CLT for Quasi-local Statistics for point processes with fast decay of Correlations.

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Joint works with

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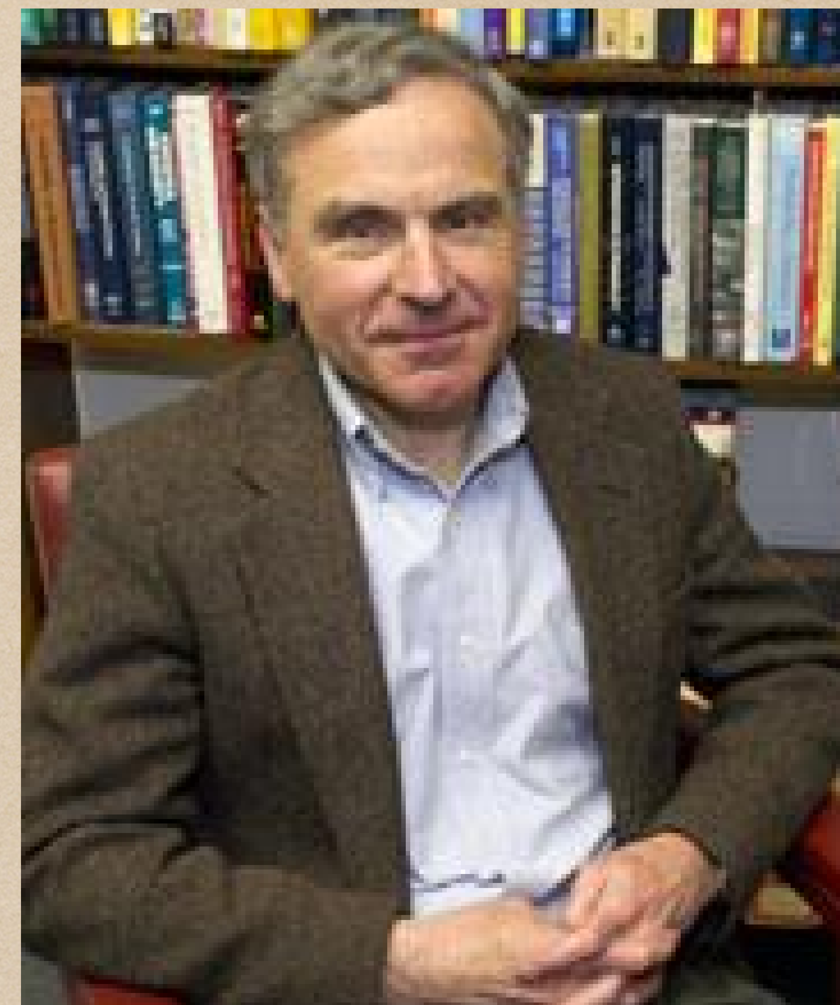
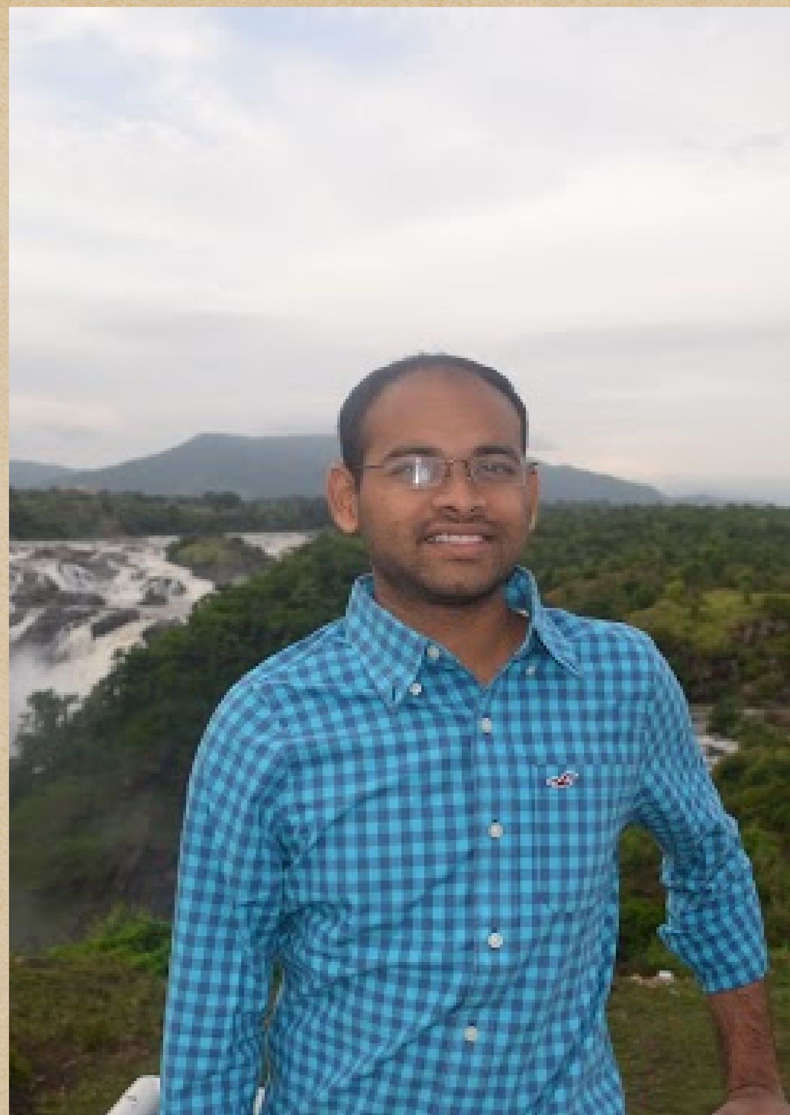
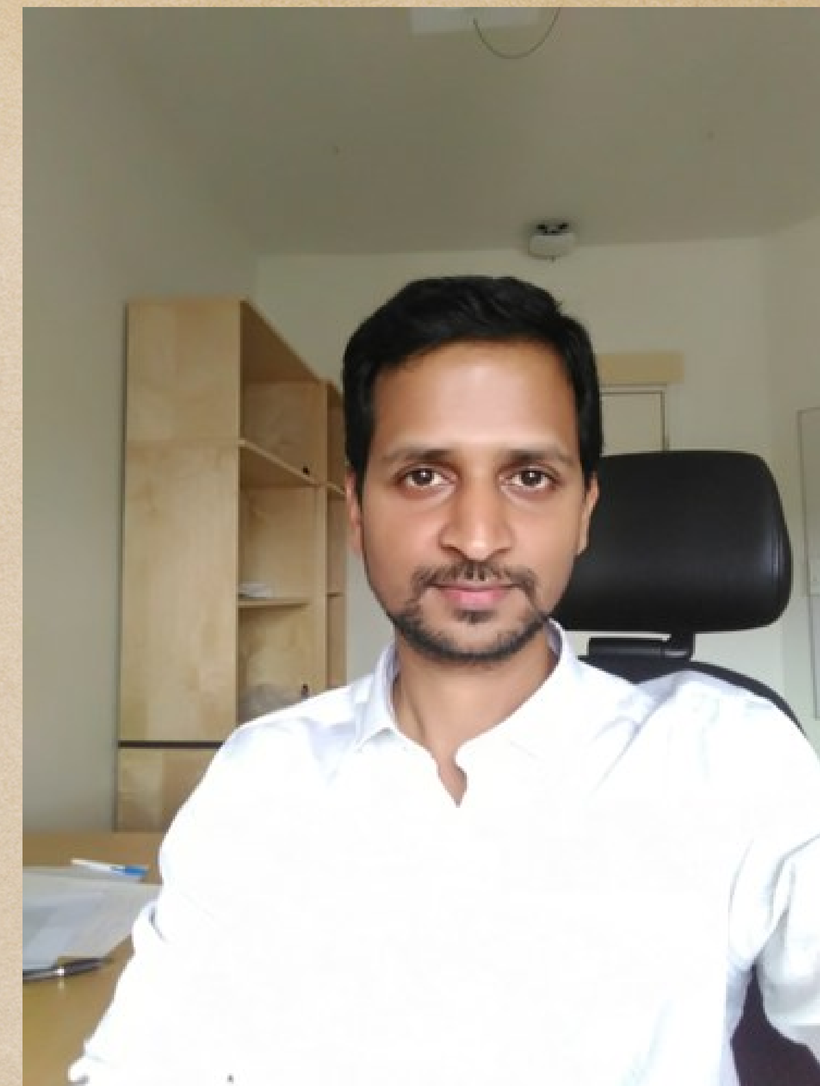


