \in V[X \cap G]$ which contradicts that $Z \notin V$. Therefore $\text{cof}(\lambda)^V > \delta$ and so $\mathbb{P}$ is not δ-cc in $V[G^*]$ for *all* V-generic filters $G^* \subset \mathbb{P}$ since (τ, λ) does not depend on G. (One could also just note that $X[G] \prec V_\gamma[G]$ and just use that $\mathbb{P}$ is not δ-cc in $V[G]$). But then $\mathbb{P} \cap X$ is not δ-cc in $X[G]$ which contradicts that $X \cap G \in V$ since $\mathbb{P}$ is δ-cc.

The theorem is an immediate corollary of (1.1) and (7.1). □

For Lemma 6.7 it is convenient to make the following definition. Note that N has the weak δ-cover property if and only if N has the δ^+-cover property.

Definition 6.6. Suppose that $\delta \geq \omega$ is a cardinal and that N is a transitive inner model of ZFC containing the ordinals. Then N has the *weak δ-cover property* if for all $\sigma \subset N$ with $|\sigma| = \delta$, there exists $\tau \in N$ such that $\sigma \subseteq \tau$ and such that $|\tau| = \delta$.

The following lemma (proved several times by Hamkins and Reitz) is useful.

Lemma 6.7 (Hamkins–Reitz). *Suppose that $\delta > \omega$ is a regular cardinal and that N is a transitive inner model of* ZFC *containing the ordinals. Then the following hold.*

(1) *Suppose that N has the δ-approximation property. Then N has the κ-approximation property for all regular cardinals $\kappa > \delta$.*

(2) *Suppose that N has the δ-approximation property and the δ-cover property. Then N has the weak κ-cover property for all cardinals $\kappa \geq \delta$.*

(3) *Suppose that N has the δ-approximation property and the δ-cover property. Then N has the κ-cover property for all regular cardinals $\kappa \geq \delta$.*

Proof. Since N has the δ-approximation property, trivially N has the κ-approximation property for all regular cardinals $\kappa > \delta$. We prove (2). Assume toward a contradiction that (2) fails and let κ_0 be the least cardinal for which (2) fails. Let $\sigma \subset N$ witness that (2) fails for κ_0.

Let $\kappa = \mathrm{cof}(\kappa_0)$. Thus there is an increasing sequence

$$\langle \sigma_\alpha : \alpha < \kappa \rangle$$

of increasing subsets of σ such that

(1.1) $\sigma = \cup\{\sigma_\alpha \mid \alpha < \kappa\}$,

(1.2) $|\sigma_{\alpha|} < \kappa_0$ for all $\alpha < \kappa$.

By induction on α, the choice of κ_0, and since $\kappa = \mathrm{cof}(\kappa_0)$, there is an increasing sequence

$$\langle \tau_\alpha : \alpha < \kappa \rangle$$

of subsets of N such that for all $\alpha < \kappa$:

(2.1) $|\tau_\alpha| < \kappa_0$ and $\sigma_\alpha \subseteq \tau_\alpha$,

(2.2) $\tau_\alpha \in N$.

Let $\tau = \cup\{\tau_\alpha : \alpha < \kappa\}$. Thus $|\tau| = \kappa_0$.

There are two cases.

Case 1: $\delta \leq \kappa$.

Thus $\tau \cap X \in N$ for all $X \in N$ such that $|X| < \delta$. This implies $\tau \in N$ by the δ-approximation property.

Case 2: $\kappa < \delta$.

Then by the δ-cover property, there exists $X \in N$ such that $|X| < \delta$ and such that

$$\{\tau_\alpha \mid \alpha < \kappa\} \subseteq X.$$

Let

$$\tau = \cup\{Z \in X \mid |Z| < \kappa_0\} = \cup\left\{Z \in X \mid |Z|^N < \kappa_0\right\}.$$

Then $\tau \in N$, $\sigma \subseteq \tau$ and $|\tau| = \kappa_0$.

Thus in each case the required cover of σ exists and this contradicts the choice of (κ_0, σ). This proves (2).

Finally (2) trivially implies (3). □

As an immediate corollary of Lemma 6.7, we obtain the following theorems on cardinals and their successors.

Theorem 6.8. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose $\gamma > \delta$ is a singular cardinal. Then γ is a singular cardinal in N and*

$$\gamma^+ = (\gamma^+)^N.$$

Proof. Let $\kappa = \max(\mathrm{cof}(\gamma), \delta) < \gamma$. By Lemma 6.7, N has the κ^+-cover property. Thus there is a cofinal set $\sigma \subset \gamma$ such that $\sigma \in N$ and such that $|\sigma| = \kappa$. Therefore

$$(\mathrm{cof}(\gamma))^N \leq |\sigma|^N < \gamma.$$

We finish by proving $\gamma^+ = (\gamma^+)^N$. Let

$$\lambda = (\gamma^+)^N$$

and assume toward a contradiction that $\lambda < \delta^+$. Then since γ is singular,

$$\mathrm{cof}(\lambda) < \gamma.$$

But then arguing exactly as above but with $\kappa = \max(\mathrm{cof}(\lambda), \delta)$,

$$(\mathrm{cof}(\lambda))^N < \gamma$$

and this is a contradiction. □

The proof of Theorem 6.8 easily adapts to prove the following version of that theorem for regular cardinals.

Theorem 6.9. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose $\gamma \geq \delta$ is a regular cardinal and that*

$$\gamma^+ \neq (\gamma^+)^N.$$

Then $\mathrm{cof}\left((\gamma^+)^N\right) = \gamma$. □

Remark 6.10. Both Theorem 6.8 and Theorem 6.9 are corollaries of the following theorem which also easily follows from Lemma 6.7(2). This is the theorem for weak extender models proved in [9] but by a very different argument. □

Theorem 6.11. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose $\lambda \geq \delta$ and that λ is a regular cardinal in N. Then*

$$(\mathrm{cof}(\lambda))^V = |\lambda|^V. \qquad \square$$

We prove the following lemma which will have a number of corollaries.

Lemma 6.12. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose* $\mathrm{Ord} \subset M \subset V$ *is an inner model of* ZFC *such that*

$$M^\delta \subset M,$$

and that $N_0 \subset M$ has the δ-cover property and the δ-approximation property in M. Suppose

$$H(\delta^+) \cap N_0 \subseteq H(\delta^+) \cap N.$$

Then $N_0 \subseteq N$.

Proof. Note that N_0 has the δ-cover property in V (since $M^{<\delta} \subset M$ and since N_0 has the δ-cover property in M).

Clearly, it suffices to prove:

(1.1) For all $\alpha \in \mathrm{Ord}$, $\mathcal{P}(\alpha) \cap N_0 \subset N$.

Fix $\alpha \in \mathrm{Ord}$ and fix $X \in \mathcal{P}(\alpha) \cap N_0$ such that $X \cap \beta \in N$ for all $\beta < \alpha$. It suffices to prove $X \in N$. Since N has the δ-approximation property we can reduce to the case that $\mathrm{cof}(\alpha) < \delta$.

Fix a strong limit cardinal $\lambda > \delta$ such that $\alpha < \lambda$. Let

$$\langle (\tau_\eta, N \cap \tau_\eta, N_0 \cap \tau_\eta) : \eta < \delta \rangle$$

be an increasing sequence of elementary substructures of $(H(\lambda), N \cap H(\lambda), N_0 \cap H(\lambda))$ together with

$$\langle (\tau_\eta^{N_0}, \tau_\eta^{N}) : \eta < \delta \rangle$$

such that for all $\eta < \delta$:

(2.1) $X \in \tau_0$ and $\tau_0 \cap \alpha$ is cofinal in α.

(2.2) $|\tau_\eta| < \delta$

(2.3) $\tau_\eta^{N_0} \prec H(\lambda) \cap N_0$ and $\tau_\eta^{N_0} \in N_0$.

(2.4) $\tau_\eta^N \prec H(\lambda) \cap N$ and $\tau_\eta^N \in N$.

(2.5) $\tau_\eta \cap N_0 \subset \tau_\eta^{N_0} \subset \tau_{\eta+1}$

(2.6) $\tau_\eta \cap N \subset \tau_\eta^N \subset \tau_{\eta+1}$

The sequence exists since N and N_0 have the δ-cover property. Let

$$\tau = \cup \left\{ \tau_\eta \mid \eta < \delta \right\}.$$

Since N has the δ-approximation property, $\tau \cap N \in N$. Since

$$(H(\lambda) \cap M)^\delta \subset M,$$

it follows that

$$\langle \tau_\eta^{N_0} : \eta < \delta \rangle \in M$$

and so $\tau \cap N_0 \in N_0$ since

$$\tau \cap N_0 = \cup \left\{ \tau_\eta^{N_0} \mid \eta < \delta \right\}$$

and since N_0 as the δ-approximation property in M.

Let M_τ be the transitive collapse of τ and let $(\lambda_\tau, \alpha_\tau)$ be the image of (λ, α) under the transitive collapse of τ. For each $\beta \in \tau \cap \alpha$, let X_β^τ be the image of $X \cap \beta$ under the transitive collapse of τ, and let

$$X_\tau = \cup \left\{ X_\beta^\tau \mid \beta \in \tau \cap \alpha \right\}.$$

Thus $X_\tau \subseteq \alpha_\tau$, $\alpha_\tau < \delta^+$, and $X_\tau \in N_0$. Finally

$$N_0 \cap H(\delta^+) \subseteq N \cap H(\delta^+)$$

and therefore $X_\tau \in N$.

For each $\xi < \alpha_\tau$,

$$X \cap \beta = \pi(X_\tau \cap \xi) = \pi(X_\beta^\tau)$$

where π inverts the transitive collapse of $\tau \cap N$ and where $\beta = \pi(\xi)$. This implies $X \in N$ since $\tau \cap \alpha$ is cofinal in α.

This proves (1.1). □

An immediate corollary of Lemma 6.12 is the uniqueness theorem of [3] for inner models with the δ-cover property and the δ-approximation property. One simply appeals to Lemma 6.12 in the case where $M = V$.

Theorem 6.13 (Hamkins Uniqueness Theorem). *Suppose $\delta > \omega$ is a regular cardinal and that N_0, N_1 are transitive inner models with the δ-cover property and with the δ-approximation property. Suppose*

$$N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+).$$

Then $N_0 = N_1$. □

Remark 6.14. A natural question is whether the Hamkins Uniqueness Theorem can be strengthened by only requiring that

$$N_0 \cap H(\delta) = N_1 \cap H(\delta).$$

Note that if either $\delta^+ = (\delta^+)^{N_0}$ or $\delta^+ = (\delta^+)^{N_1}$, then this strengthened version of the Hamkins Uniqueness Theorem must hold, since necessarily

$$N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+)$$

by the δ-approximation property.

Similarly, if

$$\mathcal{P}_\delta(\delta^+) \cap N_0 = \mathcal{P}_\delta(\delta^+) \cap N_1$$

then again $N_0 \cap H(\delta^+) = N_1 \cap H(\delta^+)$ by the δ-approximation property. □

The proof of Lemma 6.12 immediately yields the following definability theorem (also proved by Laver [5] and in [8] in the case where V is a generic extension of N) for inner models N with the δ-approximation property and the δ-cover property.

