From Hotelling to Nakamoto: The Economic Meaning of Bitcoin Mining

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jointly with
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Agenda

Introduction

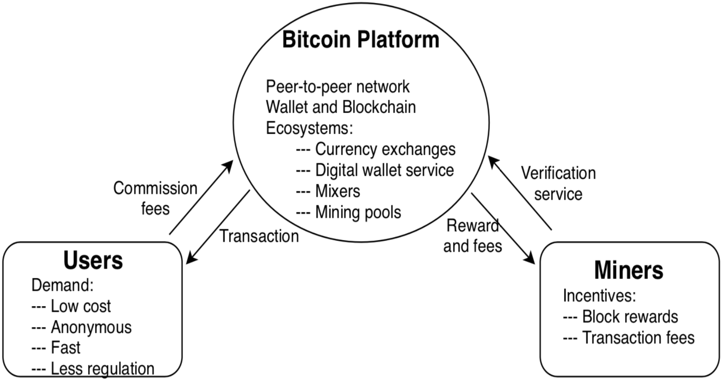
The Model

Calibration

Quantitative Analysis

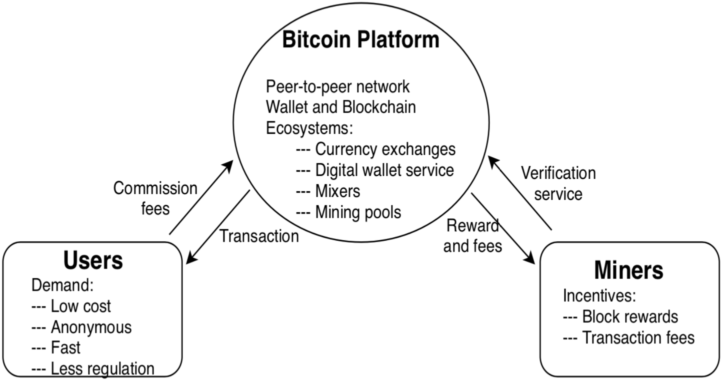
Conclusion

Bitcoin System



Basic ingredients: (a) Users, and (b) Miners

Bitcoin System



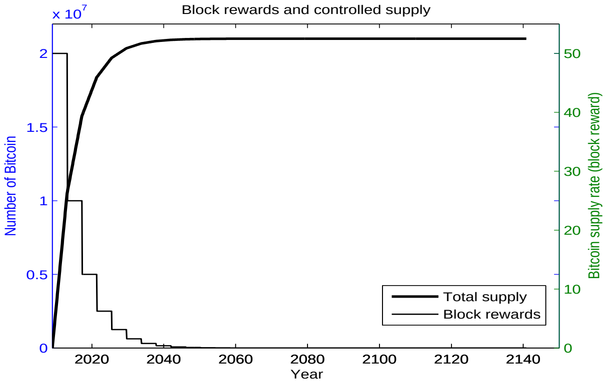
Basic ingredients: (a) Users, and (b) Miners
What is the economic meaning of Bitcoin mining?

Mining Business

Mining business consists of

- Revenue:
  - Block Bitcoin rewards (deterministic, exogenously given by the system, and are vanished after 2140)
  - Transaction fees in Bitcoin (stochastic, endogenously determined by the system)
- Cost:
  - Runing costs (e.g., mining machines, electricity, etc.)
  - Liquidation costs
  - Others
- Risk/Uncertainty:
  - Mining lottery (strong competition, low chance)
  - Exchange rate (Bitcoin/USD, extremely volatile due to adoption, policy uncertainty etc.)

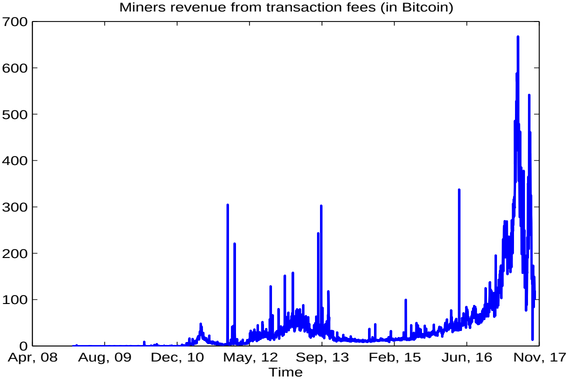
Block Rewards



Block rewards: Deterministic, Exogenous, and Scarce

Scarcity $\Longrightarrow$ Bitcoin is an exhaustible resource!