Joe Yukich

Limit theory for geometric statistics of point processes with fast decay of correlations. AOP, 2019.





Tulasi Ram Reddy

Sreekar Vadlamani

Central limit theorems for quasi-local statistics of spin models on Cayley graphs

J. STAT. PHY, 2018.

BASIC SET-UP

$S = \mathbb{R}^d$ or G, vertex transitive graph of some growth.

For this talk, will mainly focus on THE GRAPH case.

- $\mathbb{R}^d$ is technically harder. Involves Palm theory.
- Mixing, geometry of space, locality of stats are more transparent here.

VERTEX TRANSITIVE GRAPH

$G = (V, E)$ loc. finite & conn.

o – origin.

$\forall\, x \in V$, $\exists$ Graph automorphism

$\varphi_x : V \to V \ni \varphi_x(x) = o.$

$d = d_G$ = usual graph distance

Ex: Cayley graphs, Euclidean lattices

$W_n = \{ y : d(o, y) \leq n \}$

Growth: $|W_n| \leq \Delta^n$, Δ - degree

Poly. growth: $|W_n| = O(n^d)$ for some $d > 0$

For Cayley graphs, (Gromov)
poly. growth $\iff$ virtually nilpotent.

Gromov + Pansu $\Rightarrow$ $d \in \mathbb{N}$.

Point process / spin models

$$\mathcal{P} = \{X_i\}_{i \geq 1} \equiv \sum_{i \geq 1} \delta_{X_i} \; ; X_i \in V$$

$$\in \{0,1\}^V.$$

$\boxed{\text{Stationary \& non-degenerate}}$

FAST - DECAY OF CORRELATIONS:

$$\Big| \mathbb{P}(x_1, \ldots, x_{p+q} \in \mathcal{P}) - \mathbb{P}(x_1, \ldots, x_p \in \mathcal{P}) \, \mathbb{P}(x_{p+1}, \ldots, x_{p+q} \in \mathcal{P}) \Big|$$

$$\leq C_{p+q} \, \phi(c_{p+q} t) \qquad C_{p+q}, c_{p+q} \in (0, \infty)$$

$$t = d(\{x_1, \ldots, x_p\}, \{x_{p+1}, \ldots, x_{p+q}\})$$

$s^m \phi(s) \longrightarrow 0$ as $s \to \infty$ $\forall\, m \geq 1$.

(fast decreasing)

$\underline{\phi\text{-SUMMABLE:}}$ $\forall\, c > 0,\ p \geq 1$

$$\sum_{n \geq 1} |\partial W_n|\, \phi^{1/p}(cn) < \infty.$$

<u>EXPONENTIAL DECAY:</u>

$$\varliminf_{t \to \infty} t^{-b} \log \phi(t) < 0 \quad \text{for some } b > 0.$$

$b > \underline{1} \Rightarrow \phi$ - summable always.

FAST-DECAY OF CORRELATIONS

→ Weaker than most mixing conditions

→ Many $\mathbb{Z}^d$-examples (Massive Gaussian Free field, off-critical Ising model, Determinantal point process ...)

→ Many $\mathbb{R}^d$ - examples (Poisson pp, Various Cox pp, Determinantal pp, Various Gibbs pp et al ...)

→ Some Ex. in other graphs.

SCORE-FUNCTIONS

$\boxed{\text{STATISTIC}}$ $H_n := \sum_{x \in \mathcal{P} \cap W_n} \xi(x, \mathcal{P})$

ξ - Translation invariant.

LOCAL : $\xi(0, \mathcal{P}) = \xi(0, \mathcal{P} \cap W_r)$

for some $r \in \mathbb{N}$.

QUASI-LOCAL : $\xi(0, \mathcal{P}) \text{ "=" } \xi(0, \mathcal{P} \cap W_R)$ a.s.

& R has stretched sub-exponential tail.

ASSUMPTIONS

Moment conditions on ξ !!!

A1: Local score function + Summable ϕ.

A2: Quasi-local score function + Exponential decay of ϕ + Polynomial growth of G + Growth conditions on ξ

FAST-DECAY of ξ

MIXED MOMENTS:

$$m(x_1, \dots, x_p) = \mathbb{E}\left[\prod_{i=1}^{p} \xi(x_i, \mathcal{P})\right]$$

THEOREM 1: (A1) or (A2) holds. Then

$$|m(x_1, \dots, x_{p+q}) - m(x_1, \dots, x_p)\, m(x_{p+1}, \dots, x_q)|$$

$$\leq \tilde{C}_{p+q}\, \tilde{\phi}(\tilde{c}_{p+q}\, t), \quad \tilde{C}_{\circ}, \tilde{c}_{\circ} \in (0, \infty)$$

& $\tilde{\phi}$ is fast-decreasing

& $t = d(\{x_1, \dots, x_p\}, \{x_{p+1}, \dots, x_{p+q}\})$

CUMULANT METHOD & CLT

$$H_n = \sum_{x \in \mathcal{P} \cap W_n} \xi(x, \mathcal{P})$$

THEOREM 2: (A1) or (A2) holds

& $\mathrm{Var}(H_n) = \Omega(n^{\alpha})$, $\alpha > 0$

Then

$$\frac{H_n - \mathbb{E} H_n}{\sqrt{\mathrm{Var}\, H_n}} \Longrightarrow N(0,1)$$

Fast decay of correlations

$\Longrightarrow$ Decorrelation of mixed moments.

