Switching Diffusions with Mean-Field Interactions

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Joint Works with
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Outline

Examples from Finance and Risk Management

Systemic Risk Example, Garnier, Papanicolaou, Yang (2012)

- Consider a system with a large number of inter-connected components. Each component can be in a normal state or in a failed state. One wants to study the probability of overall failure of the system–systemic risk.
    - Banks cooperate and by spreading the risk of credit shocks among them can operate with less restrictive individual risk policies (capital reserves). But, this increases the risk that they may all fail–systemic risk.
    - In a power distribution system, individual components of the system are calibrated to withstand fluctuations in demand by sharing loads. But this increases the probability of an overall failure.

The risk variable $x^i(t)$ of each component $i = 1, \ldots, N$, satisfies the SDEs

$$dx^i(t) = -h\frac{\partial}{\partial y}V(x^i(t))dt + \theta(\overline{x}(t) - x^i(t))dt + \sigma dw^i(t)$$

- $V(y)$: is a potential with two stable states (normal state, failed state). Without noise, the individual risk $x^i(t)$ stays in these states. A typical V: $V(y) = -(1/4)y^4 + (1/2)y^2$.
- w^i: $i \leq N$, independent Brownian motions.
- $\overline{x}(t) := \frac{1}{N}\sum_{j=1}^{N} x^j(t)$, mean field.
- $\theta > 0$: cooperative interaction parameter.
- h: controls the probability that x^i jumps from one state to the other.

- h, σ, θ control the three effects (i) Intrinsic stability, (ii) random perturbations, and (iii) the degree of cooperation.
- Why mean field interaction? Because it is perhaps the simplest interaction that models cooperative behavior.

Example: Credit Risk, Fouque & Langsam (2008)

- Model borrowing and lending through the drifts:

$$dx^i(t) = a\Big(\frac{1}{N}\sum_{j=1}^{N} x^j(t) - x^i(t)\Big)dt + \sigma dw^i(t),\ i = 1, \dots, N.$$

- Correlated diffusions are used.
- $a > 0$. Increasing the rate a of borrowing and lending a shows "stability"
- The $x^i(t)$'s are "OU" mean-reverting to the ensemble average

$$d\Big(\frac{1}{N}\sum_{j=1}^{N} x^j(t)\Big) = d\Big(\frac{\sigma}{N}\sum_{j=1}^{N} dw^j(t)\Big).$$

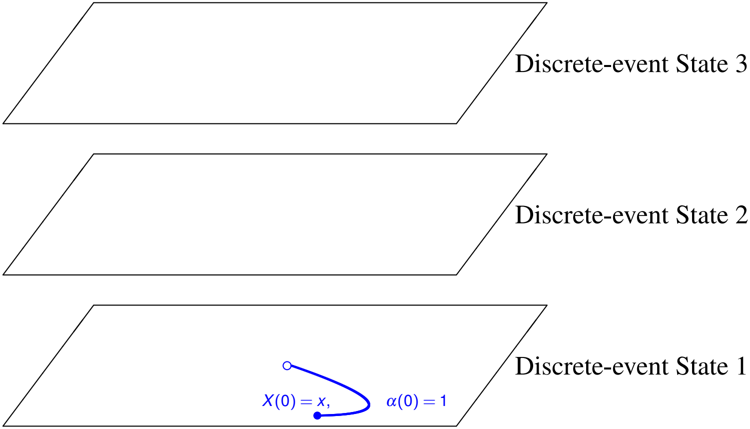
- The ensemble average is distributed as a Brownian motion with diffusion coefficient $\sigma/\sqrt{N}$.

