Optimal liquidation in spite of increasing prices

How is optimal execution affected by price trends?

Peter Frentrup

(based on joint work with Dirk Becherer and Todor Bilarev)

Singapore – 18 March 2019

How to execute large trade in face of trending prices?

- Problem: How to execute/liquidate a position of θ risky assets until a given finite time $T < \infty$ optimally?
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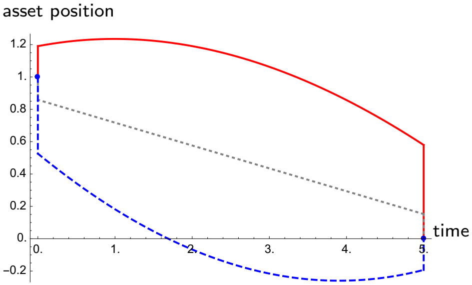

Positive asset prices with transient price impact

- Liquidate θ assets by selling/buying continuously or in blocks: bounded variation càdlàg strategy Θ_t, $t \in [0, T]$ with $\Theta_{0-} = \theta$, $\Theta_T = 0$.

- Unaffected price: $\overline{S} = e^{\mu t}\mathcal{E}(\sigma W)_t, \quad \mu \in \mathbb{R}$.

- Affected price: $S_t := f(Y_t)\overline{S}_t$ for price impact process

$$\mathrm{d}Y_t = -h(Y_t)\,\mathrm{d}t + \mathrm{d}\Theta_t, \qquad Y_{0-} = y,$$

resilience function $h(0) = 0$, $h' > 0$, e.g. $h(y) = \beta y$, $\beta > 0$.
impact function $f, f' > 0$, e.g. $f(y) = e^{\lambda y}$, $\lambda = f'/f > 0$ const.

- Maximize expected trading gains $\mathbb{E}[L_T(\Theta)]$,

$$L_T(\Theta) := -\int_0^T f(Y_t)\overline{S}_t\,\mathrm{d}\Theta_t^c$$

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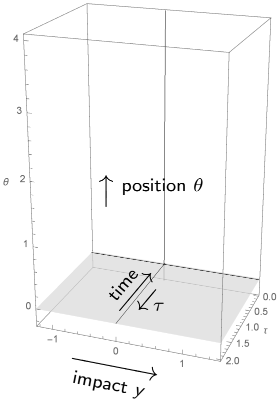
A three-dimensional free boundary problem

Value function $\quad v(\tau, y, \theta) := \sup_{\Theta} \mathbb{E}[L_\tau(\Theta) \mid Y_{0-} = y, \Theta_{0-} = \theta].$



State space $(\tau, y, \theta) \in \mathbb{R}_+ \times \mathbb{R}^2$

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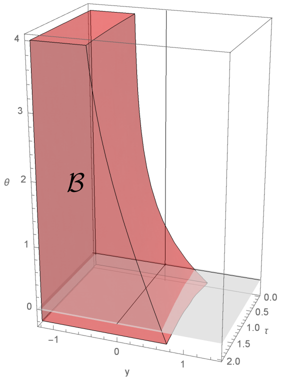

Martingale optimality principle:
State space $(\tau, y, \theta) \in \mathbb{R}_+ \times \mathbb{R}^2$ should separate into open regions $\mathcal{B}$ (buying)

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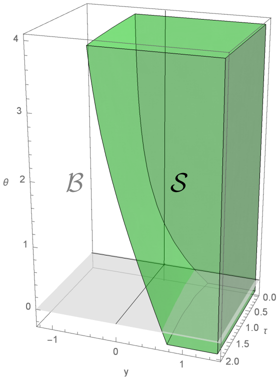

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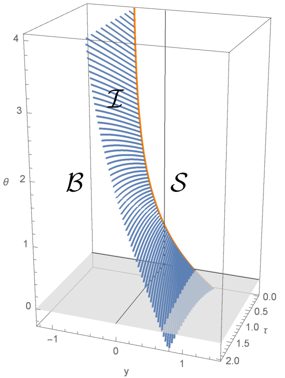