Transaction Fees



Transaction fees: Stochastic, Endogenous, and Unlimited
Key incentive to miners after the end of block rewards

Stylized Facts: Exchange Rate & Average Transaction Fee Rate

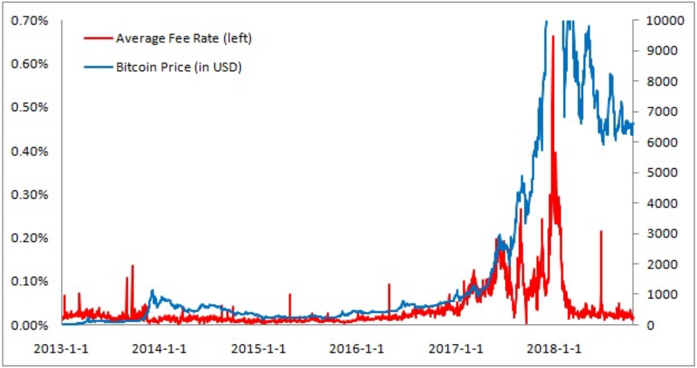


Figure: The dynamics of average transaction fee rate and Bitcoin price from 2013 to 2018.

Average fee rate at t $= \frac{\text{Total transactin fees at t}}{\text{Processed transaction volume at t}}$.

Stylized Facts: Miner's Inventory

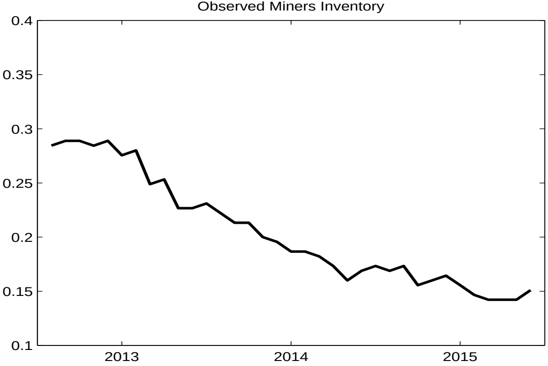


Figure: Miner's inventory proportional to the supply from 2012 to 2015 (Athey et al. 2016).

Propotional inventory $= \frac{\text{Miners' aggregate inventory at time t}}{\text{Cumulative Bitcoin supply at time t}}$.

Our Main Results

- We build a partial equilibrium model from miners' perspective by extending the classical Hotelling (1931) model with inventory and feedback supply.
- We calibrate our model to the empirical data from 2013 to 2018.
- Our model has many interesting implications including
  - A high (low) trading volume leads to a high (low) transaction fee rate.
  - High jump risk forces miners to sell their holding of Bitcoin in an early stage even when Bitcoin price is quite low.

Literature Review

- Model on transaction fees:

Easley, O'Hara, and Basu(2019, JFE)	One period	Nash equilibrium of users' fee paying strategy
Our model	Continuous time dynamic model	Transaction fees from miner's perspective incorporating declined block rewards and miners' inventory

- Resource models: Hotelling (1931, JPE); Levhari and Pindyck (1981, QJE); Pindyck (2001);
- Bitcoin as currency: Athey et al. (2016); Gandal and Halaburda (2015); Halaburda and Sarvary (2016); Bolt et al. (2016); Jermann (2018).
- Others: Cong, He, and Li (2018); Dixon (1980); Bass (2004).

A Resource Production Model

Originating from Hotelling (1931, JPE), resource mining problem can be written in general as:

$$\sup_{Q_u \geq 0} \mathbb{E}_t \Big[\int_t^{\infty} e^{-\beta\,(u-t)} \Big(\mathrm{Rev}(Q_u) - \mathrm{Cost}(Q_u) \Big) du \Big]$$

where $\beta > 0$ is a discount factor, and

- $\mathrm{Rev}(Q_u) = P_u Q_u$
- $\mathrm{Cost}(Q_u) = \lambda_1 P_u Q_u^2$ [liquidation] $+ \lambda_2 P_u Q_u^2 / H_u$ [utlity] $+ c$ [running]
- Q : Miner's selling rate
- P : Bitcoin Price
- H : Holding Inventory