Introduction

Mean-Field Models

- Dawson (1983)
- D.A. Dawson and J. Gärtner (1987), large deviations
- Gärtner (1988), McKean-Vlasov limit for interacting diffusions
- Huang, Caines, and Malhamé (2003, 2006), Mean-field games
- Lasry and Lions (2006), Mean-field games
- Bensoussan, Frehse, and Yam (2013), Springer Brief
- Kolokoltsov and Troeva (2015), common noise and the McKean-Vlasov SPDEs

Randomly Switched Dynamic System: An Illustration



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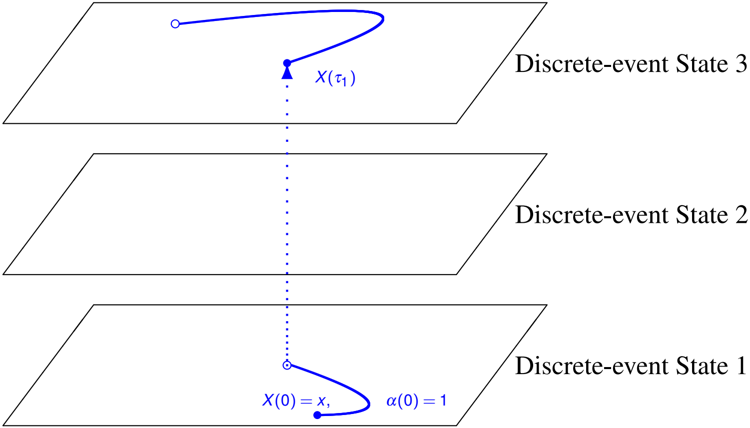

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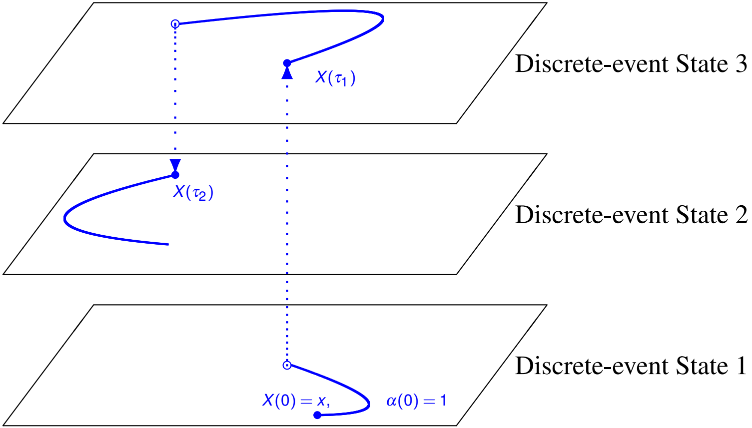

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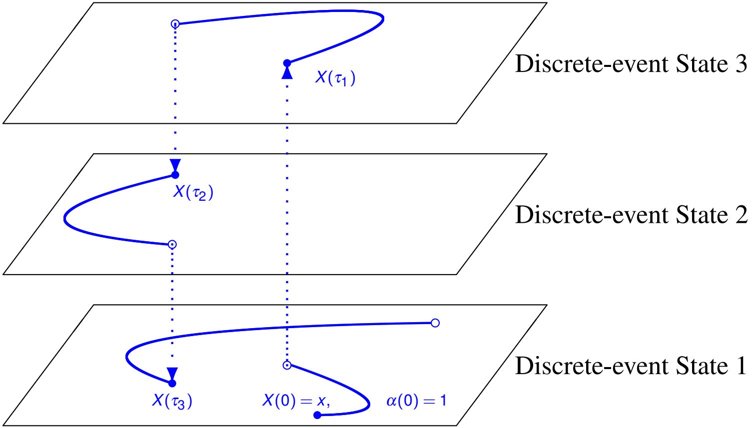

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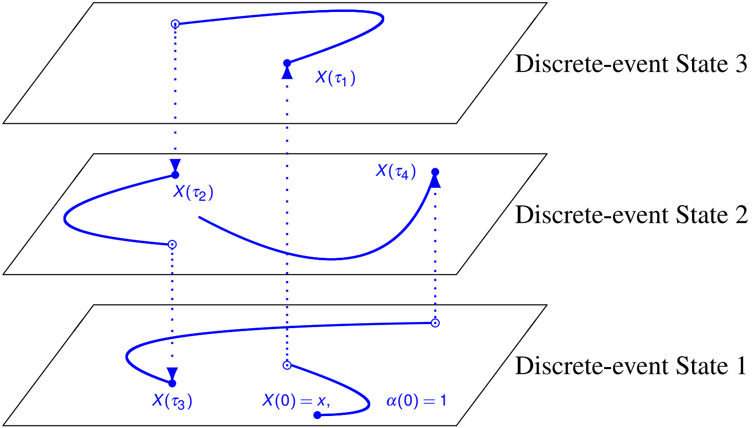

Figure 1: A "Sample Path" of A Switching Dynamic System $(X(t), \alpha(t))$.