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v_y + v_\theta - f(y) &= 0 \quad \text{everywhere}, \\
v_\tau + h(y)V_y - \mu v &> 0 \quad \text{in } \mathcal{B} \cup \mathcal{S}, \\
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with boundary condition $v(0, y, \theta) = \int_{y-\theta}^{y} f(x)\,\mathrm{d}x.$ (2)

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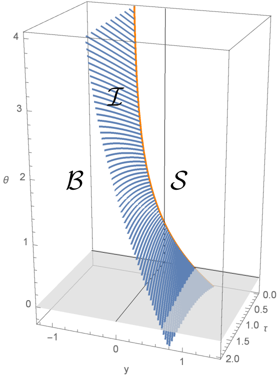

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Ansatz: $v \in C^1, \qquad \forall (\tau, y, \theta) \in \mathcal{B} \cup \mathcal{S} \ \exists! d \in \mathbb{R} \setminus \{0\} : (\tau, y + d, \theta + d) \in \mathcal{I},$

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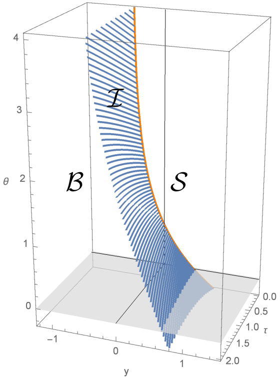

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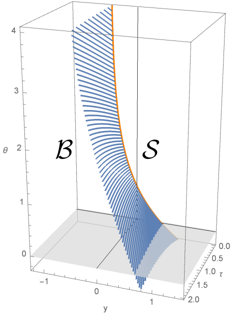
$$v(\tau, y, \theta) = \sup_{\Theta} \mathbb{E}_{y,\theta}\left[-\int_0^{\tau} e^{\mu t} f(Y_t)\, \mathrm{d}\Theta_t^c - \sum_{\substack{0 \le t \le \tau \\ \Delta\Theta_t \neq 0}} \int_0^{\Delta\Theta_t} e^{\mu t} f(Y_{t-} + x)\, \mathrm{d}x \right]$$



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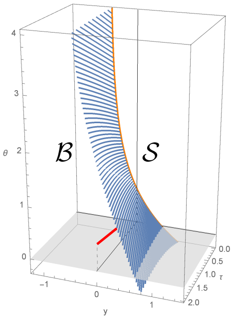

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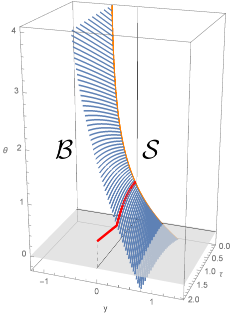

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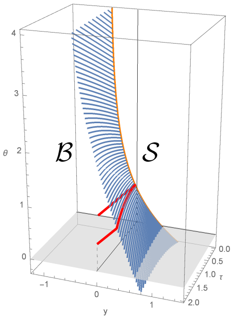

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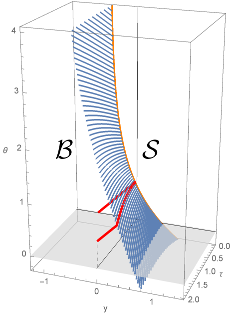

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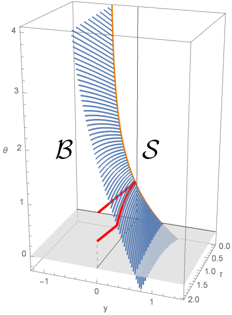

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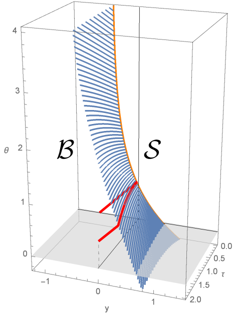

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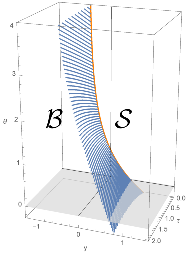
Gains for candidate optimal strategy