$\boxed{\text{THM 1}}$

$\Downarrow$ $\boxed{\text{THM 2}}$

bounds on cumulants ($\equiv$ Brillinger mixing)

$\swarrow$ $\boxed{\text{THM 2}}$

CLT

PROOF IDEA OF THEOREM 2 (mainly $\mathbb{R}^d$ or $\mathbb{Z}^d$)

Malyshev (1975), Martin & Yalcin (1980), [$\xi \equiv 1$]

Baryshnikhov & Yukich (2005), [Poisson point process]

Schreiber & Yukich (2012),

Nazarov & Sodin (2012). [$\xi \equiv 1$]

APPLICATIONS TO UNIMODULAR RANDOM GRAPHS

$\mathcal{P} \subseteq \mathbb{S}$, point process ; $\mathbb{S} = \mathbb{R}^d$ or G

$x_i \sim x_j$ if $|x_i - x_j| \leq R(x_i, \mathcal{P}) \vee R(x_j, \mathcal{P})$

$R(\cdot, \cdot)$ – Stationary & exp. tail.

$\xi(x_i, \mathcal{P}) = \xi(x_i, \mathcal{P} \cap Nbhd(x_i))$

OUR RESULTS : $\mathcal{P}$ – fast decay of correlations, $\mathbb{S}$ – 'suitable growth' & variance lower bds

$\Rightarrow$ cumulant bounds $\Rightarrow$ CLT.

SPECIFIC EXAMPLES

→ Nearest neighbour graphs

→ Random geometric graphs

→ Delaunay graphs

CAUTION ! ! !

→ Moment & growth conditions ✓✓✓

→ Variance lower bound ? ? !

In $\mathbb{R}^d$, Beck '87, Nazarov-Sodin '12, ...

PROOF SKETCH OF THEOREM 1

Ψ - fnl of point processes.

$$D_x \Psi(\mu) = \Psi(\mu|_x + \delta_x) - \Psi(\mu|_x)$$

$$\mu|_x(\cdot) = \mu(\cdot \cap W_{<x})$$ $<$ - "nice" total order.

DIFFERENCE OPERATORS:

$$x_1 < x_2 < \cdots < x_k$$

$$D^k_{x_1,\dots,x_k} \Psi(\mu) = D_{x_k}\left(D^{k-1}_{x_1,\dots,x_{k-1}} \Psi(\mu)\right)$$

For any $x_1, \dots, x_k$, order as above & apply the defn.

FACTORIAL MOMENT EXPANSION

Blaszczyszyn, Merzbach & Schmidt (1997).

$$\mathbb{E}\,\Psi(\mathcal{P}) = \Psi(\phi) + \sum_{l=1}^{\infty} \frac{1}{l!} \sum^{*}_{y_1,\dots,y_l} D^{l}_{y_1,\dots,y_l}(\phi) \times \mathbb{P}(y_1,\dots,y_l \in \mathcal{P})$$

Idea of Proof (THEOREM 1):

→ Use $\Psi(\mathcal{P}) = \prod_{i=1}^{p} \xi(x_i, \mathcal{P})$

→ If $\Psi(\mathcal{P}) = \Psi(\mathcal{P} \cap B)$ then

$$D_y \Psi(\mathcal{P}) = 0 \quad \forall\, y \notin B.$$

→ Finite Series expansion in FME for local score.

→ $\psi(x_1, \ldots, x_{p+q}; \mathcal{P}) = \prod_{i=1}^{p+q} \xi(x_i, \mathcal{P})$

$\psi(x_1, \ldots, x_{p+q}; \mathcal{P}) = \psi(x_1, \ldots, x_p; \mathcal{P})\, \psi(\ldots; \mathcal{P})$

for ξ local & $x_1, \ldots, x_p <<< x_{p+1}, \ldots, x_{p+q}$

→ For Quasi-local, truncation + FME for truncated local score.

EXTENSIONS (on-going with BB & JY)

MARKING

$\xi(X_i, Y_i, \mathcal{P})$ $\rightarrow$ extra randomness

Can depend on $\mathcal{P}$ locally.

HIGH - POLYNOMIAL STABILIZATION.

if $\mathrm{Var}(H_n) = \Omega(n)$.

POISSON APPROXIMATION

SOME QUESTIONS

Q. Point processes on more general spaces (quasi-transitive / unimodular graphs, lcsc topological groups)

Q. Potential connections to Ergodic theory ?!!

Q. Mixing for random graphs & applications ??

CUMULANT METHOD IN ERGODIC THEORY

Group G acts on a prob. space X in a measure preserving manner

CLT for $\int_{W_n} \phi(g^{-1}x)\, dg$, $\phi \in C_c^\infty(X)$

A. GORODNIK : HIGHER-ORDER CORRELATIONS FOR GROUP ACTIONS.

Björklund, Cohen, Conze et al.

CLTs for Other Unimodular random graphs

Cumulant method CLTS.

V. Feray (2018) - Statistics of Erdös-Renyi graphs

S. Janson (2019) - statistics of Configuration model.

Proofs do not seemingly involve unimodularity!