The two dimensional random field Ising model

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Based on joint work with F. Caravenna, R. Sun and N. Zygouras

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Contents

Description

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Let $\partial\Omega = \{x \in \mathbb{Z}^d \setminus \Omega : \|x - y\| = 1 \text{ for some } y \in \Omega\}$ denote the boundary.

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We then define the law over spins $\sigma \in \{\pm 1\}^\Omega$ with $+$ boundary conditions as

$$P_\Omega^+(\sigma) = \frac{1}{Z_\Omega^+} \exp\left(-\beta \mathcal{H}_\Omega(\sigma)\right) \prod_{x \in \partial\Omega} \mathbf{1}_{\{\sigma_x = 1\}}$$

where

$$\mathcal{H}_\Omega(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y, \quad \text{and} \quad Z_\Omega^+ = \sum_\sigma \exp\left(-\beta \mathcal{H}_\Omega(\sigma)\right) \prod_{x \in \partial\Omega} \mathbf{1}_{\{\sigma_x = 1\}}$$

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are the *Hamiltonian* and the *partition function* respectively.

Consider the ferromagnetic case; that is, the *inverse temperature* $\beta \geq 0$.

We can consider different boundary conditions by modifying slightly.

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It is well known that as $\Omega \uparrow \mathbb{Z}^d$ the sequence of probability measures P^+_Ω has an infinite volume limit $P^+_{\mathbb{Z}^d}$.

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1) if $d = 1$ then $\beta_c = \infty$;
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Of course, this depends on the boundary conditions as well.

A similar result holds for the model with $-$ boundary conditions and we see that

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A similar result holds for the model with $-$ boundary conditions and we see that

$$E_{\mathbb{Z}^d}^+[\sigma_0] = E_{\mathbb{Z}^d}^-[\sigma_0] \iff \beta \leq \beta_c.$$

There is a *unique infinite volume limit* if and only if $\beta \leq \beta_c$.

We say there is a *first order phase transition* for $\beta > \beta_c$.

Disorder

Is this picture changed by the addition of a small random external field?

For $\omega = (\omega_x)_{x \in \mathbb{Z}^d}$ and $\varepsilon > 0$ define the disordered Hamiltonian by

$$\mathcal{H}_\Omega^{\varepsilon,\omega}(\sigma) = -\sum_{x \sim y} \sigma_x \sigma_y - \varepsilon \sum_{x \in \Omega} \omega_x \sigma_x .$$

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For $d \geq 3$, β sufficiently large and ε sufficiently small, there is a first order phase transition.

Aizenman and Wehr (1990)

For $d \leq 2$, any $\varepsilon > 0$ and almost every ω, there is a unique infinite volume limit.

Mesh refinement

Fix a bounded, simply connected domain with piecewise smooth boundary $\Omega \subset \mathbb{R}^d$.

For $a > 0$, define $\Omega_a := \Omega \cap a\mathbb{Z}^d$ and $P_\Omega^a := P_{\Omega_a}$.

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In 1984 Belavin, Polyakov and Zamolodchikov conjectured that the scaling limit of the Ising model at criticality should be conformally invariant.

Loosely, for any conformal map $\varphi : \Omega \to \Omega'$

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In 2006 Smirnov introduced fermionic observables and established conformal covariance properties that have led to much recent progress.

Ising loops and interfaces

An Ising configuration corresponds uniquely to a loop configuration in the dual graph.

Dobrushin boundary: fix two points $u, v \in \partial\Omega$ and set $\sigma_x = -1$ for x in the boundary arc (u, v) and 1 in the boundary arc (v, u).

There is a unique Ising interface from u to v - a simple curve from u to v with $+1$ on the left and -1 on the right.

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Benoist and Hongler (2016+)

Consider the critical Ising model with $+$ boundary. The set of all Ising loops converges to CLE(3).

Spin correlations

An immediate consequence of the phase transition is that

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For any $k \geq 1$ and $x_1, ..., x_k \in \Omega$ distinct,

$$\lim_{a\to 0} a^{-\frac{k}{8}} E_\Omega^{a,+}\left[\prod_{i=1}^{k} \sigma_{x_i}\right] = \mathcal{C}^k \phi_\Omega^+(x_1, ..., x_k).$$

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We have that

i) the convergence holds uniformly over $x_1, ..., x_k$ of distance at least $\epsilon > 0$ from each other and the boundary;

ii) for any conformal map $\varphi : \Omega \to \Omega'$

$$\phi_\Omega^+(x_1, ..., x_k) = \phi_{\Omega'}^+(\varphi(x_1), ..., \varphi(x_k)) \prod_{i=1}^{k} |\varphi'(x_i)|.$$

Magnetisation field

We define the magnetisation field as

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Consider the critical Ising model with $+$ boundary. The magnetisation field Φ_Ω^a converges in law to a limiting random distribution Φ_Ω.