Theorem 6.15 (Definability Theorem). *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Then $N \cap H(\gamma)$ is uniformly definable in $H(\gamma)$ from $N \cap H(\delta^+)$, for all strong limit cardinals $\gamma > \delta$.* □

Remark 6.16. An immediate corollary of Theorem 6.15 is that if N has the δ-approximation property and the δ-cover property then N is Σ_2-definable in V from $N \cap H(\delta^+)$. Thus the theory of such inner models is part of the first order theory of V. □

The following universality theorem is a slight refinement of the universality theorems of [3]. The corollaries, such as Theorem 6.36 and Theorem 6.38, are immediate from the version of the universality theorem proved in [3]. We first prove a weak version which only requires the δ-approximation property.

Theorem 6.17 (Weak Universality Theorem: Hamkins [3]). *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-approximation property. Suppose that $\lambda \geq \delta$ and that U is a δ-complete ultrafilter on λ. Then $U \cap N \in N$.*

Proof. Fix $\sigma \in N$ with $|\sigma| < \delta$. Since N has the δ-approximation property, it suffices to prove that

$$U \cap \sigma \in N.$$

By enlarging σ is necessary we can suppose that

$$\lambda \backslash A \in \sigma,$$

for each $A \in \mathcal{P}(\lambda) \cap \sigma$.

But U is δ-complete and so there exists $\xi < \lambda$ such that

$$U \cap \sigma = \{A \in \sigma \cap \mathcal{P}(\lambda) \mid \xi \in A\}.$$

Therefore $U \cap \sigma \in N$. □

Theorem 6.18 (Universality Theorem). *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose that E is an N-extender of length η with critical point $\kappa_E \geq \delta$. Let*

$$\pi_E : N \to M_E \cong \mathrm{Ult}_0(N, E)$$

be the ultrapower embedding. Then the following are equivalent.

(1) *For each $A \subset \mathcal{P}(\eta^{<\omega}) \cap N$, $\pi_E(A) \cap \eta^{<\omega} \in N$.*

(2) *$E \in N$.*

Proof. Clearly (2) implies (1) and so it suffices to assume (1) and prove (2).

Let F be the function

$$F : \mathcal{P}(\eta^{<\omega}) \cap N \to V$$

where $F(A) = \pi_E(A) \cap \eta^{<\omega}$. To show that $E \in N$, it suffices to show that $F \in N$.

Clearly $\mathrm{dom}(F) \in N$ and by (1), $F \subset N$. Thus since N has the δ-approximation property, to show that $F \in N$, it suffices to prove:

(1.1) $F|\sigma \in N$ for all $\sigma \in N$ such that $\sigma \subset \mathrm{dom}(F)$ and $|\sigma| < \delta$.

Fix $\sigma \in N$ such that $\sigma \subset \mathrm{dom}(F)$ and $|\sigma| < \delta$. Fix a bijection

$$\rho : |\sigma|^N \to \sigma$$

such that $\rho \in N$. Choose $A \in \mathcal{P}(\eta^{<\omega}) \cap N$ such that for all $\xi < |\sigma|^N$ and for all $s \in \eta^{<\omega}$, $\xi^\frown s \in A$ if and only if $s \in \rho(\xi)$. By (1)

$$\pi_E(A) \cap \eta^{<\omega} \in N.$$

By the elementarity of π_E, and since $\kappa_E \geq \delta$, for each $\xi < |\sigma|^N$,

$$\pi_E(\rho(\xi)) = \left\{s \in \pi_E(\eta)^{<\omega} \mid \xi^\frown s \in \pi_E(A)\right\}.$$

Thus for each $\xi < |\sigma|^N$,

$$F(\rho(\xi)) = \left\{s \in \eta^{<\omega} \mid \xi^\frown s \in F(A)\right\},$$

and so $F|\sigma \in N$ since $F(A) \in N$. This proves (1.1).

Therefore $F \in N$ and so $E \in N$. □

Suppose N has the δ-cover property and the δ-approximation property. Then by Theorem 6.18, subject only to a fairly weak constraint, N contains all N-extenders E with associated critical point $\kappa_E \geq \delta$.

The following theorem greatly amplifies the utility of this by showing that for many extenders E of V, this constraint (which is condition (1) of Theorem 6.18) is necessarily satisfied for the induced N-extender. Thus

$$E \cap N \in N$$

for all such extenders $E \in V$.

Theorem 6.19. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose $\gamma > \delta$ is a strong limit cardinal with* $\mathrm{cof}(\gamma) \geq \delta$*, and that*

$$j : V \to M$$

is an elementary embedding such that $\mathrm{CRT}(j) > \delta$ *and such that $H(\gamma) \subset M$. Let E be the N-extender of length γ given by E and let*

$$j_E : N \to N_E \cong \mathrm{Ult}_0(N, E)$$

be the ultrapower embedding. Then:

(1) $N_E \cap H(\gamma) = N \cap H(\gamma)$.

(2) $E \in N$.

Proof. Let $A = N \cap H(\delta^+)$. Since $\mathrm{CRT}(j) > \delta$, $j(A) = A$. Therefore by the Definability Theorem, Theorem 6.15,

(1.1) $j(N) \cap H(\gamma) = N \cap H(\gamma)$.

This implies (1) since $j(N) \cap H(\gamma) = j_E(N) \cap H(\gamma)$.

Suppose that $X \in \mathcal{P}(\gamma^{<\omega}) \cap N$. By (1.1),

$$j_E(X) \cap \xi^{<\omega} \in N$$

for all $\xi < \gamma$. Thus since $\mathrm{cof}(\gamma) \geq \delta$,

$$j_E(X) \cap \gamma^{<\omega} \cap \sigma^{<\omega} \in N$$

for all $\sigma \in N$ with $|\sigma| < \delta$. Therefore since N has the δ-approximation property,

$$j_E(X) \cap \gamma^{<\omega} \in N.$$

Thus by the Universality Theorem, Theorem 6.18, necessarily

$$E \in N,$$

and this proves (2). □

Remark 6.20. (1) If one drops the requirement that $\operatorname{cof}(\gamma) \geq \delta$ then the conclusion of Theorem 6.19 can fail.

(2) It is not clear if one can weaken the requirement on the critical point of j to just $\text{CRT}(j) \geq \delta$, as is the case in the Universality Theorem. This seems unlikely, but the case where one also assumes both

$$H(\gamma) \prec_{\Sigma_2} V$$

and

$$H(\gamma) \prec_{\Sigma_2} M$$

is an interesting one. □

One can strengthen the conclusion of Theorem 6.19 considerably in the case where $\kappa_E > \delta$ and $\operatorname{cof}(\gamma) > \delta$, and this stronger version is a corollary of the main theorem of [3].

Remark 6.21. We will prove a stronger version of Theorem 6.22. The stronger theorem is Theorem 6.66 and this theorem shows that if the given embedding

$$j : V \to M$$

is given by an extender then the induced embedding on N is also an (internal) N-ultrapower embedding by an N-extender. □

Theorem 6.22 (Hamkins, [3]). *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose that*

$$j : V \to M$$

is an elementary embedding such that $\text{CRT}(j) > \delta$ *and such that $M^\delta \subset M$. Then*

(1) $j(N) \subset N$.

(2) *For all $\alpha \in \text{Ord}$, $j|(N \cap V_\alpha) \in N$.*

Proof. It suffices to prove:

(1.1) Suppose $\gamma > \delta$ is a strong limit cardinal such that $\operatorname{cof}(\gamma) > \delta$. Let E be the extender of length γ given by j. Then $E \cap N \in N$.

Fix a strong limit cardinal $\gamma > \delta$ such that $\operatorname{cof}(\gamma) > \delta$.
Let

$$j_E : V \to M_E \cong \text{Ult}_0(V, E)$$

be the ultrapower embedding.
By the Universality Theorem, Theorem 6.18, it suffices to prove:

(2.1) $j_E(X) \cap \gamma^{<\gamma} \in N$ for each $X \in \mathcal{P}(\gamma^{<\omega}) \cap N$.

For this it suffices to show

(3.1) $j_E(N) \subset N$.

Note that

$$M_E = \{j_E(f)(s) \mid f : [\gamma]^{<\omega} \to V \text{ and } s \in [\gamma]^{<\omega}\}$$

and so since $\mathrm{cof}(\gamma) > \delta$ and since $\mathrm{CRT}(j) > \delta$, necessarily

$$M_E^\delta \subset M_E.$$

Further by the elementarity of j_E and since $\mathrm{CRT}(j_E) > \delta$:

(4.1) $j_E(N)$ has the δ-cover and δ-approximation properties in M_E.

(4.2) $H(\delta^+) \cap j_E(N) \subseteq H(\delta^+) \cap N$ (in fact $H(\delta^+) \cap j_E(N) = H(\delta^+) \cap N$).

Therefore (3.1) is an immediate corollary of Lemma 6.12. □

Theorem 6.19 and Theorem 6.22 shows that most large cardinal notions are downward absolute from V to N above δ, where N is an inner model with the δ-cover property and the δ-approximation property. We give several examples below, all due to Hamkins [3], after first defining the notion of a weak extender model of δ is supercompact.

Whether one uses Theorem 6.19 or Theorem 6.22, for example in the case of supercompact cardinals, is more a matter of personal preference, though Theorem 6.22 is ultimately far more useful.

Weak extender models of δ is supercompact, implicitly defined in [7], are formally defined in [9].

Definition 6.23. Suppose $\delta > \omega$ is a supercompact cardinal and that N is an inner model of ZFC such that $\mathrm{Ord} \subset N$. Then N is a *weak extender* model of δ is supercompact if for all $\lambda > \delta$ there exists a δ-complete, normal, fine, ultrafiter U on $\mathcal{P}_\delta(\lambda)$ such that:

(1) $\mathcal{P}_\delta(\lambda) \cap N \in U$,

(2) $U \cap N \in N$. □

The following lemma of Magidor gives a useful alternative formulation of supercompactness.

Lemma 6.24 (Magidor). *Suppose that δ is strongly inaccessible. Then the following are equivalent.*

(1) *δ is supercompact.*

(2) *For all $\lambda > \delta$ there exist $\bar{\delta} < \bar{\lambda} < \delta$ and an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that $\mathrm{CRT}(\pi) = \bar{\delta}$ *and such that* $\pi(\bar{\delta}) = \delta$. □

The following theorem from [7] generalizes Magidor's lemma to weak extender models of δ is supercompact.

Theorem 6.25. *Suppose that N is a weak extender model of δ is supercompact. Then for all $\lambda > \delta$, for all $A \in V_\lambda$, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{A} \in V_{\bar{\lambda}}$, and there exists an elementary embedding*

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that:

(1) $\mathrm{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, *and* $\pi(\bar{A}) = A$.

(2) $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.

(3) $\pi|(N \cap V_{\bar{\lambda}}) \in N$. □

Theorem 6.26. *Suppose that N is a weak extender model of δ is supercompact. Then N has the δ-approximation property and the δ-cover property.*

Proof. The δ-cover property is immediate from Definition 6.23(1), together with the δ-completeness and fineness of U (referring to Definition 6.23).