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- Q : Miner's selling rate
- P : Bitcoin Price $P_t = \theta_p X_t$, where

$$dX_t = \mu(\xi_t, X_t)dt + \sigma(\xi_t, X_t)dW_t - (1-Z)X_t dJ_t$$

- H : Holding Inventory

$$dH_u = \{(b_u[\text{block}] + I_u[\text{transaction}])\pi[\text{probability}] - Q_u\}du$$

Modeling Bitcoin Price

- Bitcoin price satisfies an inverse demand function.
- Bitcoin price is determined by quantity equation of medium of exchange (Bolt et al. 2016, WP; Fisher 1911; Friedman 1973):

$$P_t = \theta_p X_t.$$

where the constant θ_p is determined by Bitcoin supply and velocity.

Modeling Demand Shock

Demand shock (Bass 2004; Gronwald 2015; Gandal et al. 2018):

$$dX_t = \mu(\xi_t, X_t)dt + \sigma(\xi_t, X_t)dW_t - (1 - Z)X_t dJ_t$$

where

- $\xi_t \in \{\mathbb{H}, \mathbb{L}\}$ represent two transaction states: High-active/Low-active markets, with transition intensities $\zeta = (\zeta_{\mathbb{H}}, \zeta_{\mathbb{L}})$.
- $\mu(\xi_t, X_t) = \kappa_\xi(\nu_\xi - \ln X_t)X_t$, and $\sigma(\xi_t, X_t) = \sigma_\xi X_t$ denote the adoption term and volatility term respectively in state ξ_t (Gompertz Model).
- J_t is a jump process with intensity λ_J, and $1 - Z$ is the proportional jump size (Weil 1987, QJE).

Miner's Inventory

Miner's inventory H_t satisfies

$$dH_t = [(b_t + I_t)\pi - Q_t]dt,$$

- $\pi = \frac{\omega}{D \times 2^{32}/600}$ is the probability of successful validations and D is the difficulty level (Hayes 2017).
- b_t is the block reward at t with total supply $\bar{S} = \int_0^\infty b_t dt = \int_0^T b_t dt < \infty$
- I_t is the transaction fees in candidate blocks at t.

Note. $D \times 2^{32}/600$ is also called network hash rate.

Modeling Transaction Fees

- Total volume of submitted orders by others:

$$L_t = \theta(\xi_t)(S_t - H_t)\log(1 + X_t) \text{ with } \theta(\xi_t) \in \{\theta_{\mathbb{H}}, \theta_{\mathbb{L}}\}.$$

- The distribution of orders with different fee rate:

$$f(\phi), \quad \phi \in (0, \bar{\phi}) \quad \text{with C.D.F.} \quad F(\phi).$$

- Each time, a fixed number of orders G will be processed by miners.

Transaction Fees

- The miner selects fee threshold $\mathit{\Phi}_t$ to solve

$$\max_{\mathit{\Phi}_t} I_t(\mathit{\Phi}_t) = K(\mathit{\Phi}_t)L_t$$
$$s.t. \quad k(\mathit{\Phi}_t)L_t \leq G,$$

where $k(\mathit{\Phi}_t) = \int_{\mathit{\Phi}_t}^{\bar{\phi}} f(\phi)d\phi$, and $K(\mathit{\Phi}_t) = \int_{\mathit{\Phi}_t}^{\bar{\phi}} f(\phi)\phi d\phi$.

- Optimal fee threshold satisfies:

$$\Phi_t^* = \begin{cases} F^{-1}(1 - \frac{G}{L_t}), & \text{if } L_t > G, \\ 0 & \text{if } L_t \leq G, \end{cases}$$

- The miner's average transaction fee rate:

$$r_t = \frac{K(\mathit{\Phi}_t^*)}{k(\mathit{\Phi}_t^*)}.$$

Average transaction Fee Rate

Proposition

1. *In state ξ, for demand level lower than $G/\theta(\xi)$, the average transaction fee rate is constant $K(0)/k(0)$. For demand level higher than $G/\theta(\xi)$, the average transaction fee rate is an increasing function of demand.*
2. *The above results hold for the market average transaction fee rate (aggregation).*

