Near-critical percolation with heavy-tailed impurities and forest fire processes

Pierre Nolin[1] (CityU Hong Kong)

based on j.w. with Rob van den Berg (CWI and VU, Amsterdam)

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SASI Probability Meeting, NUS

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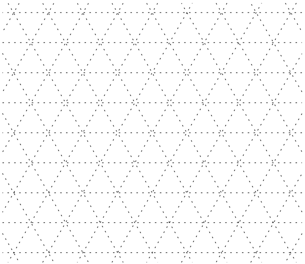
Bernoulli percolation

Model for random media: Bernoulli percolation (Broadbent, Hammersley, 1957)

Site percolation on $\mathbb{Z}^2$



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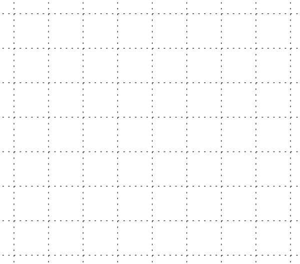

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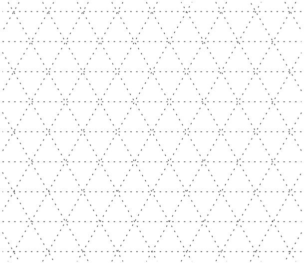

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For some parameter $p \in [0, 1]$, vertices ("sites") independently

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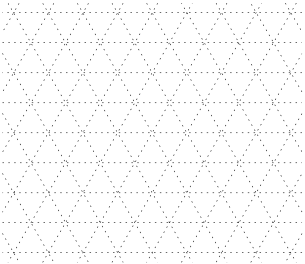

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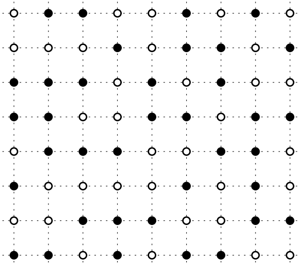

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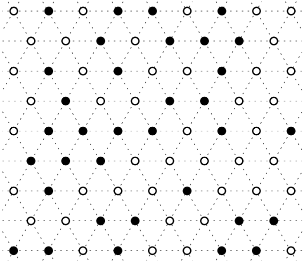

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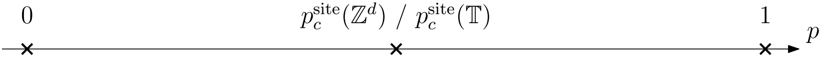


Bernoulli percolation

Percolation: phase transition as p varies

0 $\qquad$ $p_c^{\mathrm{site}}(\mathbb{Z}^d)$ / $p_c^{\mathrm{site}}(\mathbb{T})$ $\qquad$ 1 $\qquad$ p

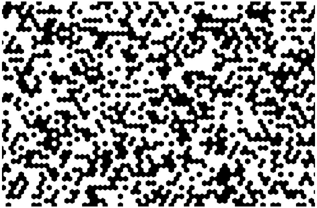
subcritical regime

no ∞ cluster

exponential decay
for cluster size

only tiny clusters

trivial large scale behavior



Bernoulli percolation

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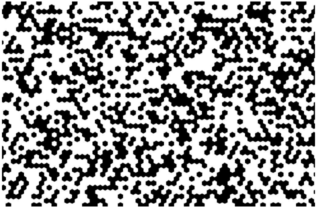
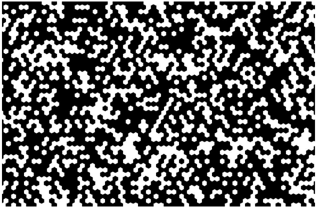
subcritical regime	*supercritical regime*
no ∞ cluster	unique ∞ cluster
exponential decay for cluster size	exponential decay for finite clusters
only tiny clusters	only tiny finite clusters
trivial large scale behavior	**trivial large scale behavior**





Forest fire processes

We consider processes on a 2D lattice ($\mathbb{Z}^2$ or $\mathbb{T}$), constructed from 2 Poisson point processes: on each vertex, **births** (rate 1) and **ignitions** (rate $\zeta > 0$, typically very small)