Law of Large Numbers

Original Dawson's Mean-Field Model with Switching Added

- $\alpha(t)$: with $\mathscr{M} = \{1, 2, \dots, m_0\}$.
- Consider an ℓ-body mean-field model For $i = 1, 2, \dots, \ell$,

$$
\begin{aligned}
dX_i(t) = & \ \big[\gamma(\alpha(t))X_i(t) - X_i^3(t) - \beta(\alpha(t))(X_i(t) - \overline{X}(t))\big]dt \\
& +\sigma_{ii}(X(t), \alpha(t))dw_i(t), \\
\overline{X}(t) = & \ \frac{1}{\ell}\sum_{j=1}^{\ell} X_j(t), \\
X(t) = & \ (X_1(t), X_2(t), \dots, X_\ell(t))',
\end{aligned} \tag{1}
$$

$\gamma(i) > 0$ and $\beta(i) > 0$ for $i \in \mathscr{M}$.

- Originated from statistical mechanics, mean-field models are concerned with many-body systems with interactions. To overcome the difficulty of interactions due to the many bodies, one of the main ideas is to replace all interactions to any one body with an average or effective interaction.

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- Kolokoltsov (2010), nonlinear Markov processes
- Kurtz and Xiong (1999), infinite system of SDEs for the locations and weights of a collection of particles, a common space-time Gaussian white noise, their model contains infinitely many exchangeable particles so ergodic theory can be applied

Our Work

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- Main difficulty: to characterize the limit using the martingale problem formulation does not work.
- Compare to Kurtz and Xiong, we no longer have infinitely many exchangeable particles thus the existing ergodic theory is not applicable.
- we characterize the limit as the unique solution to a stochastic McKean-Vlasov equation w/ Markovian switching, which is represented by the conditional distribution of the solution to a McKean-Vlasov stochastic differential equation with a Markovian switching given the history of the switching process.

Formulation: Law of Large Numbers

A mean-field system of N particles ($N \gg 1$),

$$dX_t^{i,N} = b\Big(X_t^{i,N}, \frac{1}{N}\sum_{j=1}^{N}\delta_{X_t^{j,N}}, \alpha(t_-)\Big)dt + \sigma\Big(X_t^{i,N}, \frac{1}{N}\sum_{j=1}^{N}\delta_{X_t^{j,N}}, \alpha(t_-)\Big)dW_t^i, \tag{2}$$

Digression: Empirical Distribution, Glivenko-Cantelli Theorem

- Let $Z_1, Z_2, \ldots, Z_n, \ldots$ be a sequence of i.i.d. r.v.'s
- Z_1 has d.f. $F(z)$.
- Form
$$\widehat{F}_n(z) = \frac{1}{n}\sum_{i=1}^{n}\delta_{\{Z_i \le z\}}.$$
- $\widehat{F}_n(z) \to F(z)$ as $n \to \infty$.

Setup (cont.)

- $C_b(\mathbb{R}^d)$: space of bounded continuous functions on $\mathbb{R}^d$
- $C_b^k(\mathbb{R}^d)$: C^k functions with bdd partials
- C_c^k: C^k functions with compact support
- E: metric space
- $\mathscr{B}(E)$: Borel σ-field on E
- $\mathscr{P}(E)$: space probability measures on $(E, \mathscr{B}(E))$ w/ weak topology
- $C([0,T],E)$: space of continuous functions with sup metric
- $D([0,T],E)$: space of all càdlàg functions with Skorohod topology

Setup (cont.)