HJB Equation

- Short-run case: $t < T$, there are block rewards.
- In state ξ, for $(t, X_t, H_t) = (t, x, h) \in (0, \infty)^2 \times [0, S(t)]$,

$$\frac{\partial V_\xi}{\partial t} + \mathcal{L}V_\xi + \max_{\{q \geq 0\}} \left\{ (\pi(b_t + K(\phi)L) - q) \frac{\partial V_\xi}{\partial h} + Pq - \lambda_q P q^2 - c \right\}$$
$$+\lambda_J \left[V_\xi(t, Zx, h) - V_\xi(t, x, h) \right] + \zeta_\xi \left[V_{\tilde{\xi}}(t, x, h) - V_\xi(t, x, h) \right] = \beta V_\xi$$

where

$$\mathcal{L}V_\xi = \frac{1}{2}\sigma(\xi, x)^2 \frac{\partial^2 V_\xi}{\partial x^2} + \mu(\xi, x)\frac{\partial V_\xi}{\partial x}.$$

- Long-run case: $b_t = 0$ for $t \geq T$
- $V_\xi(t, X, H) = V_\xi(T, X, H) := V_\xi^L(X, H)$ for any $t \geq T$.

Optimal Selling Strategies

- In state ξ, optimal inventory strategy q_ξ^* satisfies:

$$q_\xi^* = \max\left\{\frac{h}{2P(\lambda_1 h + \lambda_2)}\left(P - \frac{\partial V_\xi}{\partial h}\right), 0\right\}$$

Optimal Selling Strategies

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- **Holding / Selling regions:**
  - **Selling region:**

$$\{q_\xi^* > 0\} = \{P > \frac{\partial V_\xi}{\partial h}\}$$

  - **Holding region:**

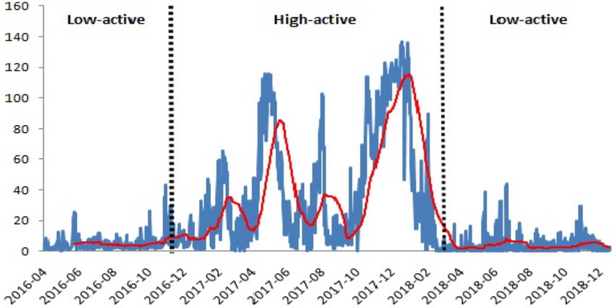
$$\{q_\xi^* = 0\} = \{P \leq \frac{\partial V_\xi}{\partial h}\}$$

Calibration: Data

- Observed data (Monthly from 2013-2018. source: https://www.blockchain.com):
  - Bitcoin price $\{P_t\}$
  - Difficulty level $\{D_t\}$
  - Miners' aggregate inventory $\{H_t^A\}$ from 2013 - 2015,
  - Market average fee rate: $\{r_t^A\}$
  - Aggregate transaction fees: $\{I_t^A\}$
- Bitcoin prices are informative to parameters $\Theta_1 = \{\kappa, \nu, \sigma_{\mathbb{H}}, \sigma_{\mathbb{L}}\}$.
- Miners' aggregate inventory, average fee rate, and aggregate fee income are informative to parameters $\Theta_2 = \{\lambda_1, \lambda_2, \theta_{\mathbb{H}}, \theta_{\mathbb{L}}\}$.

High-active/low-active Market

- Detect the high-active and low-active market by Mempool size (High-active: mempool size > 10 MB).
- Low-active: 2013Q1-2016Q3; High-active: 2016Q4-2017Q4; Low-active: 2018Q1-201018Q4



Note. Red line is the 60-day moving average.

Calibration Method

- **Step 1:** Set $\beta = 0.06; \bar{S} = 1; G = 10; \theta_p = 100;$ $\lambda_J = 57; Z = 0.9$. The $f(\cdot)$ satisfies Beta distribution with parameters $(a, b) = (0.1, 99.9)$.
- **Step 2:** Estimate $\Theta_1 = (\kappa, \nu, \sigma_{\mathbb{H}}, \sigma_{\mathbb{L}})$ with Bitcoin price data.
- **Step 3:** Given $\Theta_2 = (\lambda_1, \lambda_2, \theta_{\mathbb{H}}, \theta_{\mathbb{L}})$ and observed Bitcoin price, we can compute the path of demand shock $\{\widetilde{X}_t; t = 1, \cdots, T_1\}$. For miners start to mine in year $y \in (2013, \cdots, 2018)$, we can compute the
    - implied transaction fees $\{\widetilde{I}_{y,t}; t = 1, \cdots, T_1\}$,
    - implied average fee rate $\{\widetilde{r}_{y,t}; t = 1, \cdots, T_1\}$.
    - implied inventory $\{\widetilde{H}_{y,t}; t = 1, \cdots, T_2\}$,