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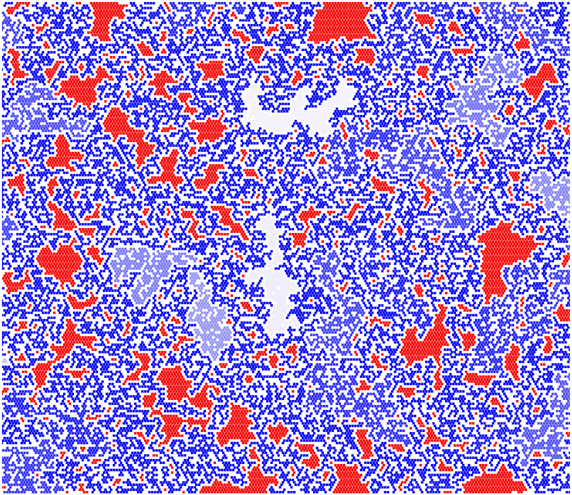
Frozen percolation



$N = 200$-volume-frozen percolation on $\mathbb{T}$

Final configuration at time $t = \infty$ (Fig. Demeter Kiss)

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  - **with recovery**: burnt vertices can become occupied again, at later birth times

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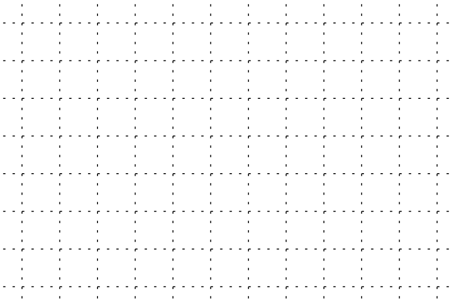
- trees appear (indep.), rate 1



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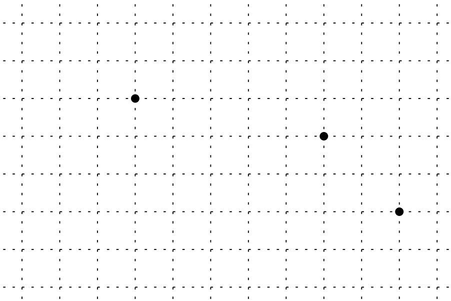

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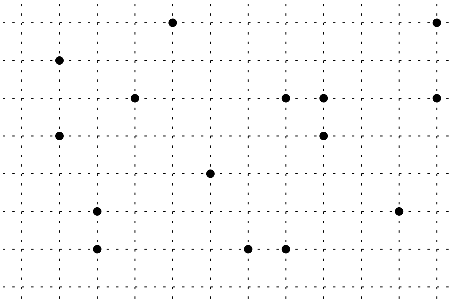

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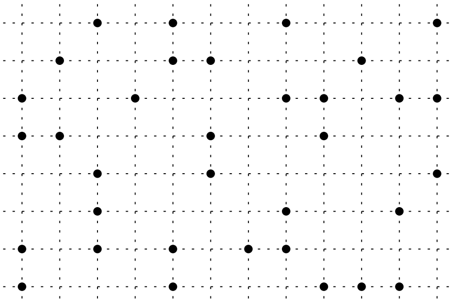

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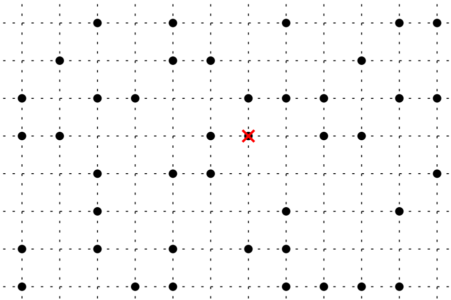

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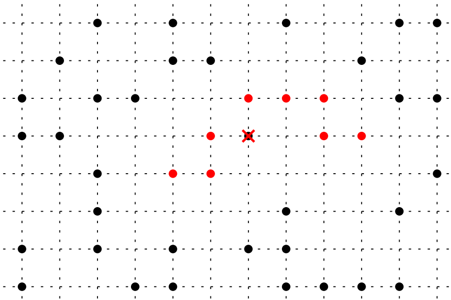

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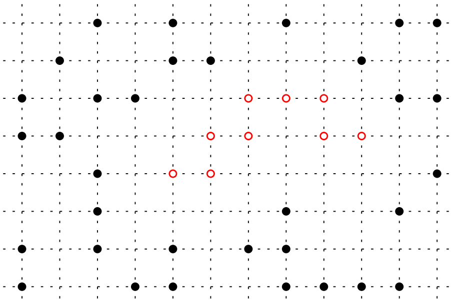