- $\mathscr{M} := \{1, 2, \ldots, m_0\}$, state space of the Markov chain
- $D_f([0,T], \mathscr{M})$: subspace of $D([0,T], \mathscr{M})$ with finite jumps
- $d_{\mathscr{M}}$: metric on $\mathscr{M}$, $d_{\mathscr{M}}(i_0, j_0) = 1 - \delta_{i_0, j_0}$ for $i_0, j_0 \in \mathscr{M}$.
- $\mathcal{M}_1$: space of probability measures on $\mathbb{R}^d$
- with $\mu \in \mathcal{M}_1$, $\langle \mu, f \rangle = \int_{\mathbb{R}^d} f(x)\mu(dx)$
- For $\mu \in \mathcal{M}_1$, $f(\cdot, \cdot, i_0) \in C_b(\mathbb{R} \times \mathbb{R}^d)$, $g(\cdot, i_0) \in C_b(\mathbb{R}^d)$, define $\langle \mu, f(t, \cdot, i_0) \rangle = \int_{\mathbb{R}^d} f(t, x, i_0)\mu(dx)$ & $\langle \mu, g(\cdot, i_0) \rangle = \int_{\mathbb{R}^d} g(x, i_0)\mu(dx)$
- $\|\cdot\|_{TV}$: total variation metric $\mathcal{M}_1$
- $\|\mu - \eta\|_{BL} = \sup \left\{ \left| \langle \mu, f \rangle - \langle \eta, f \rangle \right| : \|f\| \leq 1, \sup_{x \neq y \in \mathbb{R}^d} \frac{|f(x) - f(y)|}{|x - y|} \leq 1 \right\}$
- $(\mathcal{M}_1, \|\cdot\|_{BL})$ is a complete & separable metric space
- $d((\mu, i_0), (\eta, j_0)) = \left\|\mu - \eta\right\|_{BL} + d_{\mathscr{M}}(i_0, j_0)$, $\forall \mu, \eta \in \mathcal{M}_1, i_0, j_0 \in \mathscr{M}$.

Setup (cont.)

- $\varphi(x) = |x|$ and $\psi(x) = |x|^2$
- $\mathscr{F}^{\alpha}_{t_-} = \sigma\{\alpha(s) : 0 \le s < t\}$
- $\mathscr{F}^{N,\alpha}_{t} = \sigma\{w_i(s), \alpha(s) : 0 \le s \le t, 1 \le i \le N\}$.
- For a r.v. ς on $(\Omega, \mathscr{F}, \mathbb{P})$, denote
  - $\mathcal{L}(\varsigma)$: distribution; law of ς
  - $\eta_t = \mathcal{L}(\varsigma | \mathscr{F}^{\alpha}_{t_-})$: conditional law given $\mathscr{F}^{\alpha}_{t_-}$, meaning $\mathbb{E}(f(\varsigma)|\mathscr{F}^{\alpha}_{t_-}) = \int_{\mathbb{R}^d} f(x)\eta_t(dx)$ $\forall f \in C_b(\mathbb{R}^d)$.
- For $N \ge 1$, $t \in [0, T]$, $A \in \mathscr{B}(\mathbb{R}^d)$, define a measure-valued process

$$\mu_N(t, A) = \frac{1}{N}\sum_{j=1}^{N} \delta_{X^{j,N}_t}(A). \tag{3}$$

- $\mathcal{P}_N$: the induced probability measure of $(\mu_N(\cdot), \alpha(\cdot))$ on $D([0, T], \mathcal{M}_1 \times \mathbb{S})$ [Note $\mathcal{P}_N$ concentrates on the set $C([0, T], \mathcal{M}_1) \times D_f([0, T], \mathscr{M})$, a closed subspace of $D([0, T], \mathcal{M}_1 \times \mathscr{M})$.]

Assumption A.

(A1) For $i_0 \in \mathbb{S}$, $b(\cdot,\cdot,i_0) : \mathbb{R}^d \times \mathcal{M}_1 \to \mathbb{R}^d$ & $\sigma(\cdot,\cdot,i_0) : \mathbb{R}^d \times \mathcal{M}_1 \to \mathbb{R}^{d\times d}$ are Lipschitz, i.e., $\exists L$ s.t.

$$\Big|b(x,\mu,i_0) - b(y,\eta,i_0)\Big| + \Big|\sigma(x,\mu,i_0) - \sigma(y,\eta,i_0)\Big| \le L\Big(|x-y| + \|\mu-\eta\|_{BL}\Big),$$

$\forall x,y \in \mathbb{R}^d$ & $\mu,\eta \in \mathcal{M}_1$.