Calibration Method

- **Step 3 (continue):** Compute
  - implied aggregate transaction fees $\{\widetilde{I^A}_t; t = 1, \cdots, T_1\}$.
  - implied market average fee rate $\{\widetilde{r^A}_t; t = 1, \cdots, T_1\}$;
  - implied aggregate inventory $\{\widetilde{H^A}_t; t = 1, \cdots, T_2\}$;

We estimate $\widehat{\Theta}_2$ by minimizing:

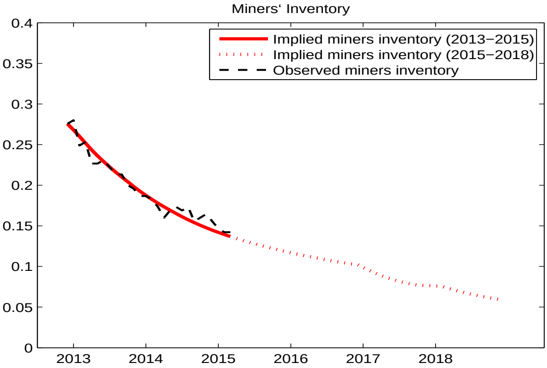
$$\min_{\Theta_2} \frac{1}{T_1} \sum_{1}^{T_1} \left\{ w_t^1 (r_t^A - \widetilde{r^A}_t)^2 + w_t^2 (I_t^A - \widetilde{I^A}_t)^2 \right\}$$
$$+ \frac{1}{T_2} \sum_{1}^{T_2} \left\{ w_t^3 (H_t^A - \widetilde{H^A}_t)^2 \right\}$$

where w_t^1, w_t^2, w_t^3 are the weight coefficients.

Summary of Parameters

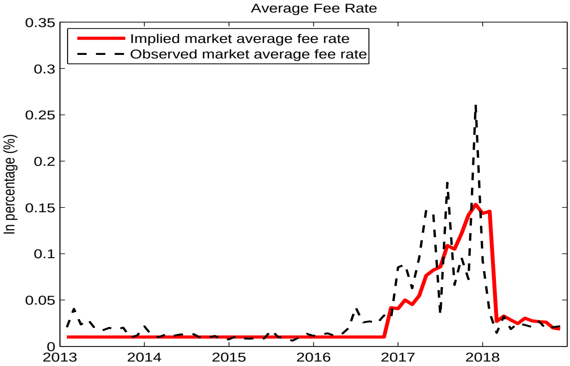
Parameters	Symbol	Value
Risk-free rate	β	0.06
Total supply of Bitcoin	$\bar{S}$	1
Capacity of blocks per unit of time	G	10
Hash rate per miner (TH/s)		5.2
Coefficient in quantity equation (Billion USD per unit)	θ_p	100
Upper bound of fee rate	$\bar{\phi}$	10%
Beta distribution parameters	(a, b)	(0.1, 99.9)
Adoption speed of Bitcoin	κ	1.1742
Log carrying capacity	ν	0.7793
Volatility of demand shock in high-active market	$\sigma_{\mathbb{H}}$	0.7910
Volatility of demand shock in low-active market	$\sigma_{\mathbb{L}}$	0.6225
State transition intensity	$(\zeta_{\mathbb{H}}, \zeta_{\mathbb{L}})$	(0.8, 0.3)
Jump parameters	(λ_J, Z)	(57, 0.9)
parameter in liquidation cost	λ_1	4.5
parameter in utility cost in liquidation	λ_2	0.6
Sensitivity of volume to demand in high-active market	$\theta_{\mathbb{H}}$	251.3
Sensitivity of volume to demand in low-active market	$\theta_{\mathbb{L}}$	30.6
Marginal cost of mining (Billion USD per TH/s)	C_m	3.61×10^{-7}