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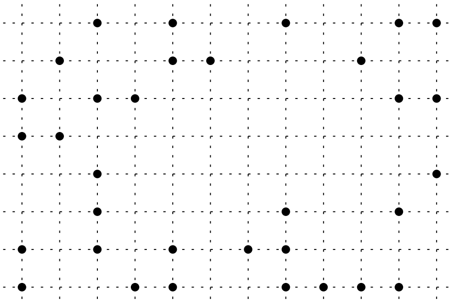

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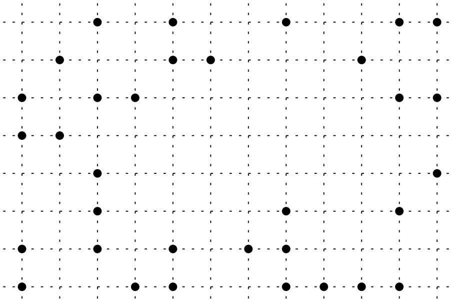

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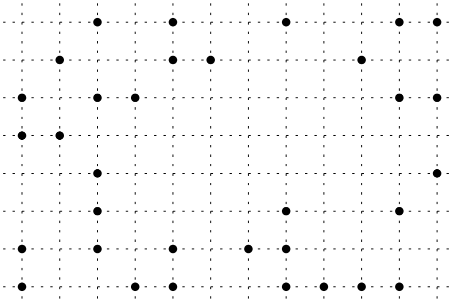

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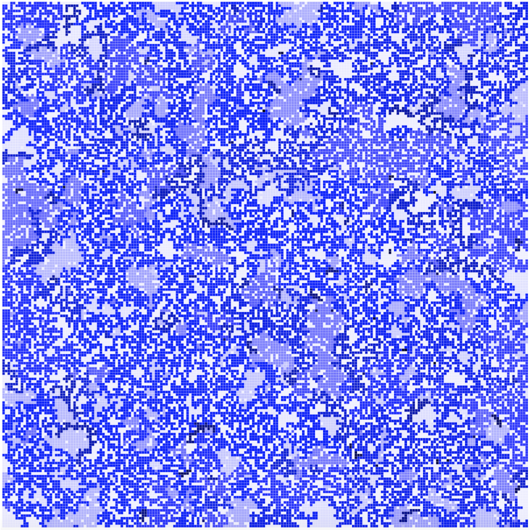
We can consider forest fire processes *with* or *without* recovery.

Forest fire processes



Forest fire process *without recovery*, rate $\zeta = 0.01$

Forest fire processes

Relevant "macroscopic" behavior starts to occur around **critical time** t_c (defined by $1 - e^{-t_c} = p_c$).

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- Here: in relation to the phase transition of Bernoulli percolation in 2D (very-well understood: Lawler, Schramm, Smirnov, Werner, 1999–2001).

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- N-volume-frozen percolation ($N \to \infty$) now well understood[1]: **deconcentration** phenomenon

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Critical regime

Percolation: phase transition as p varies

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subcritical regime	*supercritical regime*
no ∞ cluster	unique ∞ cluster
exponential decay for cluster size	exponential decay for finite clusters
only tiny clusters	only tiny finite clusters
trivial large scale behavior	**trivial large scale behavior**

critical regime

non-trivial scaling limits

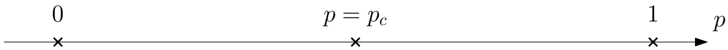
conformal invariance

connection with SLE(6)

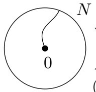
(Lawler, Schramm, Werner, Smirnov 1999-2001)

...

Near-critical regime



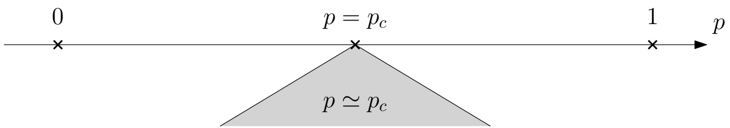
critical regime ($p = p_c$)

e.g. $\mathbb{P}_{p_c}\Bigg($ 

$\Bigg) = N^{-5/48+o(1)}$

$(N \to \infty)$

Near-critical regime



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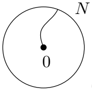
e.g. $\mathbb{P}_{p_c}\Big($  $\Big) = N^{-5/48+o(1)}$ $(N \to \infty)$

scaling relations
(Kesten 1987)