In terms of $\bar{y}$:



Maximize

$$J(\bar{y}) = \overbrace{F(Y_{0-}) - F\big(\bar{y}(T)\big)}^{\text{initial block}} + \overbrace{e^{\mu T}\big(F(y) - F\big(y - g(y)\big)\big)\Big|_{y=\bar{y}(0)}}^{\text{terminal block}} \\ + \underbrace{e^{\mu T}\int_0^T e^{-\mu\tau} f\big(\bar{y}(\tau)\big)\big(\bar{y}'(\tau) - h\big(\bar{y}(\tau)\big)\big)\,\mathrm{d}\tau}_{\text{trading in rates}}$$

for $F(y) = \int_0^y f(x)\,\mathrm{d}x$ subject to the *isoperimetric condition*

$$\Theta_{0-} - Y_{0-} \equiv K(\bar{y}) := \underbrace{\underbrace{g\big(\bar{y}(0)\big)}_{=\bar{\theta}(0)} + \int_0^T \underbrace{\big(\bar{y}' - h(\bar{y})\big)}_{=\bar{\theta}'(\tau)}\,\mathrm{d}\tau}_{=\bar{\theta}(T)} - \bar{y}(T)\,.$$

Goal: find intermediate impact $\bar{y}(\tau)$
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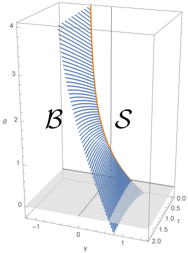

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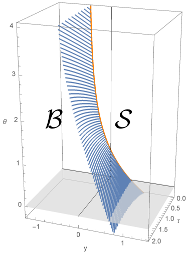

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Calculus of variations

- Maximize $J(\overline{y})$ subject to $K(\overline{y}) = \text{(const)}$ over $\overline{y} \in C^1([0,T])$ with $\overline{y}(T) = Y_0$
- Equivalent problem: $\max_{\overline{y}}\big(J(\overline{y}) + m_T K(\overline{y})\big)$ with (unknown) Lagrange multiplier $m_T \in \mathbb{R}$.
- Taylor approximation: $(J + m_T K)(\overline{y} + z)$

$$= (J + m_T K)(\overline{y}) + \underbrace{\delta(J + m_T K)(\overline{y})[z]}_{\text{first variation}} + \underbrace{\delta^2(J + m_T K)(\overline{y})[z]}_{\text{second variation}} + \mathcal{O}(\|z\|^3_{W^{1,\infty}})$$

where $\|z\|_{W^{1,\infty}} := \|z\|_\infty \vee \|z'\|_\infty$.

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Calculus of variations – candidate solution

Necessary condition $\delta(J + m_T K)(\bar{y})[z] = 0 \ \forall z \in C^1$ with $z(T) = 0$, gives

- Lagrange multiplier m_T,
- candidate terminal position

$$g(y) = y - f^{-1}\left(\left(f\frac{h\lambda + h' - \mu}{h'}\right)(y)\right), \qquad y > y_\infty,$$

where $(h\lambda + h' - \mu)(y_\infty) = 0$,

- ODE for candidate impact trajectory $\bar{y}(\tau)$:

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Non-convex problem – proving optimality

Proof of optimality for this **non-convex** problem:

1. showing local optimality with 2^{nd} variation;
2. use this to extend to global optimality.