IMPLIED INVENTORY

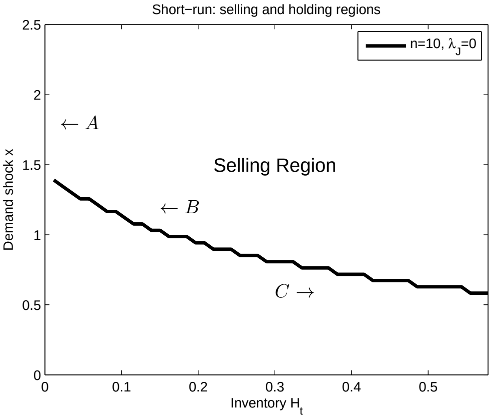


Note. Proportional inventory.

Implied Average Fee Rate



Selling boundary in short-run



Note. Bull market. $t = 2014$. $H_t \in [0, S_t], S_t = 0.5871. n = 1/\pi$.

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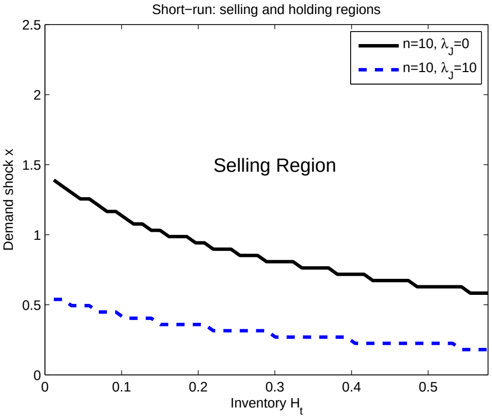

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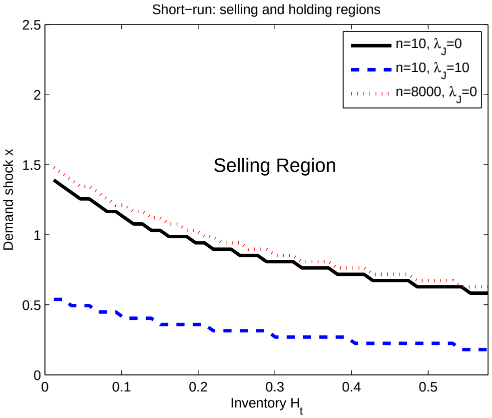

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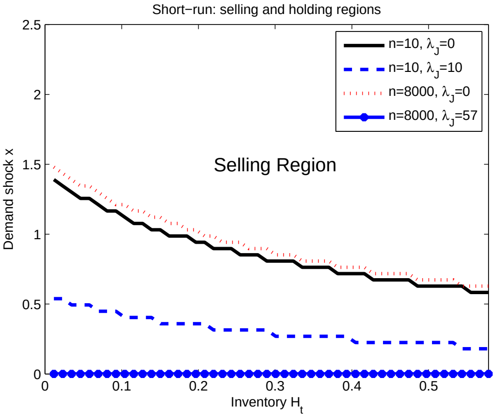

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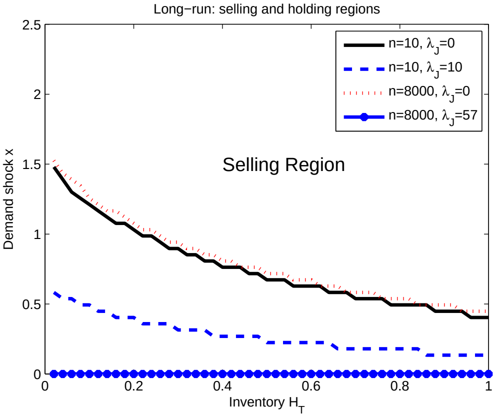
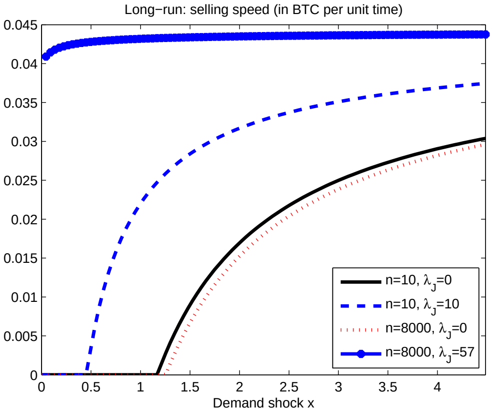
Selling boundary in long-run