$\downarrow$

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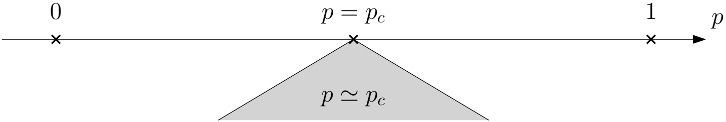
Near-critical regime



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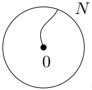
e.g. $\mathbb{P}_{p_c}\Big($ 

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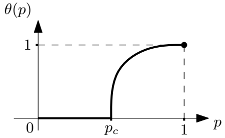
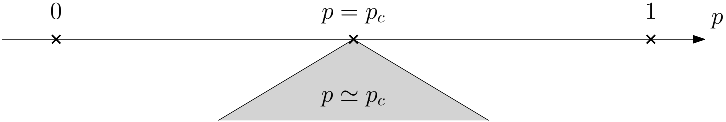


e.g. density $\theta(p) = (p - p_c)^{5/36+o(1)}$ $(p \searrow p_c)$

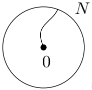
Near-critical regime



scales "below" $L(p) = |p - p_c|^{-4/3+o(1)}$ $(p \to p_c)$

(characteristic length)

critical regime $(p = p_c)$

e.g. $\mathbb{P}_{p_c}\Big($  $\Big) = N^{-5/48+o(1)}$

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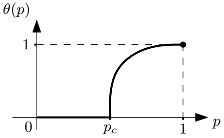

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$(p \searrow p_c)$

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Sequence of **exceptional scales**: for all $k \geq 1$,

$$m_k(\zeta) = \zeta^{-\delta_k + o(1)}, \quad \text{with } \delta_k \nearrow \delta_\infty = \frac{48}{55}$$

($\rightarrow$ highlight **non-monotonicity**, not predicted in the literature)

Forest fire processes

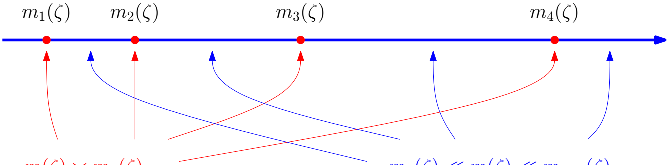
Sequence of **exceptional scales**: for all $k \geq 1$,

$$m_k(\zeta) = \zeta^{-\delta_k + o(1)}, \quad \text{with } \delta_k \nearrow \delta_\infty = \frac{48}{55}$$

($\rightarrow$ highlight **non-monotonicity**, not predicted in the literature)

Theorem (van den Berg, N., 2018)

For forest fire process without recovery, in box $B_{m(\zeta)}$: as $\zeta \to 0$,



$m(\zeta) \asymp m_k(\zeta)$

$\liminf_{\zeta \to 0} \mathbb{P}_\zeta^{B_{m(\zeta)}}(0 \text{ burns before } t) > 0$ $\quad (t > t_c)$

burning on (t_c, ∞)

$m_k(\zeta) \ll m(\zeta) \ll m_{k+1}(\zeta)$

$\mathbb{P}_\zeta^{B_{m(\zeta)}}(0 \text{ burns before } t) \underset{\zeta \to 0}{\longrightarrow} 0$

burning only near t_c

clusters in final configuration:

macroscopic (volume $\asymp \zeta^{-1}$)

microscopic (volume $O(1)$)

mesoscopic (volume $\zeta^{-\delta + o(1)}$) $\quad (0 < \delta < 1)$

Forest fire processes

- Related to behavior established earlier for volume-frozen percolation[3].

[3] van den Berg, N., *Two-dimensional volume-frozen percolation: exceptional scales*, Ann. Appl. Probab. **27**, 91–108 (2017)

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- $\rightarrow$ We have to understand the effect of these **"impurities"** on the connectedness of the lattice. (not clear that they do not perturb too much the "near-critical picture"!)