Moreover, for a conformal map $\varphi : \Omega \to \Omega'$ with inverse $\psi : \Omega' \to \Omega$, the pushforward distribution $\varphi * \Phi_\Omega$ has the same law as the random distribution $|\psi'|^{15/8} \Phi_{\Omega'}$.

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The field can be represented as

$$\Phi_\Omega = \sum_j \eta_j \mu_j^{FK}$$

where η_j are i.i.d. signs and μ_j^{FK} are rescaled area measures associated to FK-Ising clusters.

Random field Ising model

Let $\omega = (\omega_x)_{x\in\mathbb{Z}^2}$ be i.i.d. with,

- $\mathbb{E}[\omega_x] = 0$;
- $\mathrm{Var}_{\mathbb{P}}(\omega_x) = 1$;
- $\mathbb{E}[e^{u\omega_x}] < \infty$ for all $u \in \mathbb{R}$.

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For $x \in \Omega_a$, let $\lambda^a = (\lambda^a_x)_{x\in\Omega_a}$ and $h^a = (h^a_x)_{x\in\Omega_a}$ be deterministic.

For ω fixed we define the *random field Ising model* as

$$P^{\omega,a,+}_{\Omega;\lambda,h}(\sigma) = \frac{\exp\left(\sum_{x\in\Omega_a}(\lambda^a_x\omega_{xa^{-1}} + h^a_x)\sigma_x\right)}{Z^{\omega,a,+}_{\Omega;\lambda,h}} P^{a,+}_{\Omega}(\sigma)$$

where

$$Z^{\omega,a,+}_{\Omega;\lambda,h} = E^{a,+}_{\Omega}\left[\exp\left(\sum_{x\in\Omega_a}(\lambda^a_x\omega_{xa^{-1}} + h^a_x)\sigma_x\right)\right]$$

is the *random partition function*.

Partition functions

Using a high temperature expansion,

$$Z^{\omega,a,+}_{\Omega;\lambda,h} = \cosh(\lambda^a_{\cdot}\omega_{\cdot a^{-1}} + h^a_{\cdot})^{\Omega_a} \sum_{I\subseteq\Omega_a} E^{a,+}_{\Omega}[\sigma^I_{\cdot}] \tanh(\lambda^a_{\cdot}\omega_{\cdot a^{-1}} + h^a_{\cdot})^I$$

$$\approx e^{\frac{1}{2}\left(\|\lambda^a\|^2_{l^2}+\|h^a\|^2_{l^2}\right)} \sum_{I\subseteq\Omega_a} \phi^+_{\Omega}(x^I_{\cdot}) a^{I/8} (\lambda^a_{\cdot}\omega_{\cdot a^{-1}} + h^a_{\cdot})^I.$$

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Choose $\lambda^a_x := a^{7/8}\lambda(x)$ and $h^a_x := a^{15/8}h(x)$ where $\lambda, h \in C^1(\Omega)$ are fixed with $\lambda > 0$ and $\tilde{Z}^{\omega,a,+}_{\Omega;\lambda,h} := \theta_a Z^{\omega,a,+}_{\Omega;\lambda,h}$ where $\theta_a := e^{-\frac{1}{2}a^{-1/4}\|\lambda\|^2_{L^2}}$.

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Caravenna, Sun and Zygouras (2017)

The rescaled partition function $\tilde{Z}^{\omega,a,+}_{\Omega;\lambda,h}$ converges in $\mathbb{P}$-distribution to the *Wiener chaos expansion*

$$\mathcal{Z}^{W,+}_{\Omega;\lambda,h} = 1 + \sum_{n=1}^{\infty} \frac{\mathcal{C}^n}{n!} \int \cdots \int_{\Omega^n} \phi^+_\Omega(x_1, ..., x_n) \prod_{i=1}^{n} (\lambda(x_i) W(\mathrm{d}x_i) + h(x_i)\mathrm{d}x_i)$$

where W is *white noise* and ϕ^+_Ω is the *spin correlation function*.