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Local optimality

Theorem (strict local maximizer $\bar{y}$)

Under technical conditions[†] *on impact and resilience functions f and h,* $\exists \varepsilon > 0$ *s.t. for all $y \in C^1$ with $y(T) = \bar{y}(T)$, $\|y - \bar{y}\|_{W^{1,\infty}} \in (0, \varepsilon)$:*

$$(J + m_T K)(\bar{y}) > (J + m_T K)(y).$$

Proof: 2^{nd} variation $\delta^2(J + m_T K)(\bar{y})[z] < 0$,
higher order terms are $O(\|z\|^3_{W^{1,\infty}})$. □

- Candidate buy-sell boundary $\mathcal{I} = \{(\tau, \bar{y}(\tau; z), \bar{\theta}(\tau; z)) \mid \tau \in [0, \infty), z > y_\infty\}$
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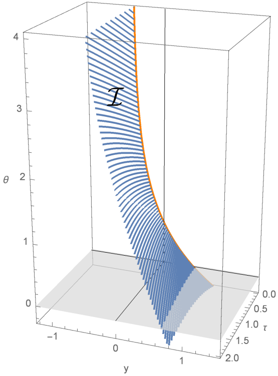

Corollary (variational inequality near $\mathcal{I}$)

We have
$V_\tau + h(y)V_y - \mu V > 0$ *in a neighborhood of* $\mathcal{I}$
with equality on $\mathcal{I}$,
and $V_y + V_\theta = f$ *everywhere.*

Proof: Otherwise, construct a strategy given by $\hat{y}$, with $0 < \|\hat{y} - \bar{y}\|_{W^{1,\infty}} < \varepsilon$ and $K(\hat{y}) = K(\bar{y})$ which would give $J(\hat{y}) \geq J(\bar{y})$. □

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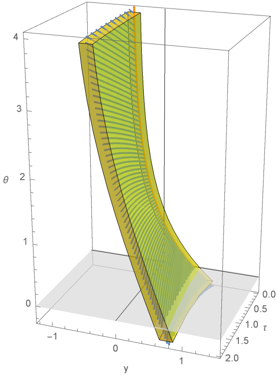

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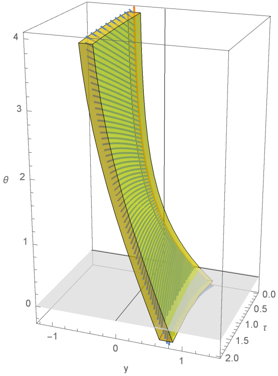

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Global optimality

Question: How to conclude from local to global optimality?

- Let $k(d) := \big(V_\tau + h(y)V_y - \mu V\big)(\tau; y_b + d, \theta_b + d)$
  for fixed $(\tau, y_b, \theta_b) = (\tau, \bar{y}(\tau; z), \bar{\theta}(\tau; z)) \in \mathcal{I}$.
- Previous corollary: $k(0) = 0$, $k'(0-) \le 0 \le k'(0+)$;
- Now, can show analytically the inequalities:

$$k'(-d) < 0 < k'(d) \text{ for } d > 0.$$

- Hence, $k(d) > 0$ for $d \in \mathbb{R} \setminus \{0\}$, giving strict global optimality.

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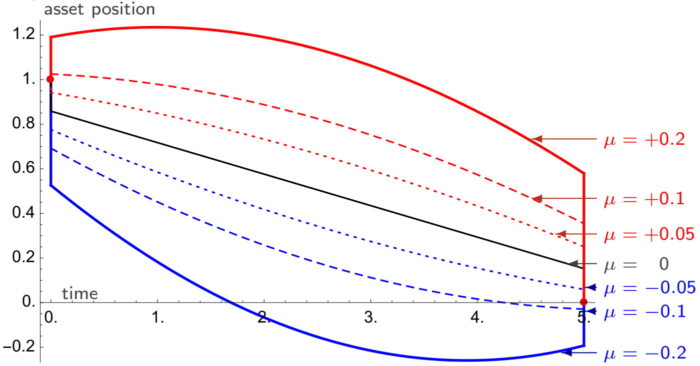
Optimal strategy: dependence on price trend

(comparative statics)



- **no trend** in fundamental price $\overline{S}$ (martingale, Obizhaeva/Wang situation): constant rate of trading in time $(0, T)$.
- increasing $\overline{S}$: defer asset sales to later times; for strong upwards trend: temporary buying.
- decreasing $\overline{S}$: sell asset at earlier times; for string downwards trend: go short.