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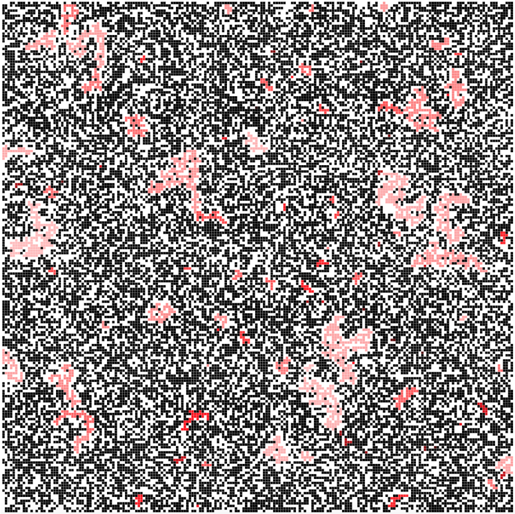

"Impurities" created by fires before time $t_c - \varepsilon \quad (\varepsilon = 0.1)$

Heavy-tailed impurities

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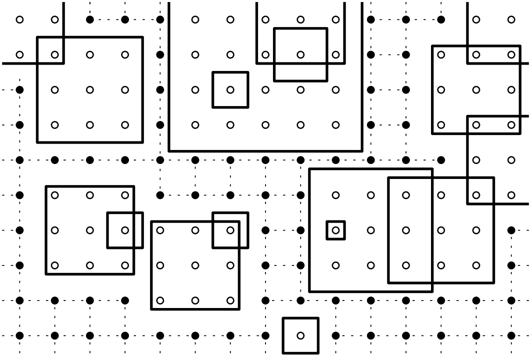
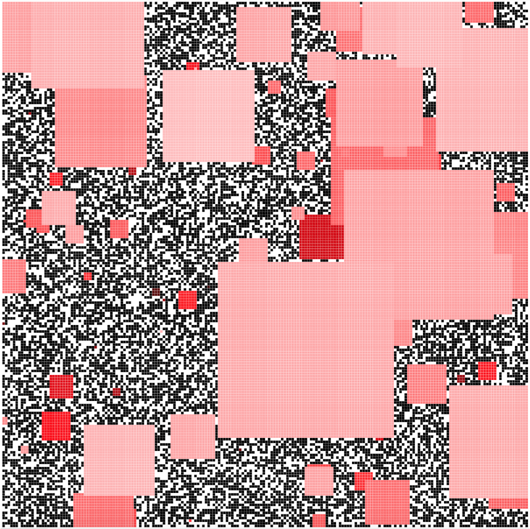


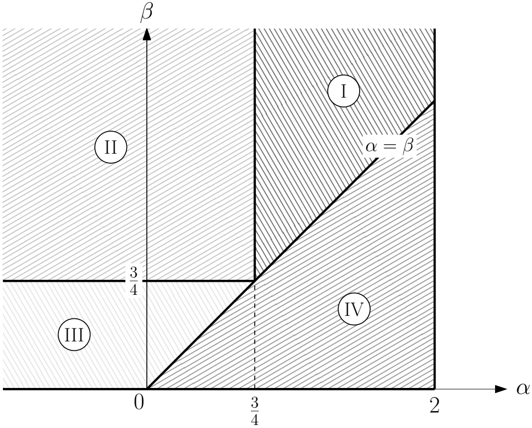
Heavy-tailed impurities

Percolation with heavy-tailed impurities: random environment



Heavy-tailed impurities

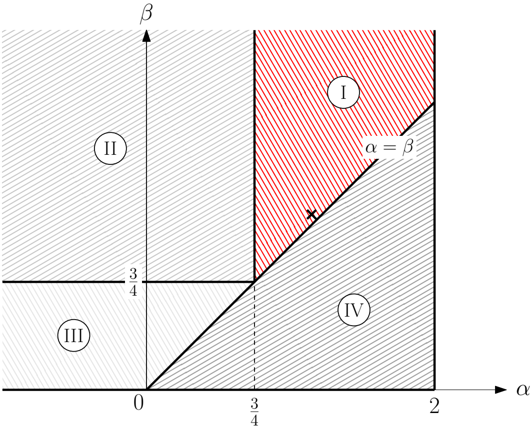
We manage to obtain the **full "phase diagram"** as α, β vary:



Heavy-tailed impurities

For forest fires, $\alpha = \frac{55}{48}$ and $\beta > \alpha$ (most interesting regime)

Note: impurities have density $m^{-(\beta-\alpha)}$, $\beta - \alpha$ arbitrarily small