(A2) The following conditions hold.

- For some constant C & $\varphi : \mathbb{R}^d \to \mathbb{R}$, $\varphi(x) = |x|$,

$$\Big|b(x,\mu,i_0)\Big| \le C\Big(1 + |x| + \langle \mu, \varphi\rangle\Big), \quad (x,\mu,i_0) \in \mathbb{R}^d \times \mathcal{M}_1 \times \mathscr{M}.$$

- $\sigma(\cdot,\cdot,\cdot)$ is bounded.

Assume (A1), (A2), and

$$\sup_{N\in\mathbb{N}} \mathbb{E}\langle \mu_N(0), \psi \rangle < \infty, \quad \mathcal{L}\big(\mu_N(0)\big) \Rightarrow \delta_{\mu_0} \text{ in } \mathscr{P}\big(\mathcal{M}_1, \|\cdot\|_{BL}\big),$$

where $\psi : \mathbb{R}^d \to \mathbb{R}$ with $\psi(x) = |x|^2$. Then we will show that $\big(\mu_N(\cdot), \alpha(\cdot)\big) \Rightarrow \big(\mu_\alpha(\cdot), \alpha(\cdot)\big)$.

Proposition

The sequence $\big\{\big(\mu_N(\cdot), \alpha(\cdot)\big), N \geq 1\big\}$ is weakly compact in the topology of weak convergence of probability measure on $D\big([0,T], \mathcal{M}_1 \times \mathscr{M}\big)$.

To characterize the limit, use martingales associate to the switching process

- $(i_0, j_0) \in \mathscr{M} \times \mathscr{M}$,
- $1\!\!1$: indicator function

$$\begin{aligned} [M_{i_0 j_0}](t) &= \sum_{0 \le s \le t} 1\!\!1(\alpha(s_-) = i_0) 1\!\!1(\alpha(s) = j_0), \\ \langle M_{i_0 j_0} \rangle(t) &= \int_0^t q_{i_0 j_0} 1\!\!1(\alpha(s_-) = i_0) ds, \end{aligned} \tag{4}$$

$$M_{i_0 j_0}(t) = [M_{i_0 j_0}](t) - \langle M_{i_0 j_0} \rangle(t) \tag{5}$$

$M_{i_0 j_0}(t)$ is a square integrable martingale w.r.t. $\mathscr{F}_t^\alpha$, $M_{i_0 j_0}(0) = 0$.

Operator Associate to the Limiting System

- $f(\cdot, i_0) \in C_c^2(\mathbb{R}^d)$,
- $i_0 \in \mathscr{M}$,
- $(x, \mu, i_0) \in \mathbb{R}^d \times \mathcal{M}_1 \times \mathscr{M}$

denote the operator

$$\begin{aligned}\mathscr{L}(\mu)f(x,i_0) = b'(x,\mu,i_0)\nabla_x f(x,i_0) + \frac{1}{2}\Big(a(x,\mu,i_0)\nabla_x\Big)'\nabla_x f(x,i_0)\\ + \sum_{j_0\in\mathbb{S}} q_{i_0j_0}\Big(f(x,j_0) - f(x,i_0)\Big),\end{aligned} \tag{6}$$

$$a(x,\mu,i_0) = \sigma(x,\mu,i_0)\sigma'(x,\mu,i_0) \in \mathbb{R}^{d\times d}. \tag{7}$$