Fix a set $X \subset N$ such that

(1.1) $X \cap \sigma \in N$ for all $\sigma \in N$ with $|\sigma| < \delta$.

We must prove that $X \in N$.

Fix λ large enough such that $X \in V_\lambda$. By Theorem 6.25, there exist $\bar{\delta} < \bar{\lambda} < \delta$, $\bar{X} \in V_{\bar{\lambda}}$, and there exists an elementary embedding

$$\pi : V_{\bar{\lambda}+1} \to V_{\lambda+1}$$

such that:

(2.1) $\mathrm{CRT}(\pi) = \bar{\delta}$, $\pi(\bar{\delta}) = \delta$, and $\pi(\bar{X}) = X$.

(2.2) $\pi(N \cap V_{\bar{\lambda}}) = N \cap V_\lambda$.

(2.3) $\pi|(N \cap V_{\bar{\lambda}}) \in N$.

Let $\sigma = \pi[V_{\bar{\lambda}} \cap N]$. Thus $|\sigma| < \delta$ and by (2.3), $\sigma \in N$. Thus by (1.1), $X \cap \sigma \in N$. But

$$\bar{X} = \{a \in V_{\bar{\lambda}} \cap N \mid \pi(a) \in X \cap \sigma\}.$$

Therefore by (2.3), $\bar{X} \in N$ and so $\pi(\bar{X}) \in N$. This proves that $X \in N$. □

As an immediate corollary of Theorem 6.8 and Theorem 6.26 we obtain the following theorems on cardinals and their successors, relative to weak extender models.

Theorem 6.27. *Suppose that N is a weak extender model of δ is supercompact. Suppose $\gamma > \delta$ is a singular cardinal. Then γ is a singular cardinal in N and*

$$\gamma^+ = (\gamma^+)^N. \qquad \square$$

Theorem 6.28. *Suppose that N is a weak extender model of δ is supercompact. Suppose $\gamma \geq \delta$ is a regular cardinal and that*

$$\gamma^+ \neq (\gamma^+)^N.$$

Then $\mathrm{cof}\left((\gamma^+)^N\right) = \gamma$. $\square$

Further, as an immediate corollary of Theorem 6.18 and Theorem 6.26 we obtain the universality theorems for weak extender models.

Theorem 6.29 (Weak Universality Theorem for weak extender models). *Suppose that N is a weak extender model of δ is supercompact. Suppose that $\lambda \geq \delta$ and that U is a δ-complete ultrafilter on λ. Then $U \cap N \in N$.* $\square$

Theorem 6.30 (Universality Theorem for weak extender models). *Suppose that N is a weak extender model of δ is supercompact. Suppose that E is an N-extender of length η with critical point $\kappa_E \geq \delta$. Let*

$$\pi_E : N \to M_E \cong \mathrm{Ult}_0(N, E)$$

be the ultrapower embedding. Then the following are equivalent.

(1) *For each $A \in \mathcal{P}(\eta^{<\omega}) \cap N$, $\pi_E(A) \cap \eta^{<\omega} \in N$.*

(2) *$E \in N$.* $\square$

Theorem 6.31. *Suppose that N is a weak extender model of δ is supercompact. Suppose $\gamma > \delta$ is a strong limit cardinal with $\mathrm{cof}(\gamma) \geq \delta$, and that*

$$j : V \to M$$

is an elementary embedding such that $\mathrm{CRT}(j) > \delta$ and such that $H(\gamma) \subset M$. Let E be the extender of length γ given by E and let

$$j_E : N \to N_E \cong \mathrm{Ult}_0(N, E)$$

be the ultrapower embedding. Then:

(1) *$N_E \cap H(\gamma) = N \cap H(\gamma)$.*

(2) *$E \cap N \in N$.* $\square$

Remark 6.32. Theorem 6.27, Theorem 6.28, and Theorem 6.30 were proved in [7] by somewhat different arguments. □

The following corollary of the HOD Dichotomy Theorem, Theorem 3.29 on page 25 of [10], shows the utility of the HOD Hypothesis. We give the proof and then use this theorem to give a proof of the HOD Dichotomy Theorem.

Theorem 6.33. *Suppose that δ is an extendible cardinal. Then the following are equivalent.*

(1) *The* HOD *Hypothesis.*

(2) HOD *is a weak extender model of δ is supercompact.*

Proof. We sketch the proof. If suffices to show that (1) implies (2), since if (2) holds then by Theorem 6.27,

$$\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$$

for all singular cardinals $\gamma > \delta$.

Fix $\lambda > \delta$ such that $V_\lambda \prec_{\Sigma_3} V$. Thus the following hold.

(1.1) $\mathrm{HOD}^{V_\lambda} = \mathrm{HOD} \cap V_\lambda$.

(1.2) For all $\gamma < \lambda$, there exists a regular cardinal κ_γ and a partition

$$\langle S^\gamma_\alpha : \alpha < \gamma\rangle \in \mathrm{HOD}$$

of $\left\{\eta < \kappa_\gamma \mid \mathrm{cof}(\eta) = \omega\right\}$ into stationary sets, such that $\gamma < \kappa_\gamma < \lambda$.

Since δ is an extendible cardinal there exists and elementary embedding

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.

We prove that for all $\gamma < \lambda$,

(2.1) $\pi[\gamma] \in (\mathrm{HOD})^{V_{\pi(\lambda)}}$.

Let $S = \left\{\eta < \kappa_\gamma \mid \mathrm{cof}(\eta) = \omega\right\}$. Thus

$$\pi(S) = \left\{\eta < \pi(\kappa_\gamma \mid \mathrm{cof}(\eta) = \omega\right\}.$$

Let $\theta = \sup(\pi[\kappa_\gamma])$ and let

$$\langle T_\alpha : \alpha < \pi(\gamma)\rangle = \pi(\langle S^\gamma_\alpha : \alpha < \gamma\rangle).$$

Thus $\langle T_\alpha : \alpha < \pi(\gamma)\rangle$ is a partition of $\pi(S)$ into stationary sets and

$$\langle T_\alpha : \alpha < \pi(\gamma)\rangle \in \pi(\mathrm{HOD} \cap V_\lambda) = \pi(\mathrm{HOD}^{V_\lambda}) = \mathrm{HOD}^{V_{\pi(\lambda)}}.$$

The key point is that for all $\eta < \lambda$, if $\mathrm{cof}(\eta) = \omega$ then

$$\pi(\eta) = \sup(\pi[\eta]).$$

Thus for all $\alpha < \pi(\gamma)$, the following are equivalent.

(3.1) $T_\alpha \cap C \neq \emptyset$ for all closed cofinal sets $C \subset \theta$.

(3.2) $\alpha = \pi(\beta)$ for some $\beta < \gamma$.

But this implies $\pi[\gamma]$ is definable in $V_{\pi(\lambda)}$ from θ and

$$\langle T_\alpha : \alpha < \pi(\gamma) \rangle \in (\mathrm{HOD})^{V_\lambda}.$$

This proves (2.1).

Since $|V_\lambda| = \lambda$, for all $\epsilon < \lambda$, there exists $\gamma < \lambda$ and a surjection

$$F : \gamma \to \mathrm{HOD} \cap V_\epsilon$$

such that $F \in \mathrm{HOD} \cap V_\lambda$. Therefore

$$\pi(F)[\pi[\gamma]] = \pi[\mathrm{HOD} \cap V_\epsilon]$$

and $\pi(F) \in (\mathrm{HOD})^{V_\lambda}$. Thus:

(4.1) For all $X \in \mathrm{HOD} \cap V_\lambda$, $\pi[X] \in \mathrm{HOD}^{V_\lambda}$.

For each $\delta < \gamma < \lambda$, let U_γ be the δ-complete normal fine ultrafilter on $\mathcal{P}_\delta(\gamma)$ given by π:

$$U_\gamma = \{X \subseteq \mathcal{P}_\delta(\gamma) \mid \pi[\gamma] \in \pi(X)\}.$$

By (4.1), for all $\gamma < \lambda$,

(5.1) $\mathrm{HOD} \cap \mathcal{P}_\delta(\gamma) \in U_\gamma$.

(5.2) $U_\gamma \cap \mathrm{HOD} \in \mathrm{HOD}$.

Finally $V_\lambda \prec_{\Sigma_3} V$ and so for all cardinals $\gamma > \delta$, there exists a δ-complete normal fine ultrafilter U on $\mathcal{P}_\delta(\gamma)$ such that

(6.1) $\mathrm{HOD} \cap \mathcal{P}_\delta(\gamma) \in U$.

(6.2) $U \cap \mathrm{HOD} \in \mathrm{HOD}$.

This proves (2). □

As a corollary we obtain the HOD Dichotomy Theorem.

Theorem 6.34 (HOD Dichotomy Theorem). *Suppose that δ is an extendible cardinal. Then one of the following hold.*

(1) *Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD *and there is no inner model $N \subseteq$* HOD *such that N has the δ-approximation property and the δ-cover property.*

(2) *No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in* HOD *and* HOD *has the δ-approximation property and the δ-cover property.*

Proof. We first show that the following are equivalent.

(1.1) The HOD Hypothesis.

(1.2) There exists a regular cardinal $\kappa \geq \delta$ such that κ is not ω-strongly measurable in HOD.

It suffices to assume (1.2) and prove (1.1) Fix $\kappa \geq \delta$ such that κ is not ω-strongly measurable in HOD. Suppose $\eta > \kappa$ and let

$$V_\lambda \prec_{\Sigma_3} V$$

be such that $\eta < \lambda$. Let

$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$

be an elementary embedding such that $\mathrm{CRT}(\pi) = \delta$ and $\delta > \lambda$. Thus

$$\mathrm{HOD} \cap V_\lambda = (\mathrm{HOD})^{V_\lambda}$$

and this implies that κ is not ω-strongly measurable in $(\mathrm{HOD})^{V_\lambda}$. Therefore $\pi(\kappa)$ is not ω-strongly measurable in

$$\pi\,(\mathrm{HOD} \cap V_\lambda) = (\mathrm{HOD})^{V_{\pi(\lambda)}} \subset \mathrm{HOD}.$$

Therefore $\pi(\kappa)$ is not ω-strongly measurable in HOD. But $\pi(\kappa) > \eta$ and this proves (1.1).

Thus by Theorem 6.33, the following are equivalent.

(2.1) There exists a regular cardinal $\kappa \geq \delta$ such that κ is not ω-strongly measurable in HOD.

(2.2) HOD is a weak extender model of δ is supercompact.

By Theorem 6.26, if HOD is a weak extender model of δ is supercompact then HOD has the δ-approximation property and the δ-cover property. Therefore by Theorem 6.27 and the equivalence of (2.1) with (2.2), the following are equivalent.

(3.1) There exists an inner model $N \subseteq \mathrm{HOD}$ such that N has the δ-approximation property and the δ-cover property.

(3.2) There exists a regular cardinal $\kappa \geq \delta$ such that κ is not ω-strongly measurable in HOD.