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- effect on **pivotal sites**: quite subtle balance (impurities "help" vacant arm / "hinder" occupied arms)
  $\rightarrow$ relies on inequality between arm exponents

$$\alpha_4 \leq \alpha_2 + 1$$

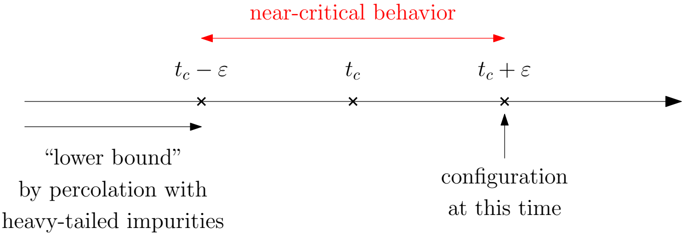
(hence, specific to $\mathbb{T}$ so far).

Forest fire processes

Forest fire process at time $t_c + \varepsilon$, in a box with side length

$$M \gg m = L(t_c - \varepsilon) \asymp L(t_c + \varepsilon)$$

(typically, $m = \hat{M}$)



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Theorem (van den Berg, N., 2019+)

For forest fire process without recovery, in full plane $\mathbb{T}$: for all $t > 0$,

$$\mathbb{P}_{\zeta}^{\mathbb{T}}(0 \textit{ burns before } t) \underset{\zeta \to 0}{\longrightarrow} 0$$

+ qualitative description of what happens right after t_c ("avalanche" of successive fires surrounding 0, more and more localized).

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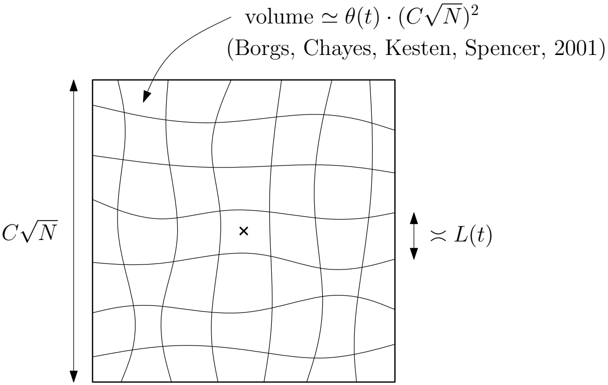

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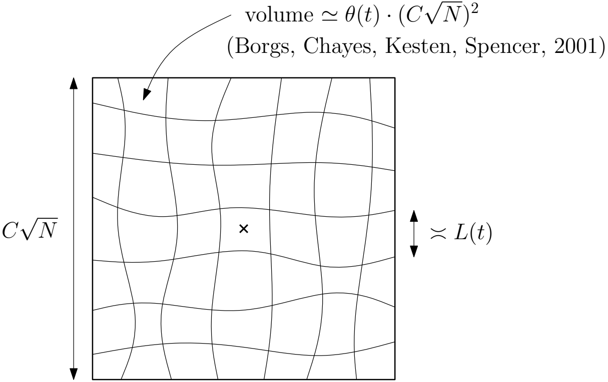

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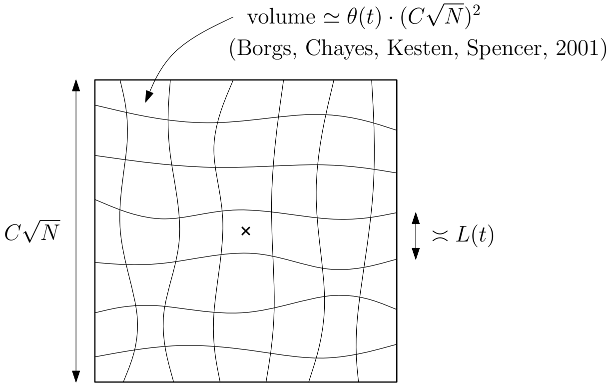

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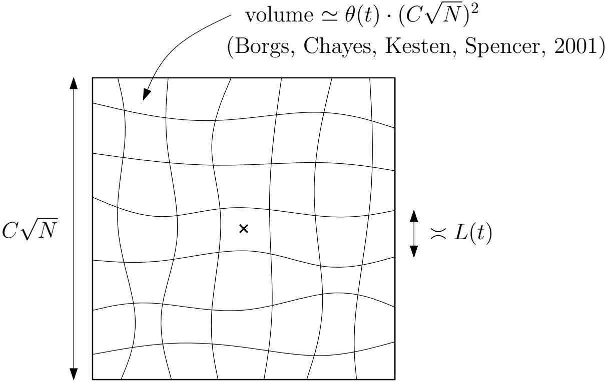