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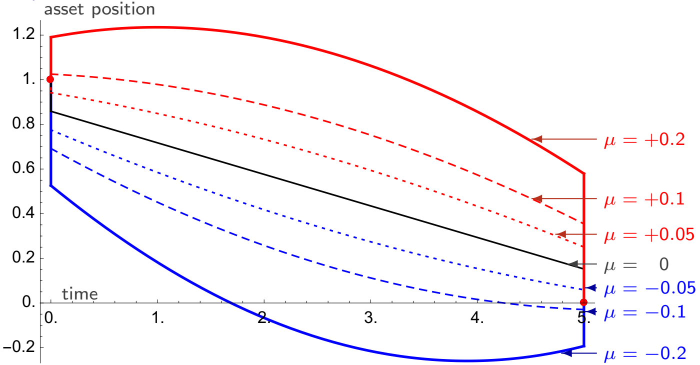

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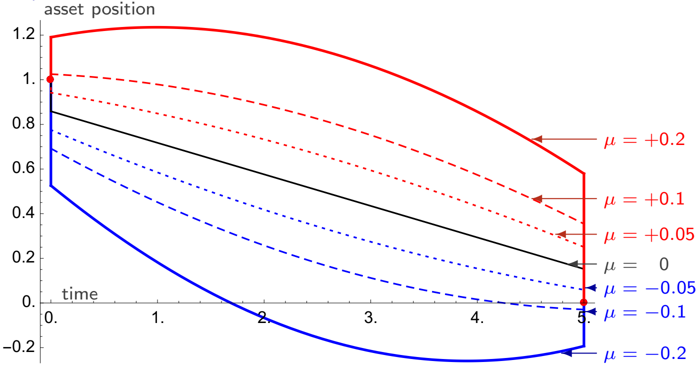

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If there is still some time...

Stochastic liquidity and transaction costs

- Stochasticity in the impact/signal:

$$\mathrm{d}Y_t = -\beta Y_t \,\mathrm{d}t + \hat{\sigma}\,\mathrm{d}B_t + \mathrm{d}\Theta_t, \qquad Y_{0-} = y,$$

  for correlated Brownian motion B with $\mathrm{d}[B,W]_t = \rho\,\mathrm{d}t$.
  [Becherer/Bilarev/F., FS 2018]
  [Lehalle/Neuman – *Incorporating Signals into Optimal Trading*, arXiv:1704.00847]

- Bid-ask spread through proportional transaction costs:

$$\text{sell at } S^{\text{bid}} := f(Y_t)\overline{S}_t\,, \qquad \text{buy at } S^{ask} := \kappa f(Y_t)\overline{S}_t\,,$$

  for transaction cost factor $\kappa > 1$.

- Maximize over Θ : càdlàg, adapted, bounded variation, ≥ 0,
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  Infinite time horizon eases analysis: (non-convex) free boundary problem in $\mathbb{R}^2$.

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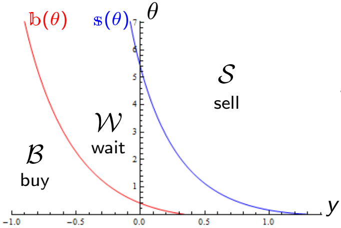
Variational (in-)equalities



State space $(y, \theta) \in \mathbb{R} \times \mathbb{R}_+$ should separate into open regions $\mathcal{B}$, $\mathcal{W}$, $\mathcal{S}$ with corresponding HJB **variational (in-)equalities** ...