(3.3) HOD is a weak extender model of δ is supercompact.

Therefore it suffies to prove that if HOD is a weak extender model of δ is supercompact then there is no regular cardinal $\kappa > \delta$ which is ω-strongly measurable in HOD.

Assume toward a contradiction that $\kappa > \delta$ and κ is ω-strongly measurable in HOD. Let

$$S = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \omega\}.$$

There must exist a stationary set $T \subseteq S$ such that $T \in \mathrm{HOD}$ and such that T cannot be partitioned into 2 stationary sets, each of which is in HOD. Let *calF* be the club filter at κ. Thus $\mathcal{F} \cap \mathrm{HOD} \in \mathrm{HOD}$ and so $\mathcal{F}|T$ yields and ultrafilter U on κ in HOD such that $S \in U$ and such that in HOD, U a κ-complete uniform normal ultrafilter on κ.

But HOD has the δ-cover property and so for all $\alpha \in S$, $\mathrm{cof}(\alpha)^{\mathrm{HOD}} < \delta$. But this contradicts that $S \in U$ since $\kappa > \delta$. □

The following theorem is a curious corollary of Theorem 6.33.

Theorem 6.35. *Suppose that δ is an extendible cardinal. Then every measurable cardinal $\kappa \geq \delta$ is a measurable cardinal in* HOD*.*

Proof. By the (weak) HOD Dichotomy Theorem, Theorem 6.34, one of the following hold.

(1.1) Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.

(1.2) No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD.

If (1.1) holds then every regular cardinal $\kappa \geq \delta$ is a measurable cardinal and if (1.2) holds then by Theorem 6.33, HOD is a weak extender model of δ is supercompact. But then by Theorem 6.29, every measurable cardinal $\kappa \geq \delta$ is a measurable cardinal in HOD. □

By Theorem 6.19, extendible cardinals above δ are downward absolute to inner models N with the δ-cover property and the δ-approximation property, and hence to weak extender models of δ is supercompact.

Theorem 6.36 (Hamkins [3]). *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose that $\kappa > \delta$ and that κ is an extendible cardinal. Then*

$$N \vDash \text{“}\kappa \text{ is an extendible cardinal”}. \quad \square$$

Theorem 6.19 also shows that the downward absoluteness of large cardinals to weak extender models of supercompactness is really quite a local result.

For example it applies to ω-extendible cardinals (with an extra step appealing to absoluteness), and using Magidor's lemma, Lemma 6.24, it also applies to supercompact cardinals.

Lemma 6.37. *Suppose that N is a weak extender model of δ is supercompact, $\kappa > \delta$, and that κ is a Vopěnka cardinal. Then*

$$N \vDash \text{“}\kappa \text{ is a Vopěnka cardinal”}. \quad \square$$

By Lemma 6.24 and Theorem 6.19 one easily obtains the version of Theorem 6.36 for supercompact cardinals, reformulated below in terms of weak extender models.

Theorem 6.38 (Hamkins [3]). *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose that $\kappa > \delta$ and that κ is a supercompact cardinal. Then N is a weak extender model of κ is supercompact.* □

Thus we obtain the following equivalence.

Theorem 6.39. *Suppose that there is a proper class of supercompact cardinals and that N is a transitive inner model of* ZFC *containing the ordinals. Then the following are equivalent.*

(1) *N is a weak extender model of δ is supercompact, for some δ.*

(2) *N has the δ-cover property and the δ-approximation property, for some regular cardinal $\delta > \omega$.* □

Remark 6.40. Theorem 6.39 suggests the following questions. Suppose that δ is supercompact and that N has the δ-cover property and the δ-approximation property.

(1) Must N be a weak extender model of δ is supercompact?

(2) Suppose δ is supercompact in N. Must N be a weak extender model of δ is supercompact?

The following theorem of Goldberg, Theorem 6.41, gives a strong negative answer to the first question. The proof of Theorem 6.44 below shows that N must be a weak extender model for δ is strongly compact (where this is defined in the obvious fashion), and so in some sense Theorem 6.41 is as strong a counter-example as possible.

However, if one modifies the first question by also requiring that N has the δ-genericity property (Definition 6.46) then the proof of Theorem 6.41 no longer applies, see Remark 6.48. □

Theorem 6.41 (Goldberg). *Suppose that δ is supercompact. Then there exists an inner model N such that:*

(1) *N has the δ-approximation property and the δ-cover property.*

(2) *δ is the least measurable cardinal of N.* □

Definition 6.42. Suppose $\lambda > \omega$ is a cardinal and that U is an ultrafilter on $P(\lambda)$ (and so $U \subset \mathcal{P}(\mathcal{P}(\lambda))$).

(1) U is *fine* if for all $\alpha < \lambda$, $\{\sigma \subset \lambda \mid \alpha \in \sigma\} \in U$.

(2) U is *normal* if for all functions
$$f : P(\lambda) \to \lambda,$$
if $\{\sigma \subseteq \lambda \mid f(\sigma) \in \sigma\} \in U$ then there exists $\alpha < \lambda$ such that
$$\{\sigma \subseteq \lambda \mid f(\sigma) = \alpha\} \in U.$$
□

The following corollary of Theorem 6.22 covers the downward absoluteness for essentially all large cardinal notions expressed in terms of normal fine ultrafilters. This includes supercompact, huge, n-huge for $n < \omega$, etc. It also covers the related notions such as almost huge etc. This theorem is implicit in [3].

Theorem 6.43. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose that $\lambda > \delta$ and that U is a normal fine (δ^+)-complete ultrafilter on $\mathcal{P}(\lambda)$. Then:*

(1) $N \cap \mathcal{P}(\lambda) \in U$.

(2) $U \cap N \in N$.

Proof. Let
$$j_U : V \to M_U \cong \mathrm{Ult}_0(V, U)$$
be the ultrapower embedding. Thus

(1.1) $j_U[\lambda] \in M_U$,

(1.2) $\mathrm{CRT}(j_U) > \delta$,

(1.3) $(M_U)^\delta \subset M_U$.

By Theorem 6.22:

(2.1) $j_U(N) \subseteq N$,

(2.2) $j_U|(N \cap V_\gamma) \in N$ for all $\gamma \in \mathrm{Ord}$.

By (1.2) and (2.2):

(3.1) $j_U(N)$ has the δ-approximation property in M_U.

(3.2) $j_U[\lambda] \in N$.

We claim:

(4.1) $j_U[\lambda] \in j_U(N)$.

Suppose $\sigma \in j_U(N)$ and $|\sigma| < \delta$. Then since $j_U[\lambda] \in N$, $j_U(N) \subset N$, and since

$$\mathcal{P}(\delta) \cap N = \mathcal{P}(\delta) \cap j_U(N),$$

necessarily,

$$\mathcal{P}(\sigma) \cap N = \mathcal{P}(\sigma) \cap j_U(N).$$

Thus $\sigma \cap j_U[\lambda] \in j_U(N)$.

Therefore by (3.1), $j_U[\lambda] \in j_U(N)$ and this proves (4.1). Thus by (2.2), this also proves (1). (2) follows by the Weak Universality Theorem, or alternatively by (2.2) again. □

As an immediate corollary of the Weak Universality Theorem (Theorem 6.17) and Lemma 6.7, we obtain the downward absoluteness of strongly compact cardinals, even at δ itself.

Theorem 6.44. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose that $\kappa \geq \delta$ and that κ is a strongly compact cardinal. Then κ is strongly compact in N.*

Proof. By Lemma 6.7, N has the κ-cover property.

Fix $\lambda > \kappa$ and let

$$I = \mathcal{P}_\kappa(\lambda) \cap N.$$

Let $\mathcal{F}$ be the filter on I generated by the sets

$$A_\sigma = \{\tau \in I \mid \sigma \subset \tau\}$$

where $\sigma \in \mathcal{P}_\kappa(\lambda)$. Since N has the κ-cover property, $\mathcal{F}$ is a κ-complete filter, and since κ is strongly compact, there is a κ-complete ultrafilter, U, on I such that $\mathcal{F} \subset U$.

By the Weak Universality Theorem, and since N has the δ-approximation property, $U \cap N \in N$, and so κ is λ-strongly compact in N. □

The local version of Theorem 6.44 (but only for $\kappa > \delta$) is a corollary of Lemma 6.7 and Theorem 6.22. This verifies a conjecture of [3], but Lemma 6.7 is the only additional ingredient here (exactly as it is for Theorem 6.44 in the case that $\kappa > \delta$).

Theorem 6.45 (Hamkins–Reitz). *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose that $\lambda > \kappa > \delta$ and that κ is a λ-strongly compact cardinal. Then κ is λ-strongly compact in N.*

Proof. Fix a κ-complete fine ultrafilter U on $\mathcal{P}_\kappa(\lambda)$. Let

$$j_U : V \to M_U \cong \mathrm{Ult}_0(V, U)$$

be the ultrapower embedding. Thus there exists $\sigma \in M_U$ such that

$$j_U[\lambda] \subset \sigma$$

and such that $|\sigma|^{M_U} < j_U(\kappa)$.

By Theorem 6.22:

(1.1) $j_U(N) \subset N$,

(1.2) $j_U|(N \cap V_\alpha) \in N$ for all $\alpha <$ Ord.

By Lemma 6.7:

(2.1) For all regular cardinals $\epsilon > \delta$, N has the ϵ-approximation property and the ϵ-cover property.

Therefore by the elementarity of j_U, $j_U(N)$ has the $j_U(\kappa)$-cover property in M_U. This implies that there exists $\tau \in j_U(N)$ such that

(3.1) $j_U[\lambda] \subset \sigma \subset \tau$,

(3.2) $|\tau|^{M_U} < j_U(\kappa)$.

Therefore $j_U|N$ witnesses in N that κ is λ-strongly compact. □

The main focus of [7] is on weak extender models N of δ is supercompact which satisfy a key additional requirement, see the definition of a Suitable Extender Model, Definition 161 in [7]. This requirement implies that if $\sigma \in \mathcal{P}_\delta(N)$ then $N[\sigma]$ is a generic extension of N for some partial order $\mathbb{P} \in N \cap V_\delta$. This motivates the following definition.

Definition 6.46. Suppose that $\delta > \omega$ is a regular cardinal and that N is a transitive inner model of ZFC containing the ordinals. Then N has the *δ-genericity property* if for each $\sigma \subset \delta$ with $|\sigma| < \delta$, there exists a partial order $\mathbb{P} \in N$ and an N-generic filter $G \subset \mathbb{P}$ such that

$$\sigma \in N[G]$$

and such that $|\mathbb{P}| < \delta$. □

Remark 6.47. The Ultimate-L Conjecture of [7] is arguably in essence just the conjecture that if there is an extendible cardinal then (provably) there is an inner model N such that for some δ:

(1) N has the δ-approximation property, the δ-cover property, and the δ-genericity property;

(2) $N \vDash V =$ Ultimate-L.

Further this version is equivalent to the version where one requires in addition that $N \subseteq \text{HOD}$, if one assumes there is a proper class of extendible cardinals.