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- leaves holes with volume $\lesssim L(\bar{t})^2 \ll N$
- nothing else freezes: only 1 giant cluster freezes, "spanning" the box

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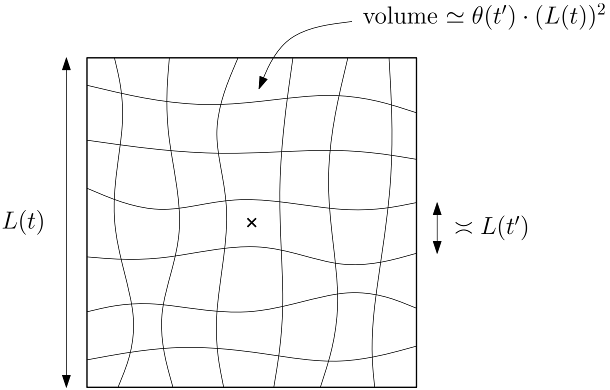
In a box with side length $m = L(t)$ ($t = t(N) \searrow t_c$): for t' just above t,



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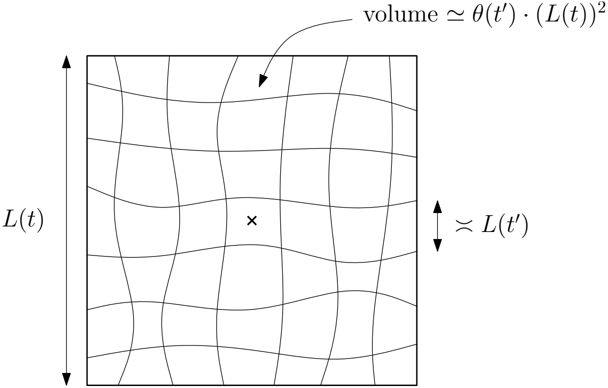

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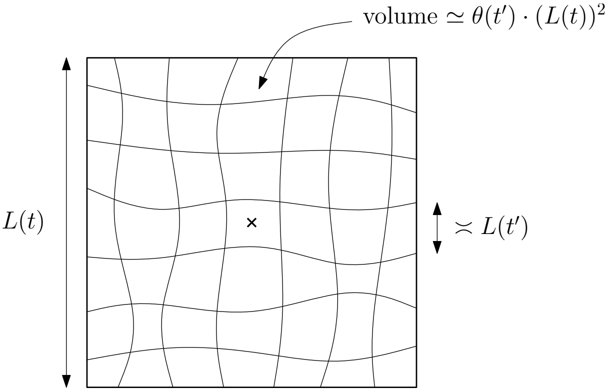

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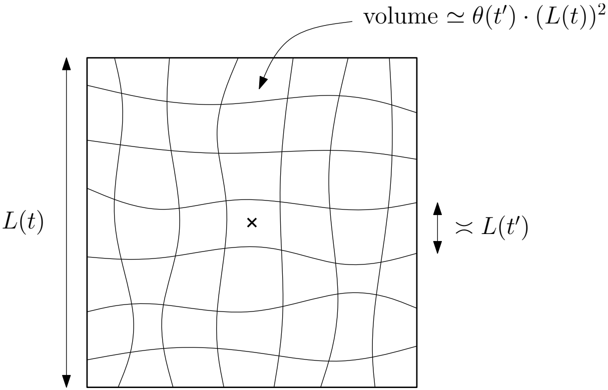

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- $\rightarrow$ next scale $\hat{m} = L(\hat{t})$.

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$\to$ condition $m^2 \pi_1(m) \ll N$, i.e.

$$m \ll m_\infty(N) = N^{\delta_\infty + o(1)}$$

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In order to make this reasoning rigorous, we use the model with impurities.

Forest fire processes

Conclusion:

- By studying **percolation with heavy tailed impurities**, we show that early fires do not perturb too much connectedness of the forest.

[4] Kiss, Manolescu, Sidoravicius, *Planar lattices do not recover from forest fires*, Ann. Probab. **43**, 3216–3238 (2015)

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- This should improve our understanding of the long-term ($t \to \infty$) behavior, but limited progress so far.

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End

Thank you!