Magnetisation field I

Denote by $\mu^{\omega,a,+}_{\Omega;\lambda,h}$, the quenched law over the magnetisation

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Let $\varphi \in C^\infty_c(\Omega)$ and write $\varphi^a_x := a^{15/8}\varphi(x)$ then

$$E^{\omega,a,+}_{\Omega;\lambda,h}\left[\exp\left(i\left\langle \varphi, \Phi^a_\Omega \right\rangle\right)\right] = \frac{E^{a,+}_\Omega\left[\exp\left(\sum_{x\in\Omega_a}\left(\lambda^a_x\omega^a_x + h^a_x + i\varphi^a_x\right)\sigma_x\right)\right]}{Z^{\omega,a,+}_{\Omega;\lambda,h}} = \frac{Z^{\omega,a,+}_{\Omega;\lambda,h+i\varphi}}{Z^{\omega,a,+}_{\Omega;\lambda,h}}.$$

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We want to consider joint convergence of $\tilde{Z}^{\omega,a,+}_{\Omega;\lambda,h+i\varphi}$ for $\varphi \in \mathcal{C}^\infty_c$. We should have marginal limits

$$\mathcal{Z}^{W,+}_{\Omega;\lambda,h+i\varphi} = 1 + \sum_{n=1}^\infty \frac{\mathcal{C}^n}{n!} \int\cdots\int_{\Omega^n} \phi^+_\Omega(x_1,\dots,x_n) \prod_{j=1}^n (\lambda(x_j)W(\mathrm{d}x_j) + (h(x_j) + i\varphi(x_j))\,\mathrm{d}x_j).$$

Magnetisation field II

Write

$$W^{\omega,a} = a \sum_{x \in \Omega_a} \omega_{xa^{-1}} \delta_x \quad \text{and} \quad W_\psi^{\omega,a} = \langle W^{\omega,a}, \psi \rangle .$$

Then, $W^{\omega,a}$ converges in $\mathbb{P}$-distribution to white noise W.

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B, Caravenna, Sun and Zygouras (2019+)

Let $A, B \subset C_c^\infty(\Omega)$ be finite. Then, $\left((\tilde{Z}^{\omega,a,+}_{\Omega;\lambda,h+i\varphi})_{\varphi \in A}, (W_\psi^{\omega,a})_{\psi \in B} \right)$ converges in $\mathbb{P}$-distribution to $\left((\mathcal{Z}^{W,+}_{\Omega;\lambda,h+i\varphi})_{\varphi \in A}, (W_\psi)_{\psi \in B} \right)$ as $a \to 0$.

In particular, $\mu^{\omega,a,+}_{\Omega;\lambda,h}$ converges in $\mathbb{P}$-distribution to a random probability measure $\mu^{W,+}_{\Omega;\lambda,h}$ as $a \to 0$.

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In particular, $\mu^{\omega,a,+}_{\Omega;\lambda,h}$ converges in $\mathbb{P}$-distribution to a random probability measure $\mu^{W,+}_{\Omega;\lambda,h}$ as $a \to 0$.

Furthermore, this shows that $W \longmapsto \mu^{W,+}_{\Omega;\lambda,h}$ is a well defined probability map.

Relation to the case without disorder I

B, Caravenna, Sun and Zygouras (2019+)

For $\mathbb{P}$-a.e. W, the probability measure $\mu^{W,+}_{\Omega;\lambda,h}$ is singular with respect to μ^{+}_{Ω}.

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Let $\mathcal{F}_N := \sigma\left(\left\{\left\langle \Phi, \lambda^2 \mathbf{1}_{B^N_{i,j}}\right\rangle\right\}^N_{i,j=1}\right)$ where $\{B^N_{i,j}\}^N_{i,j=1}$ is a partition of Ω.

It suffices to show that for $\mathbb{P}$-a.e. W,

$$\mathcal{R}_N := \mathcal{Z}^{W,+} \frac{\mathrm{d}\mu^{W,+}}{\mathrm{d}\mu^{+}}\bigg|_{\mathcal{F}_N}$$

converges to 0 in μ^{+} probability.

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In particular, it suffices to show that

$$\lim_{N\to\infty} \mathbb{E}\left[E^+\left[(\mathcal{R}_N)^{1/2}\right]\right] = 0.$$

Relation to the case without disorder II

Using Fatou's lemma and Skorohod's representation it suffices to show that

$$\lim_{N\to\infty}\lim_{a\to 0}\mathbb{E}\left[E^{+}\left[(\mathcal{R}_N^a)^{1/2}\right]\right]=0.$$

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Let $g_N^a = g_N^a(W^{\omega,a},\Phi^a) > 0$ be such that for each fixed $W^{\omega,a}$ we have that $g_N^a \in \mathcal{F}_N$.