In any case, this weaker version of the Ultimate-L Conjecture suffices for essentially all of the main absoluteness results between N and V, and so for essentially all of the main applications.

The only possible exception is the strong absoluteness of Ω-logic, see Theorem 6.67. For this one really seems to need to match δ-genericity with being a weak extender model of δ is supercompact, or at the very least with δ is supercompact in both N and V, and not just with δ-approximation and δ-cover. □

Remark 6.48. Suppose that δ is supercompact and that N has the δ-cover property and the δ-approximation property. By Theorem 6.41, N need not be a weak extender model of δ is supercompact.

However if one adds either that N has the δ-genericity property or that δ is supercompact in N, then that counterexample no longer apples.

This is illustrated by Lemma 6.55 which strongly suggests that completely different approach is needed to generalize Theorem 6.41 to the case where N also satisfies the δ-genericity property. The reason is that the proof of Theorem 6.41 uses (linear) iterations. □

By the results of [7]:

Theorem 6.49. *Suppose that N is a suitable extender model. Let δ be least such that N is a weak extender model of δ is supercompact. Then N has the δ-genericity property.* □

Remark 6.50. Suppose that δ is strongly inaccessible. Then by Vopěnka's theorem, HOD has the δ-genericity property. □

Remark 6.51. Suppose that δ is supercompact and that there is a proper class of strongly inaccessible cardinals. Let $V[G]$ be the backward Easton extension of V where at each strongly inaccessible cardinal κ of V,

$$V[G_{\kappa+1}] = V[G_\kappa][g]$$

where g is $V[G_\kappa]$-generic for adding a generic subset of κ (by initial segments).

Then in $V[G]$, V is a weak extender model of δ is supercompact and V has the δ-genericity property. But $V[G]$ is not a set-generic extension of V. □

We note the following trivial lemmas.

Lemma 6.52. *Suppose that N is a transitive inner model of* ZFC *containing the ordinals and that δ is an uncountable regular cardinal. Then the following are equivalent.*

(1) *N has the δ-cover property and the δ-genericity property.*

(2) *For each* $\sigma \subset N$ *with* $|\sigma| < \delta$*, there exists a partial order* $\mathbb{P} \in N$ *and an* N*-generic filter* $G \subset \mathbb{P}$ *such that*
$$\sigma \in N[G]$$
and such that $|\mathbb{P}| < \delta$. □

Lemma 6.53. *Suppose that* N *is a weak extender model of* δ *is supercompact. Then the following are equivalent.*

(1) N *has the* δ*-genericity property.*

(2) *For each* $\sigma \subset \delta$ *with* $|\sigma| < \delta$*, there exists a partial order* $\mathbb{P} \in N$ *and an* N*-generic filter* $G \subset \mathbb{P}$ *such that*
$$\sigma \in N[G].$$
□

Note that if r is a random real over L, then in $L[r]$, L has the ω_1-cover property and the ω_1-approximation property, but L does not have the ω_1-genericity property in $L[r]$ since in $L[r]$ there are no L-generic Cohen reals.

The following lemma follows from the results of [7] and this lemma shows that the δ-genericity property does not follow from δ-cover property together with the δ-approximation property, even if one assumes that N is a weak extender model of δ is supercompact.

Remark 6.54. Lemma 6.55(2) identifies a new class of examples of weak extender models of δ is supercompact, and hence of inner models N with the δ-approximation property and the δ-cover property.

We show in Lemma 6.57 below that the conclusion Lemma 6.55(1) can fail without the assumption that δ is supercompact. □

Suppose E is an extender and that $j_E : V \to M_E \cong \mathrm{Ult}(V, E)$ is the associated elementary embedding. Then $\kappa_E = \mathrm{CRT}(j_E)$ and ι_E denotes the least cardinal γ such that
$$M_E = \{j_E(f)(s) \mid f \in V, s \in [\gamma]^{<\omega}\}.$$

Lemma 6.55. *Suppose that* δ *is supercompact and that* $E \in V_\delta$ *is an extender. Let*
$$N = \mathrm{Ult}_0(V, E)$$
Then:

(1) N *has the* δ*-cover property and the* δ*-approximation property.*

(2) N *is a weak extender model of* δ *is supercompact.*

(3) *If* $\kappa_E = \iota_E$ *and if* $N^{\kappa_E} \subset N$ *then* E *is not set generic over* N*, and so* N *does not have the* δ*-genericity property.*

Proof. (2) implies (1) by Theorem 6.26 and (2) follows immediately from Lemma 6.59 below, which is Corollary 148 on page 157 of [7]. Thus it suffices to prove (3). We prove:

(1.1) $V = N[E]$.

Let

$$k_E : N[E] \to \mathrm{Ult}_0(N[E], E)$$

be the ultrapower embedding. Since $N^{\kappa_E} \subset N$ and since $\kappa_E = \iota_E$,

$$j_E|\mathrm{Ord} = k_E|\mathrm{Ord}.$$

Thus for each $A \subset \mathrm{Ord}$,

$$A = \{\alpha \in \mathrm{Ord} \mid k_E(\alpha) \in j_E(A)\}$$

and this implies $A \in N[E]$. Therefore $V = N[E]$ and this proves (1.1).

By (1.1), E is not set-generic over N since there are cofinally many cardinals of N which are not cardinals in V. By the hypothesis of the lemma, $E \in V_\delta$, and so N does not have the δ-genericity property. □

Remark 6.56. Suppose that N is a weak extender model of δ is supercompact, $\kappa < \delta$ is a measurable cardinal, and that U is a normal uniform ultrafilter on κ. Let (M_ω, U_ω) be the ω-th iterate of (V, U) and let

$$j : V \to M_\omega$$

be the associated iterated ultrapower embedding. Then

$$j_U(M_\omega) = M_\omega$$

where

$$j_U : V \to M_U \cong \mathrm{Ult}_0(V, U)$$

is the ultrapower embedding.

There is an extender $E \in V_\delta$ such that

$$M_\omega = \mathrm{Ult}_0(V, E).$$

Thus by Lemma 6.55, M_ω is a weak extender model of δ is supercompact.

Let $\langle \kappa_i : i < \omega \rangle$ be the critical sequence of j_U. Thus:

(1) $\langle \kappa_i : i < \omega \rangle$ is an M_ω-generic Prikry sequence for $j(U)$;

(2) $(M_\omega[\langle \kappa_i : i < \omega \rangle])^\kappa \subset M_\omega[\langle \kappa_i : i < \omega \rangle]$.

By (2) and the proof of Lemma 6.55(3), N does not have the δ-genericity property where

$$N = M_\omega[\langle \kappa_i : i < \omega \rangle].$$

This in turn implies that M_ω does not have the δ-genericity property.

This example, which is from [7], shows that there can exist elementary embeddings

$$j : N \to N$$

where N is a weak extender model of δ is supercompact, and so shows that the restriction $\kappa_E \geq \delta$ is necessary in the Universality Theorem. In fact with N as above one also has $N^\omega \subset N$ and much more.

An interesting question in the case that N is a weak extender model of δ is supercompact for which there is an elementary embedding

$$j : N \to N,$$

is whether N can ever have the δ-genericity property. □

The following lemma shows that Lemma 6.55(1) essentially requires the hypothesis that δ be supercompact. This raises a second question. How strong is the existence of a nontrivial elementary embedding

$$j : N \to N$$

such that N has the δ-cover property and the δ-approximation property, for some regular cardinal $\delta > \omega$?

The only examples we know use Lemma 6.55(1), as described above.

Lemma 6.57. *Suppose that U is a normal (uniform) ultrafilter on κ and that $\eta > \kappa$. Then there is a generic extension $V[G]$ of V such that the following hold.*

(1) *$V^\eta \subset V$ in $V[G]$.*

(2) *Let $M_U^G = \mathrm{Ult}_0(V[G], U)$. Then M_U^G does not have the δ-approximation property in $V[G]$ for any $\delta < \eta$.*

Proof. Let $\lambda > \eta$ be a strong limit cardinal such that $\mathrm{cof}(\lambda) = \kappa$ and such that

$$\lambda = |V_\lambda|.$$

Let

$$\langle \lambda_\alpha : \alpha < \kappa \rangle$$

be an increasing cofinal closed sequence of cardinals below λ. For each $\alpha < \kappa$, let $\mathbb{P}_\alpha$ be the partial order for adding a generic subset of λ_α^+. Let

$$\mathbb{P} = (\Pi_{\alpha<\kappa} \mathbb{P}_\alpha) / U.$$

The key claim is:

(1.1) $\mathbb{P}$ is (λ, ∞)-distributive.

Since λ is singular it suffices to prove that for each $\gamma < \lambda$:

(2.1) $\mathbb{P}$ is (γ, ∞)-distributive.

Fix γ and clearly we can reduce to the case that $\gamma^+ < \lambda_0$ by replacing the sequence $\langle \lambda_\alpha : \alpha < \kappa \rangle$ with the sequence $\langle \lambda_{\alpha_0+\alpha} : \alpha < \kappa \rangle$ were $\lambda_{\alpha_0} > \gamma^+$.

Fix a sequence $\langle D_\alpha : \alpha < \gamma \rangle$ of open dense subsets of $\mathbb{P}$ and fix

$$f \in \Pi_{\xi<\kappa} \mathbb{P}_\xi.$$

Each partial order $\mathbb{P}_\xi$ is γ^+-closed and so there a sequence

$$\langle f_\alpha : \alpha \leq \gamma \rangle$$

of functions in $\Pi_{\xi<\kappa} \mathbb{P}_\xi$ such that:

(3.1) $f_0(\xi) \leq f(\xi)$ in $\mathbb{P}_\xi$ for all $\xi < \kappa$.

(3.2) For all $\alpha < \beta \leq \gamma$, $f_\beta(\xi) \leq f_\alpha(\xi)$ in $\mathbb{P}_\xi$ for all $\xi < \kappa$.

(3.3) For all $\alpha < \gamma$, $f_\alpha/U \in D_\alpha$.

Thus

$$f_\gamma/U \in D_\alpha$$

for all $\alpha < \gamma$ and

$$f_\gamma/U \leq f/U$$

This proves (1.1).

Let

$$j : V \to M_U \cong \mathrm{Ult}_0(V, U)$$

be the ultrapower embedding.

Let $G \subset \mathbb{P}$ be V-generic and let

$$j_G : V[G] \to M_U^G \cong \mathrm{Ult}_0(V[G], U)$$

be the ultrapower embedding as computed in $V[G]$.

Since $V^\lambda \subset V$ in $V[G]$:

(4.1) $j_U = j_U^G | V$.

(4.2) $H(\lambda^+)^V = H(\lambda^+)^{V[G]}$.

(4.3) $\lambda = \sup(j_U[\lambda]) = \sup(j_U^G[\lambda])$.

(4.4) $(\lambda^+)^V = (\lambda^+)^{V[G]} = (\lambda^+)^{M_U} = (\lambda^+)^{M_U^G}$.

Therefore by the definition of $\mathbb{P}$, in $V[G]$ there is an M_U^G-generic filter for adding a generic subset of λ^+.