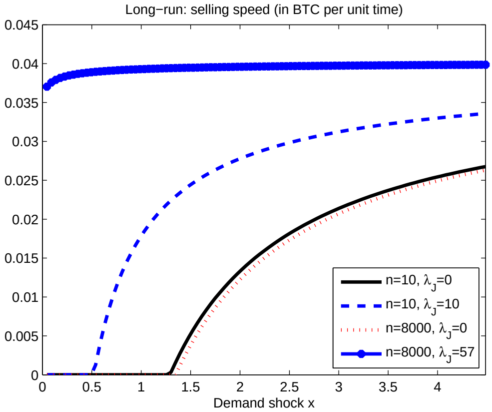
Note. Bull market.

Short-run: Optimal Selling



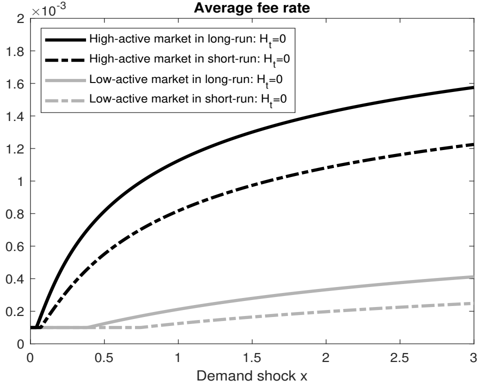
Note. Bull market. $t = 2014$. We fix the miner's holding to be $H_t = 0.1$. $n = 1/\pi$.

Long-run: Optiomal Selling



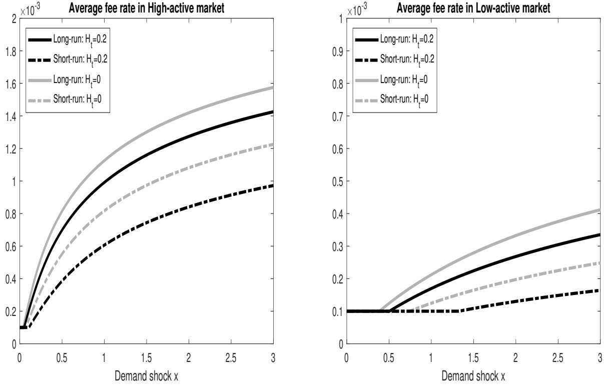
Note. Bull market. We fix the miner's holding to be $H_T = 0.1$. $n = 1/\pi$.

Average fee rate to Demand



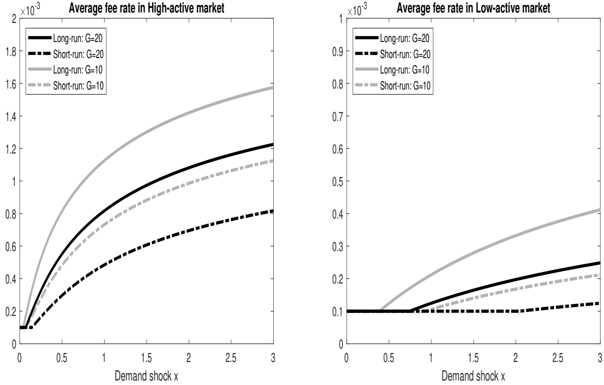
Note. $H = 0$.

Average fee rate to Inventory



Note. Here we assume short-run case is about at $t = 2014$. Left figure shows the average fee rate under different inventory for both long-run and short-run in high-active market, while the right figure shows that in low-active market.

Average fee rate to Capacity



Note. Here we assume short-run case is about at $t = 2014$. Left figure shows the average fee rate under different system capacity for both long-run and short-run in high-active market, while the right figure shows that in low-active market.

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Conclusion

- We build a partial equilibrium model from miners' perspective by extending the classical Hotelling model with inventory and feedback supply, and calibrate our model to the empirical data from 2013 to 2018.
- The model can simultaneously generate the dynamics of average transaction fee rate and miners' inventory holdings consistent with the observed data.
- We find trading volume and jump risk are respectively key factors to understand the dynamics of average transaction fee rate and miners' inventory holdings.

Thanks for your attention!