Characterization of Limit: Stochastic McKean-Vlasov Equations

Theorem

Under (A1) *and* (A2), *for* $f(\cdot, i_0) \in C_c^2(\mathbb{R}^d)$ *and* $i_0 \in \mathbb{S}$, *the system*

$$\begin{aligned}\langle \mu(t), f(\cdot,\alpha(t))\rangle =& \langle \mu_0, f(\cdot,\alpha(0))\rangle + \int_0^t \Big\langle \mu(s), \mathscr{L}(\mu(s)) f(\cdot,\alpha(s_-))\Big\rangle ds \\ &+ \sum_{i_0, j_0 \in \mathbb{S}} \int_0^t \Big\langle \mu(s), f(\cdot, j_0) - f(\cdot, i_0)\Big\rangle dM_{i_0 j_0}(s),\end{aligned} \tag{8}$$

has a *unique solution* $\mathcal{L}(y(t) | \mathscr{F}_{t_-}^{\alpha})$ *in* $D\big([0,T], \mathcal{M}_1\big)$ *for all* $0 \le t \le T$.

$y(t)$ is the unique solution of

$$\begin{cases} dy(t) &= b\Big(y(t), \mu_\alpha(t), \alpha(t_-)\Big) dt + \sigma\Big(y(t), \mu_\alpha(t), \alpha(t_-)\Big) d\tilde{w}(t), \\ \mu_\alpha(t) &= \mathcal{L}\big(y(t) \big| \mathscr{F}_{t_-}^{\alpha}\big), \quad \mathcal{L}(y(0)) = \mu_0, \end{cases}$$

where $\tilde{w}(\cdot)$ is a standard Brownian motion independent of $\alpha(\cdot)$.

Maximum Principle

Motivation: Some References

- Treating mean-field game problem with many particles or multi-agents and random switching, the formulation in Wang and Zhang (2012) require having a Markov chain for each particle.

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- Treating mean-field game problem with many particles or multi-agents and random switching, the formulation in Wang and Zhang (2012) require having a Markov chain for each particle.
- Zhang, Sun, and Xiong (2018) studied a mean-field control problem for a general model including both switching and jump. Even though it was not explicitly mentioned in the reference, the system considered there is the limit of finite population of weakly interacting jump-diffusion systems with Markovian switching. In addition, in the prelimit of the weakly interacting systems, all the particles are coupled by independent Markov chains.

Motivation: Some References (cont.)

- Although mean-field games and mean-field-type of controls have received much needed attention in recent years, the study for mean-field control and games with regime-switching remains largely in an early stage.
  - Wang and Zhang (2017) dealt with a LQG social optimal problem with mean-field term.
  - The effort was to approximate the mean-field term $x_t^{(N)}$ in the finite population model. By taking N large enough, the Brownian motion is averaged out and the limit becomes a switched ODE than switching diffusion.
  - The switching is frozen to approximate the mean-field term in the limit control problem. It only plays a role as a random coefficient in the limit control problem and is not affected by the control.

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Motivation: Our work

- Our work focuses on obtaining maximum principles for regime-switching diffusions with mean-field interactions
- Combine mean-field-type controls and switching diffusions
- The difficulties involve not only the mean-field interactions, but also the "correlation" due to the modulating Markov chain.
- It is often regarded that a maximum principle is largely of theoretical value. Here we show that our result, in fact, leads to computable control strategies for LQG problems.
- We make crucial use of conditional mean field.

Regarding the use of conditional mean fields

- In Buckdahn, Li, and Ma, mean-field control with partial observations is treated. The idea of converting a partially observable system into a fully observable systems is used. Then naturally, a conditional mean-field is used conditioning on the observations.

- In Carmona and Zhu, a probabilistic approach for mean-field games with major and minor players is provided. A conditional mean-field with condition on the major player is used.

Formulation: Maximum Principle

For simplicity, we work out the scalar case. The generalization to multi-dimensional cases is straightforward, but the notation is more complex.