This implies that M_U^G does not even have the λ^+-approximation property in $V[G]$, and so M_U^G does not have the δ-approximation property in $V[G]$ for any $\delta < \lambda^+$.

The point of course is that for any inner model N, if N has the δ-approximation property then N has the κ-approximation property for all $\kappa > \delta$. □

A key question arises. Does the assumption that N is an inner model of ZFC which satisfies the δ-approximation property and the δ-cover property, imply that N is "genuinely" close to V? In particular:

(1) Are all large cardinals downward necessarily downward absolute to N, if there is a proper class of such (or even stronger) large cardinals in V?

(2) Must every elementary embedding $j : N \to N$ be trivial?

(3) Is Ω-logic absolute between V and N?

If V is a generic extension of N then for all three questions, the answer is yes.

We have already answered the second question (negatively) if there is a supercompact cardinal. Nevertheless, the universality theorems would seem to suggest that the answer to the first question is still yes. But there is a serious issue here at the upper regions of the large cardinal hierarchy.

For example, it is not clear if the existence of a proper class of λ for which the Axiom I_0 holds at λ must be downward absolute to N, or even that the Axiom I_1 must hold in N for some λ.

The δ-genericity property allows one to prove (assuming in addition that the δ-cover and δ-approximation properties also hold) that the existence of a proper class of λ for which the Axiom I_0 holds at λ is downward absolute to N. It is open whether this downward absolutness holds for all weak extender models of δ is supercompact.

The subtle aspect of this downward absoluteness is that the Axiom I_0 can hold at $\lambda > \delta$ while the Axiom I_3 *fails* to hold at λ in N. This can happen *even* if N is a weak extender model of δ is supercompact with the δ-genericity property.

By unpublished results of Scott Cramer, if Axiom $\mathrm{I}_0^\#$ holds at λ then the Axiom I_0 holds at γ for cofinally many $\gamma < \lambda$, where the Axiom $\mathrm{I}_0^\#$ holds at ϵ if there is an elementary embedding

$$j : L\Big((V_{\epsilon+1})^\#\Big) \to L\Big((V_{\epsilon+1})^\#\Big)$$

such that $\mathrm{CRT}(j) < \epsilon$.

Therefore the following theorem yields the situation where one can have N with the δ-approximation, δ-cover, and δ-genericity properties, the Axiom I_0 holds at λ for a proper class of λ, and for *all* λ if the Axiom I_0 holds at λ then λ is not even ω-huge in N, or even that for *all* cardinals λ, λ is not ω-huge in N.

Theorem 6.58. *Suppose that Axiom $\mathrm{I}_0^\#$ holds at λ. Then there is a generic extension $V[G]$ such that in $V[G]$ the following hold.*

(1) *ω_1^V is countable and $\omega_2^V = \omega_1^{V[G]}$.*

(2) *Axiom $\mathrm{I}_0^\#$ holds at λ.*

(3) *V has the* $\omega_1^{V[G]}$*-approximation property and the* $\omega_1^{V[G]}$*-cover property.*

(4) *V has the* $\omega_1^{V[G]}$*-genericity property.*

(5) *Suppose* $\gamma < \lambda$ *and* γ *is an uncountable limit cardinal. Then* $\mathrm{cof}(\gamma)^V > \omega$. ☐

Suppose that N is a weak extender model of δ is supercompact for some $\delta < \lambda$. Note that if Axiom I_0 holds at λ but fails to hold in N at λ, then necessarily

$$N \cap H(\lambda^+) \notin L(V_{\lambda+1}).$$

This is because if $N \cap H(\lambda^+) \in L(V_{\lambda+1})$ then $N \cap H(\lambda^+)$ must be definable in $L(V_{\lambda+1})$ from $N \cap H(\delta^+)$, in which case

$$j(N \cap H(\lambda^+)) = N \cap H(\lambda^+)$$

for any elementary

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

with $\mathrm{CRT}(j) > \delta$, and by Theorem 6.30 this implies $j|(N \cap V_\lambda) \in N$. This in turn implies that there is an elementary embedding

$$j_N : L(N \cap V_{\lambda+1}) \to L(N \cap V_{\lambda+1})$$

which is $\underset{\sim}{\Sigma}_2$-definable in N, [7]. This contradicts that the Axiom I_0 does not hold at λ in N.

This example also shows that while the Definability Theorem can be refined to show, assuming N is a weak extender model of δ is supercompact and that $\delta < \lambda$, that $N \cap H(\lambda^{++})$ is definable in $H(\lambda^{++})$ from $N \cap H(\delta^+)$, this improvement is best possible.

The conclusion (2) of Lemma 6.55 is a special case of the following lemma which is an immediate corollary of Lemma 147 on page 156 in [7]. Lemma 6.59 is the generic version of Lemma 6.55.

If δ is supercompact then δ is a limit of Woodin cardinals. Therefore $(E, \mathbb{P})$ can be chosen to satisfy the hypothesis of Lemma 6.59 with *any* given uncountable regular cardinal of V_δ as the critical point associated to E, and much more.

Lemma 6.59. *Suppose that* δ *is supercompact,* G *is* V*-generic for some partial order* $\mathbb{P} \in V_\delta$*, and that* $E \in V[G]_\delta$ *is a* V*-extender. Let*

$$N = \mathrm{Ult}_0(V, E)$$

Then in $V[G]$*,* N *is a weak extender model of* δ *is supercompact.* ☐

A stronger version of Lemma 6.59 is the following and the proof is the same.

Lemma 6.60. *Suppose that* N *is a weak extender model of* δ *is supercompact and that* N *has the* δ*-genericity property. Suppose* $E \in V_\delta$ *is an* N*-extender. Then* $\mathrm{Ult}_0(N, E)$ *is is a weak extender model of* δ *is supercompact.* ☐

Lemma 6.59 (applied to V_λ) combined with the Universality Theorem, Theorem 6.30, yields the following theorem, which is Theorem 177 on page 179 in [7].

Theorem 6.61. *Suppose λ is a singular strong limit cardinal, $(V_{\lambda+1})^\#$ exists, $\mathbb{P} \in V_\lambda$, and that*

$$V[G] \vDash \text{“The Axiom } \mathrm{I}_0 \text{ holds at } \lambda\text{”}$$

where $G \subset \mathbb{P}$ is V-generic. Then one of the following hold.

(1) *The Axiom I_0 holds in V at λ.*

(2) *For cofinally many $\lambda^* < \lambda$, the Axiom I_0 holds in V at λ^*.*

Proof. We sketch the proof which appeals to the generic elementary embeddings given by the stationary tower $\mathbb{Q}_{<\delta}$ associated to δ where δ is a Woodin cardinal, [4].

Fix a Woodin cardinal $\delta < \lambda$ such that $\mathbb{P} \in V_\delta$. Let $g \subset \mathbb{Q}_{<\delta}$ and let

$$J_g : V \to M_g \subset V[g]$$

be the associated generic elementary embedding. Thus

(1.1) $\mathrm{CRT}(J_g) = \omega_1^V$.

(1.2) $(M_g)^\omega \subset M_g$ in $V[g]$.

(1.3) $M_g \cap V[g]_{\lambda+1} \in L(V[g]_{\lambda+1})$.

Thus since $(V_{\lambda+1})^\#$ exists in V:

(2.1) $(M_g \cap V[g]_{\lambda+1})^\#$ exists in $V[g]$,

(2.2) $(M_g \cap V[g]_{\lambda+1})^\# \in M_g$.

Choose $G \subset \mathbb{P}$ such that

(3.1) $G \in V[g]$ and G is V-generic,

(3.2) the Axiom I_0 holds at λ in $V[G]$.

Let

$$j : L(V[G]_{\lambda+1}) \to L(V[G]_{\lambda+1})$$

witness that Axiom I_0 holds at λ in $V[G]$. By replacing j by a finite iterate if necessary, we can assume that $\mathrm{CRT}(j) > \delta$. By factoring, $V[G][g]$ is a generic extension of $V[G]$ for some $V[G]$-generic filter on some partial order $\mathbb{P} \in V[G]_{\mathrm{CRT}(j)}$.

Thus j lifts in $V[G]$ to an elementary embedding

$$j_g : L(V[G]_{\lambda+1})[g] \to L(V[G]_{\lambda+1})[g].$$

Let $\kappa_0 = \mathrm{CRT}(j)$. Since $\kappa > \delta$, V has the κ_0-cover and κ_0-approximation property in $V[g]$, and so V_λ has the κ_0-cover and κ_0-approximation property in $V[g]_\lambda$.

The elementary embedding j_g witnesses that κ_0 is supercompact in $V[g]_\lambda$ and so by Hamkin's Universality theorem, κ_0 is supercompact in V_λ.

But V_λ is definable in $V[g]_\lambda$ from parameters in $V[g]_{\kappa_0}$ and so κ_0 is a limit of supercompact cardinals. Fix $\delta < \kappa < \kappa_0$ such that κ is supercompact in V_λ.

Therefore by Lemma 6.59,

(4.1) $M_g \cap V[g]_\lambda$ is a weak extender model of κ is supercompact in $V[g]_\lambda$.

By Theorem 6.19,

(5.1) For all $\alpha < \lambda$, $j_g|(M_g \cap V[g]_\alpha) \in M_g$.

(5.2) For all $\alpha < \lambda$, $j_g(M_g \cap V[g]_\alpha) = M_g \cap V[g]_{j_g(\alpha)}$.

Since $M_g^\omega \subset M_g$ in $V[g]$

$$j_g|(M_g \cap V[g]_\lambda) \in M_g.$$

Thus by (1.3),

$$j_g(M_g \cap V[g]_{\lambda+1}) = M_g \cap V[g]_{\lambda+1}$$

and so j_g restricts to define an elementary embedding

$$k_g : L\Big(M_g \cap V[g]_{\lambda+1}\Big) \to L\Big(M_g \cap V[g]_{\lambda+1}\Big)$$

such that

$$k_g|\Big(M_g \cap V[g]_{\lambda+1}\Big) \in M_g.$$

For each $n < \omega$, let T_n be the theory of

$$L\Big(M_g \cap V[g]_{\lambda+1}\Big)$$

with parameters from $M[g] \cap V[g]_{\lambda+1}$ and first n-many Silver indiscernables of $L\Big(M_g \cap V[g]_{\lambda+1}\Big)$ (coded naturally as a subset of $M_g \cap V[g]_{\lambda+1}$).

Clearly T_n does not depend on the choice of Silver indiscernibles and so $k_g(T_n) = T_n$ for all $n < \omega$. Thus

$$k_g|\,(M[g] \cap V[g]_{\lambda+1})$$

together $\langle T_n : n < \omega\rangle$ induces and elementary embedding

$$k_g^* : L\Big(M_g \cap V[g]_{\lambda+1}\Big) \to L\Big(M_g \cap V[g]_{\lambda+1}\Big)$$

such that $k_g^* \in M_g$.

Thus

(6.1) The Axiom I_0 holds at λ in M_g.