- **Ansatz:** there exist free boundary curves $\mathbb{b}, \mathbb{s} \in C^1(\mathbb{R}_+)$ s.t. $\mathcal{B} = \{y < \mathbb{b}(\theta)\}$, $\mathcal{W} = \{\mathbb{b}(\theta) < y < \mathbb{s}(\theta)\}$, $\mathcal{S} = \{y > \mathbb{s}(\theta)\}$.
- $V(y, 0) = 0$ for $y \geq \mathbb{s}(0)$,
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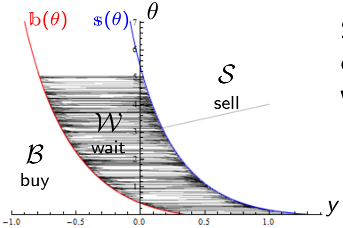
Variational (in-)equalities



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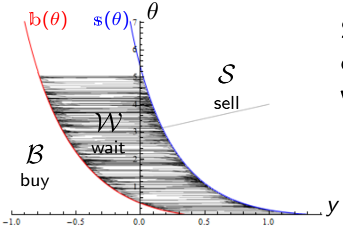
- $V(y, \infty) = \tilde{V}(y)$ via corresponding "$\theta \to \infty$" infinite fuel limit.

Variational (in-)equalities



State space $(y, \theta) \in \mathbb{R} \times \mathbb{R}_+$ should separate into open regions $\mathcal{B}$, $\mathcal{W}$, $\mathcal{S}$ with corresponding HJB **variational (in-)equalities** ...

- **Ansatz:** there exist free boundary curves $\mathbb{b}, \mathbb{s} \in C^1(\mathbb{R}_+)$ s.t.
  $\mathcal{B} = \{y < \mathbb{b}(\theta)\}$, $\mathcal{W} = \{\mathbb{b}(\theta) < y < \mathbb{s}(\theta)\}$, $\mathcal{S} = \{y > \mathbb{s}(\theta)\}$.
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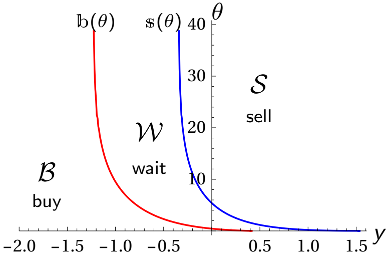
Open ODE problem for boundary curves



Smooth pasting gives

- Candidate free boundary curves $\mathbb{b}(\theta), \mathbb{s}(\theta)$ as ODE

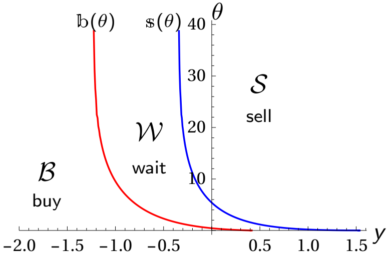
$$(\mathbb{b}', \mathbb{s}') = \text{function}(\mathbb{b}, \mathbb{s}),$$

- with asymptotes $\mathbb{b}(\infty) = b_\infty$, $\mathbb{s}(\infty) = s_\infty$, and (implicitly) given $\mathbb{s}(0)$.

Open questions:

- Existence of ODE solution for $\mathbb{b}, \mathbb{s}$;
- Proof of variational inequalities.

Open ODE problem for boundary curves



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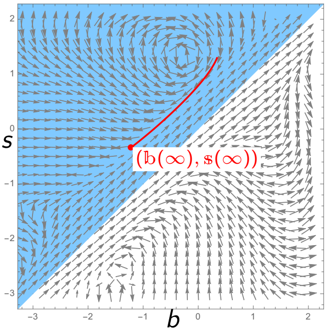

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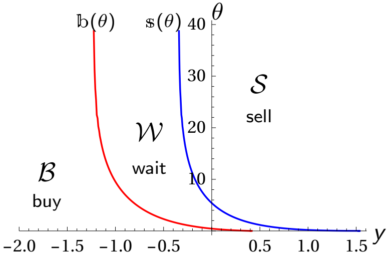

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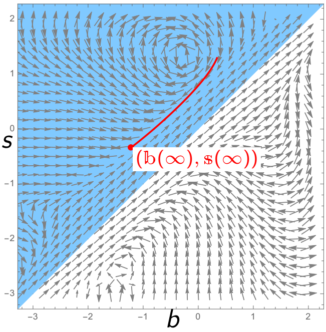

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Thank you!