By the Cauchy-Schwarz inequality we have

$$\begin{aligned}
\mathbb{E}\left[E^+\left[(\mathcal{R}_N^a)^{1/2}\right]\right] &= \mathbb{E}\left[E^+\left[(g_N^a)^{1/2}(\mathcal{R}_N^a)^{1/2}(g_N^a)^{-1/2}\right]\right]\\
&\leq \mathbb{E}\left[E^+\left[g_N^a\mathcal{R}_N^a\right]\right]^{1/2}\mathbb{E}\left[E^+\left[(g_N^a)^{-1}\right]\right]^{1/2}\\
&\leq E^+\left[\mathbb{E}\left[g_N^a\tilde{Z}^{\omega,a,+}\frac{\mathrm{d}\mu^{\omega,a,+}}{\mathrm{d}\mu^{a,+}}\right]\right]^{1/2}E^+\left[\mathbb{E}\left[(g_N^a)^{-1}\right]\right]^{1/2}.
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We want to choose g_N^a such that the first term converges to 0 and the second term is bounded.

Relation to the case without disorder III

We now consider the change of measure $\tilde{\mathbb{P}}$ on the disorder defined by

$$\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}(\omega^a) = (1 + o(1))\,\tilde{Z}^{\omega,a,+}\frac{\mathrm{d}\mu^{\omega,a,+}}{\mathrm{d}\mu^{a,+}}.$$

By universality, we choose ω_x^a to be i.i.d. $N(0,1)$ with respect to $\mathbb{P}$.

Under $\tilde{\mathbb{P}}$ for σ fixed, ω_x^a are independent $N(\lambda_x^a \sigma_x, 1)$ random variables.

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For some $M_N^a, K_N^a \nearrow \infty$, we choose

$$g_N^a(W^{\omega,a}, \Phi^a) := \exp\left(-K_N^a \mathbf{1}_{\{X_N^a \geq M_N^a\}}\right)$$

where $X_N^a = X_N^a(W^{\omega,a}, \Phi^a) \approx \tilde{Z}^{\omega,a,+} \frac{\mathrm{d}\mu^{\omega,a,+}}{\mathrm{d}\mu^{a,+}}$ and belongs to $\mathcal{F}_N$ for each $W^{\omega,a}$ fixed.

Relation to the case without disorder IV

We have

$$\tilde{Z}^{\omega,a,+}\frac{\mathrm{d}\mu^{\omega,a,+}}{\mathrm{d}\mu^{a,+}} = \exp\left(\sum_{i,j=1}^{N}\sum_{x\in B_{i,j}^{N,a}}\sigma_x\lambda_x^a\omega_x^a\right).$$

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We wish to approximate σ_x by a random variable in $\mathcal{F}_N$; for $x \in B_{i,j}^{N,a}$ we choose

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Reduces the problem to showing that

$$\lim_{N\to\infty}\lim_{a\to 0}\sum_{i,j=1}^{N}N^{-15/8}\left|\sum_{x\in B_{i,j}^{N,a}}(Na)^{15/8}\lambda(x)^2\sigma_x\right| = \infty.$$

Thank you for listening

- Aizenman, M. and Wehr, J.: Rounding effects of quenched randomness on first-order phase transitions, *Comm. Math. Phys.* (1990)
- Benoist, S., Hongler, C.: The scaling limit of critical Ising interfaces is CLE(3), *arXiv preprint* (2016)
- Bricmont, J., Kupiainen, A.: Phase transition in the 3d random field Ising model, *Comm. Math. Phys.* (1988)
- Camia, F., Garban, C., Newman, C.: Planar Ising magnetization field I: Uniqueness of the critical scaling limit, *Ann. Probab.* (2015)
- Caravenna, F., Sun, R., Zygouras, N.: Polynomial chaos and scaling limits of disordered systems, *J. Eur. Math. Soc. (JEMS)* (2017)
- Chelkak, D., Duminil-Copin, H., Hongler, C., Kemppainen, A., Smirnov, S.: Convergence of Ising interfaces to Schramms SLE curves, *C. R. Math. Acad. Sci. Paris* (2014)
- Chelkak, D., Hongler, C., Izyurov, K.: Conformal invariance of spin correlations in the planar Ising model, *Ann. of Math. (2)* (2015)