We now appeal to the generic elementary embedding

$$J_g : V \to M_g \subset V[g].$$

If $J_g(\lambda) = \lambda$ then the Axiom I_0 holds at λ in V, and if

$$J_g(\lambda) > \lambda$$

then since

$$\lambda = \sup(J_G[\lambda]),$$

necessarily, for cofinally many $\lambda^* < \lambda$, the Axiom I_0 holds in V at λ^*. This proves the theorem. □

We prove a preliminary and technical version of the theorem on the downward absoluteness of the Axiom I_0 to inner models with the δ-cover property, the δ-approximation property, and the δ-genericity property.

Theorem 6.62. *Suppose that N has the δ-cover property, the δ-approximation property, and the δ-genericity property. Suppose that $\lambda > \delta$, the Axiom I_0 holds at λ, and that $(N_{\lambda+1})^\#$ exists. Then one of the following hold.*

(1) $N \vDash$ *"$(V_{\lambda+1})^\#$ exists and the Axiom I_0 holds at λ".*

(2) $N \vDash$ *"For cofinally many $\lambda^* < \lambda$, the Axiom I_0 holds at λ^*".*

Proof. Let

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

be an elementary embedding with $\mathrm{CRT}(j) < \lambda$. By replacing j with a finite iterate we can assume $\delta < \mathrm{CRT}(j)$.

Let

$$G \subset \mathrm{Coll}(\omega, {<}\delta)$$

be V-generic and let

$$\mathbb{R}_G = (\mathbb{R})^{V[G]}.$$

Let H be $V(\mathbb{R}_G)$-generic for $\left(\mathrm{Coll}(\omega_1^{V(\mathbb{R}_G)}, \mathbb{R}_G)\right)^{V(\mathbb{R}_G)}$. Thus

$$V(\mathbb{R}_G)[H] \vDash \text{Axiom of Choice}$$

(and so $V(\mathbb{R}_G)[H] = V[G^*]$ where G^* is V-generic for a (canonical) partial order $\mathbb{P} \in V$).

Thus since N has the δ-cover property and the δ-genericity property:

(1.1) $N(\mathbb{R}_G)$ is a symmetric generic extension of N for $\mathrm{Coll}(\omega, {<}\delta)$;

(1.2) $N(\mathbb{R}_G)$ is closed under ω-sequences in $V[G]$;

(1.3) H is $N(\mathbb{R}_G)$-generic for $\left(\mathrm{Coll}(\omega_1^{N(\mathbb{R}_G)}, \mathbb{R}_G)\right)^{N(\mathbb{R}_G)}$;

(1.4) $N(\mathbb{R}_G)[H]$ is closed under ω-sequences in $V(\mathbb{R}_G)[H]$.

The elementary embedding j lifts to an elementary embedding

$$j_H^G : L(V_{\lambda+1})(\mathbb{R}_G)[H] \to L(V_{\lambda+1})(\mathbb{R}_G)[H].$$

We come to the key points:

(2.1) Since $\mathrm{CRT}(j) > \delta$, by the Definability Theorem, $j(N \cap V_\lambda) = N \cap V_\lambda$.

(2.2) Since $N(\mathbb{R}_G)[H]$ is closed under ω-sequences in $V(\mathbb{R}_G)[H]$,

$$N(\mathbb{R}_G)[H]_{\lambda+1} \in L(V_{\lambda+1})(\mathbb{R}_G)[H].$$

Thus

$$j\,(N(\mathbb{R}_G)[H]_{\lambda+1}) = j\,(N(\mathbb{R}_G)[H]_{\lambda+1})$$

and so j restricts to define an embedding

$$k^G_H : L\,(N(\mathbb{R}_G)[H]_{\lambda+1}) \to L\,(N(\mathbb{R}_G)[H]_{\lambda+1})\,.$$

Here the distinction between

$$L\,(N(\mathbb{R}_G)[H]_{\lambda+1}) = (L(V_{\lambda+1}))^{N(\mathbb{R}_G)[H]}$$

versus $L(N_{\lambda+1})[\mathbb{R}_G][H]$ is critical.

Since N has the δ-cover property, $\delta < \lambda$, and since $(N_{\lambda+1})^\#$ exists, necessarily

$$(N_{\lambda+1})^\# \in N.$$

This implies

$$(N(\mathbb{R}_G)[H]_{\lambda+1})^\# \in N(\mathbb{R}_G)[H].$$

Now arguing exactly as on page 69 in the proof of Theorem 6.61, letting T_n be the theory of

$$L\,(N(\mathbb{R}_G)[H]_{\lambda+1})$$

with parameters from

$$N(\mathbb{R}_G)[H]_{\lambda+1} \cup I$$

where I is the set of the first n Silver indiscernibles of $L\,(N(\mathbb{R}_G)[H]_{\lambda+1})$, the Axiom I_0 holds at λ in $N(\mathbb{R}_G)[H]$.

Therefore again since $(N_{\lambda+1})^\# \in N$, by Theorem 6.61, one of the following must hold.

(3.1) $N \vDash$ "$(V_{\lambda+1})^\#$ exists and the Axiom I_0 holds at λ".

(3.2) $N \vDash$ "For cofinally many $\lambda^* < \lambda$, the Axiom I_0 holds at λ^*". □

As an immediate corollary to Theorem 6.62 we obtain the following theorem. As we have already noted, without the assumption that N has the δ-genericity property, it is open whether the conclusion holds even if one assumes that N is a weak extender model of δ is supercompact.

Theorem 6.63. *Suppose that N has the δ-cover property, the δ-approximation property, and the δ-genericity property. Suppose that the Axiom* I_0 *holds at λ, for a proper class of λ. Then:*

$$N \vDash \text{“The Axiom } \mathrm{I}_0 \text{ holds at } \lambda\text{, for a proper class of } \lambda\text{”}. \qquad \square$$

Another application of the δ-genericity property concerns the correctness of N with respect to Ω-logic and we prove a very strong version of this.

The proof requires yet another variation of the universality theorem. The following definition simplifies the statement of that variation.

Definition 6.64. Suppose that E is an extender, δ is a cardinal, and that

$$M_E \cong \text{Ult}_0(V, E).$$

Then E is *δ-closed* if $(\text{LTH}(E))^\delta \subset M_E$. □

Note that if $\delta \le \kappa_E$ and E is δ-closed then $M_E^\delta \subset M_E$ where $M_E = \text{Ult}_0(V, E)$. Therefore if E^* is the extender of length γ given by the ultrapower embedding

$$j_E : V \to M_E \cong \text{Ult}_0(V, E),$$

then E^* is δ-closed.

One can strengthen the properties of Goldberg's counterexample, see Theorem 6.41, and obtain the following. Note that assuming δ is supercompact, conclusion (3) in Theorem 6.65 implies that if N has the δ-cover property then N must have the δ-approximation property. Given that δ is supercompact, it is a little surprising that N can satisfy (3) and yet have δ be the least measurable cardinal of N.

Theorem 6.65. *Suppose that δ is supercompact. Then there exists an inner model N such that:*

(1) *N has the δ-approximation property and the δ-cover property.*

(2) *δ is the least measurable cardinal of N.*

(3) *Suppose E is an extender with $\kappa_E \ge \delta$ such that E is κ-closed for all $\kappa < \delta$. Then $E \cap N \in N$.* □

Theorem 6.66(1) is just Theorem 6.22. Theorem 6.66(2) improves Theorem 6.66(1) by showing that the induced embedding on N is itself an (internal) N-ultrapower embedding by an N-extender. Finally Theorem 6.66(3) and Theorem 6.66(4) generalize this to towers of ultrapower embeddings by extenders, defined in the natural sense. The latter is the main part of the theorem.

Theorem 6.66. *Suppose $\delta > \omega$ is a regular cardinal, N is an inner model of* ZFC *containing the ordinals, and that N has the δ-cover property and the δ-approximation property. Suppose:*

(i) *E is a δ-closed extender such that $\kappa_E > \delta$ and that*

$$j_E : V \to M_E \cong \text{Ult}_0(V, E)$$

is the ultrapower embedding.

(ii) *F is a δ-closed extender such that $\kappa_F > \delta$ and that*

$$j_F : V \to M_F \cong \mathrm{Ult}_0(V, F)$$

is the ultrapower embedding.

(iii) *Suppose that*

$$\pi : M_E \to M_F \cong \mathrm{Ult}_0(V, F)$$

is an elementary embedding with $\mathrm{CRT}(\pi) > \delta$ *such that*

$$j_F = \pi \circ j_E.$$

Then:

(1) $j_E(N) \subset N$ *and* $j_E|(N \cap V_\gamma) \in N$ *for all* $\gamma <$ Ord.

(2) *Suppose that $\gamma >$* $\mathrm{LTH}(E)$ *is a strong limit cardinal. Let E^* be the N-extender of length γ given by $j_E|N$ and let*

$$j^N_{E^*} : N \to N_{E^*} \cong \mathrm{Ult}_0(N, E^*)$$

be the ultrapower embedding. Then

(a) $N_{E^*} = j_E(N)$,

(b) $j^N_{E^*} = j_E|N$.

(3) $\pi \circ j_E(N) \subset N$ *and* $\pi|(j_E(N) \cap V_\gamma) \in N$ *for all* $\gamma <$ Ord.

(4) *Suppose that $\gamma >$* $\mathrm{LTH}(F)$ *is a strong limit cardinal such that*

$$\gamma > \sup(\pi[\mathrm{LTH}(E)]).$$

Let E^* be the N-extender of length γ given by $j_E|N$ and let

$$j^N_{E^*} : N \to N_{E^*} \cong \mathrm{Ult}_0(N, E^*)$$

be the ultrapower embedding. Let F^* be the N extender of length γ given by $j_F|N$ and let

$$j^N_{F^*} : N \to N_{F^*} \cong \mathrm{Ult}_0(N_{E^*}, F^*)$$

be the ultrapower embedding. Let H be the N_{E^*}-extender of length γ given by $\pi|N_{E^*}$ and let

$$\pi^* : N_{E^*} \to N^{E^*}_H \cong \mathrm{Ult}_0(N_{E^*}, H)$$

be the ultrapower embedding. Then

(a) $N_{E^*} = j_E(N)$ *and* $j^N_{E^*} = j_E|N$.

(a) $N_{F^*} = j_F(N)$ *and* $j^N_{F^*} = j_F|N$.

(c) $H \in N$, $N^{E^*}_H = j_F(N)$, *and* $\pi^* = \pi|N_{E^*}$.

Proof. (1) follows immediately by Theorem 6.22. We prove (2). Let

$$k : N_{E^*} \to j_E(N)$$

be the natural factor embedding. To prove (2), it suffices to prove:

(1.1) k is the identity.

Assume toward a contradiction that (1.1) fails. Then

$$\text{CRT}(k) \geq \gamma.$$

By Lemma 6.7, since N has the δ-cover property and the δ-approximation property, $\mathcal{N}$ has the ϵ-cover property and the ϵ-approximation property, for all regular cardinals $\epsilon > \delta$. Therefore $\mathcal{N}$ has the (ι_E^+)-cover property.

We have that

$$j_E(N) = \{j_E(f)(\alpha) \mid f : \iota_E \to N, \alpha < j_E(\iota_E^+)\}.$$

Further $j_E(\iota_E) < \gamma$ since $\text{LTH}(E) < \gamma$ and since γ is a strong limit cardinal. Therefore, there exists

$$f : \iota_E \to \text{Ord}$$

such that $\text{CRT}(k) = j_E(f)(\alpha)$ for some $\alpha < \gamma$. Let

$$\sigma = \text{range}(f)$$

and choose $\tau \in N$ such that $\sigma \subset \tau$ and $|\tau| \leq \iota_E$. Thus

$$\text{CRT}(k) \in j_E(\tau).$$

But

$$\sup(j_E[\gamma]) = \gamma$$

and so

$$j_E(\tau) = k[j^N_{E^*}(\tau)]$$

and this is a contradiction. This proves (1.1).

We finish by proving (4) which implies (3). By (2) applied to both E and F:

(2.1) $N_{E^*} = j_E(N)$ and $j^N_{E^*} = j_E|N$.

(2.2) $N_{F^*} = j_F(N)$ and $j^N_{F^*} = j_F|N$.

Thus it suffices to just prove (4)(c). Since

$$j_F = \pi \circ j_E$$

and since

$$\gamma = \sup(j_E[\gamma]) = \sup(j_F[\gamma]),$$

by (2.1) and (2.2) it suffices to just prove that $H \in N$.

Note that since $\gamma = \sup(j_F[\gamma])$, necessarily $H \subset N_{E^*} \cap H(\gamma)$. Thus since N has the δ-approximation property and since $N_{E^*} \subset N$, it suffices to prove

(3.1) $H \cap \sigma \in N$ for all $\sigma \subset N_{E^*} \cap H(\gamma)$ such that $|\sigma| < \delta$.

Fix $\sigma \subset N_{E^*} \cap H(\gamma)$ such that $|\sigma| < \delta$ and such that $\sigma \in N$. Choose $\tau \in N_{E^*}$ such that $\sigma \subset \tau$ and $|\tau| < \delta$. The covering set τ exists since $N_E = N_{E^*} = j_E(N)$,

$$M_E^\delta \subset M_E,$$

and since $j_E(N)$ has the δ-cover property in M_E.

Thus $\pi(\tau) = \pi[\tau]$. Therefore $\pi[\tau] \in N_{F^*}$ and so $\pi[\sigma] \in N$ since $\tau \in N$. This implies that $H \cap \sigma \in N$ which proves (3.1). □

The following theorem follows from Theorem 185 of [7], but only by assuming in addition that N is a suitable extender model for which there is a cardinal $\kappa > \delta_N$ such that κ is $(\omega+2)$-extendible, where δ_N is the least δ such that N is a weak extender model of δ is supercompact.

Thus Theorem 6.67 strengthens Theorem 185 of [7] by reducing the hypothesis to essentially the best possible. Note that the conclusion Theorem 6.67(1) implies that N has the δ-genericity property.

Theorem 6.67 implies, by the generic invariance of the Ω-proof relation, that for all sentences ϕ and for all theories $T \in N$, $V \vDash$ "$T \vdash_\Omega \phi$" if and only if $N \vDash$ "$T \vdash_\Omega \phi$".

Theorem 6.67. *Suppose that there is a proper class of Woodin cardinals, N is a weak extender model of δ is supercompact and that N has the δ-genericity property. Suppose*

$$G \subset \mathrm{Coll}(\omega, {<}\delta)$$

is V-generic. Then in $V[G]$ the following hold where $\mathbb{R}_G = \mathbb{R}^{V[G]}$ and $\Gamma^\infty_G = (\Gamma^\infty)^{V[G]}$.

(1) *$N(\mathbb{R}_G)$ is a symmetric extension of N for* $\mathrm{Coll}(\omega, {<}\delta)$.

(2) $\Gamma^\infty_G = (\Gamma^\infty)^{N(\mathbb{R}_G)} = (\Gamma^\infty)^{N(\mathbb{R}_G)[H]}$ *where $H \subset \mathrm{Coll}(\omega_1^{V[G]}, \mathbb{R}_G)$ is $V[G]$-generic.*

Proof. (1) is an immediate consequence of the assumption that N has the δ-genericity property, and the characterization of when $N(\mathbb{R}_G)$ is a symmetric extension of N for $\mathrm{Coll}(\omega, {<}\delta)$.

Thus since N has the δ-cover property:

(1.1) $N^\omega \subset N(\mathbb{R}_G)$ in $V[G]$.

Since δ is supercompact in V and since there is a proper class of Woodin cardinals:

(2.1) Γ^∞_G is the set of all $A \in \mathcal{P}(\mathbb{R}_G) \cap V(\mathbb{R}_G)$ such that A is Suslin and co-Suslin in $V(\mathbb{R}_G)$.

Similarly, since δ is supercompact in N and since in N there is a proper class of Woodin cardinals, (3.1) $(\Gamma^\infty)^{N(\mathbb{R}_G)}$ is the set of all $A \in \mathcal{P}(\mathbb{R}_G) \cap N(\mathbb{R}_G)$ such that A is Suslin and co-Suslin in $N(\mathbb{R}_G)$.

(3.2) $(\Gamma^\infty)^{N(\mathbb{R}_G)} = (\Gamma^\infty)^{N(\mathbb{R}_G)[H]}$ where $H \subset \text{Coll}(\omega_1^{V[G]}, \mathbb{R}_G)$ is $V[G]$-generic.

(2.1) and (3.1) follow by the Derived Model Theorem which is from the general theory of AD^+, and (3.2) follows by the Martin-Steel Theorem, [6].

For (3.2), one can also simply use the characterization of the universally Baire sets as those sets which are κ-weakly homogeneously Suslin for all κ. This characterization holds whenever there is a proper class of Woodin cardinals.

Thus to prove (2), it suffices to prove:

(4.1) Suppose $A \in \Gamma^\infty_G$. Then A is Suslin and co-Suslin in $N(\mathbb{R}_G)$.

Suppose that
$$A \in \Gamma^\infty_G.$$
By the Martin-Steel Theorem [6], and since there is a proper class of Woodin cardinals:

(5.1) For all $\gamma > \delta$, A is γ-homogeneously Suslin in $V[G]$.

Fix $\gamma > \delta$. Let
$$\pi_A : \omega^{<\omega} \to V[G]$$
witness that A is γ-homogeneously Suslin. Thus:

(6.1) For each $s \in \omega^{<\omega}$, $\pi_A(s)$ is a γ-complete ultrafilter on $\eta^{\text{dom}(s)}$ for some η.

(6.2) For each $s \in \omega^{<\omega}$ and for each $k < \text{dom}(s)$, $\pi_A(s)$ projects to $\pi_A(s|k)$.

(6.3) For each $x \in \mathbb{R}_G$, $x \in A$ if and only if the tower $\langle \pi_A(x|k) : k < \omega \rangle$ is wellfounded.

The key point is:

(7.1) For each $s \in \omega^{<\omega}$, $\pi_A(s)$ is generated by $\pi_A(s) \cap V$ and $\pi_A(s) \in V$.

For each $s \subset t$ in $\omega^{<\omega}$ define:

(8.1) $M_s \cong \text{Ult}_0(V, \pi_A(s))$

(8.2) $j_s : V \to M_s$ is the ultrapower embedding.

(8.3) $j_{(s,t)} : M_s \to M_t$ is the factor embedding.

(8.4) $N_s = j_s(N)$.

(8.5) $k_{(s,t)} = j_{(s,t)}|N_s$.

Thus by Theorem 6.66, there exists

$$e_A : \omega^{<\omega} \to N$$

and

$$\rho_A : \omega^{<\omega} \times \omega^{<\omega} \to N$$

such that for all $s \subset t$ in $\omega^{<\omega}$:

(9.1) $e_A(s)$ is an N-extender, $N_s = \mathrm{Ult}_0(N, e_A(s))$, and $j^N_{e_A(s)} = j_s|N$.

(9.2) Let $j^N_{(s,t)} : N_s \to N_t \cong \mathrm{Ult}_0(N_s, \rho_A(s,t))$ be the ultrapower embedding. Then:

$$j^N_{s,t} = k_{(s,t)}.$$

By (1.1),

$$(e_A, \rho_A) \in N(\mathbb{R}_G).$$

By (6.3) and the agreements specified in (9.1)–(9.2), this implies that $\mathbb{R}_G \backslash A$ is κ-Suslin in $N(\mathbb{R}_G)$ for some κ.

Similarly since $\mathbb{R}_G \backslash A \in \Gamma^\infty_G$, A is κ-Suslin in $N(\mathbb{R}_G)$ for some κ.

This proves (4.1). □

Remark 6.68. Suppose that δ is supercompact, there is a proper class of Woodin cardinals, and that $G \subset \mathrm{Coll}(\omega, <\delta)$ is V-generic. Then by Theorem 4.9, Γ^∞ is very nearly sealed in $V[G]$.

As a corollary, if N is a weak extender model of δ is supercompact and N has the δ-genericity property, then by Theorem 4.9 and Theorem 6.67, the first order theory of $L(\Gamma^\infty, \mathbb{R})$ after sealing, is the *same* computed in V or in N.

This absoluteness is the starting point for the formulation of stronger versions of the Ultimate-L Conjecture. □

References

[1] Lev Bukovský. Characterization of generic extensions of models of set theory. *Fund. Math.*, 83(1):35–46, 1973.

[2] Sy-David Friedman, Sakaé Fuchino, and Hiroshi Sakai. On the set-generic multiverse. In *Sets and computations*, volume 33 of *Lect. Notes Ser. Inst. Math. Sci. Natl. Univ. Singap.*, pages 25–44. World Sci. Publ., Hackensack, NJ, 2018.

[3] Joel David Hamkins. Extensions with the approximation and cover properties have no new large cardinals. *Fund. Math.*, 180(3):257–277, 2003.

[4] Paul Larson. *The Stationary Tower: Notes on a Course by W. Hugh Woodin*. University Lecture Series. American Mathematical Society, Providence, 2004.

[5] R. Laver. Certain very large cardinals are not created in small forcing extensions. *Ann. Pure Appl. Logic*, 149(1-3):1–6, 2007.

[6] D. A. Martin and J. Steel. A proof of projective determinacy. *J. Amer. Math. Soc.*, 2:71–125, 1989.

[7] W. Hugh Woodin. Suitable Extender Models I. *Journal of Mathematical Logic*, 10(1-2):101–341, 2010.

[8] W. Hugh Woodin. The Continuum Hypothesis, the generic-multiverse of sets, and the Ω Conjecture. In *Set Theory, Arithmetic and Foundations of Mathematics: Theorems, Philosophies*, volume 36 of *Lecture Notes in Logic*, pages 13–42. Cambridge University Press, New York, NY, 2011.

[9] W. Hugh Woodin. The Weak Ultimate L Conjecture. In *Infinity, computability, and metamathematics*, volume 23 of *Tributes*, pages 309–329. Coll. Publ., London, 2014.

[10] W. Hugh Woodin. In Search of Ultimate-*L*: The 19th Midrasha Mathematicae Lectures. *Bull. Symbolic Logic*, 23(1):1–109, 2017.