- $[0, T]$: finite time horizon
- W_t: an 1-dimensional standard Brownian motion
- α_t: a continuous-time Markov chain; W_t is independent of α_t
- $\mathscr{M} = \{1, 2, \ldots, m_0\}$: state space of the Markov chain
- $Q = \left(q_{i_0, j_0}\right)_{i_0, j_0 \in \mathscr{M}}$: generator of α_t
- $$\begin{aligned} \mathscr{F}_t^W &= \sigma\{W_s : 0 \le s \le t\}, \\ \mathscr{F}_t^{W,\alpha} &= \sigma\{W_s, \alpha_s : 0 \le s \le t\}, \\ \mathscr{F}_t &= \mathscr{F}_t^{W,\alpha} \end{aligned}$$
- U: the action space, non-empty & convex subset of $\mathbb{R}$
- $\mathscr{U}$: the class of measurable, $\mathscr{F}_t$-adapted and square integrable processes $u(\cdot, \cdot) : [0, T] \times \Omega \to U$.
- $b, \sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathscr{M} \to \mathbb{R}$, $\psi, \varphi, \phi, \eta : \mathbb{R} \to \mathbb{R}$.

Set Up (cont.)

Consider a system of n interacting particles of the form

$$dX_t^{i,N} = b\Big(t, X_t^{i,N}, \frac{1}{N}\sum_{j=1}^{N}\psi(X_t^{j,N}), u_t, \alpha_{t-}\Big)dt + \sigma\Big(t, X_t^{i,N}, \frac{1}{N}\sum_{j=1}^{N}\varphi(X_t^{j,N}), u_t, \alpha_{t-}\Big)dW_t^i,$$

$(W_t^i, i \geq 1)$: collection of independent standard Brownian motions.

Taking mean-square limit as $N \to \infty$ of the above system, we obtain

$$\begin{aligned} dX_t &= b\Big(t, X_t, \mathbb{E}\big(\psi(X_t)\,|\,\mathscr{F}_{t-}^{\alpha}\big), u_t, \alpha_{t-}\Big)dt + \sigma\Big(t, X_t, \mathbb{E}\big(\varphi(X_t)\,|\,\mathscr{F}_{t-}^{\alpha}\big), u_t, \alpha_{t-}\Big)dW_t, \\ X_0 &= x_0 \in \mathbb{R}, \end{aligned} \tag{9}$$

Cost function:

$$J(u) = \mathbb{E}\left[\int_0^T h\Big(t, X_t, \mathbb{E}\big(\phi(X_t)\,|\,\mathscr{F}_{t-}^{\alpha}\big), u_t, \alpha_{t-}\Big) + g\Big(X_T, \mathbb{E}\big(\eta(X_T)\,|\,\mathscr{F}_{T-}^{\alpha}\big), \alpha_T\Big)\right]. \tag{10}$$

Conditions

(C1) The functions $\psi(\cdot)$, $\phi(\cdot)$, $\varphi(\cdot)$, and $\eta(\cdot)$ are continuously differentiable; $g(\cdot,\cdot,i_0)$ is continuously differentiable with respect to (x,y); $b(\cdot,\cdot,\cdot,\cdot,i_0), \sigma(\cdot,\cdot,\cdot,\cdot,i_0)$, and $h(\cdot,\cdot,\cdot,\cdot,i_0)$ are continuous in t and continuously differentiable with respect to (x,y,u).

(C2) In (C1), for each t and i_0, all derivatives of $\psi(\cdot)$, $\phi(\cdot)$, $\varphi(\cdot)$, $g(\cdot,\cdot,i_0)$, $b(t,\cdot,\cdot,\cdot,i_0)$, $\sigma(t,\cdot,\cdot,\cdot,i_0)$, and $h(t,\cdot,\cdot,\cdot,i_0)$ with respect to x, y, and u are Lipschitz continuous and bounded.

Some Estimates

For an admissible control u, denote the corresponding trajectory by X^u. If $\overline{u}_t$ is an optimal control, then $X^{\overline{u}}$ is the associated optimal trajectory. Define the perturbed control as follow

$$u_t^\theta = \overline{u}_t + \theta\left(v_t - \overline{u}_t\right), \quad v_t \in \mathscr{U}.$$

Denote

$$\begin{aligned}
\overline{b}(t) &= b\Big(t, X_t^{\overline{u}}, \mathbb{E}\big(\psi\big(X_t^{\overline{u}}\big)\big|\mathscr{F}_{t-}^\alpha\big), \overline{u}_t, \alpha_{t-}\Big),\\
\overline{\sigma}(t) &= \sigma\Big(t, X_t^{\overline{u}}, \mathbb{E}\big(\varphi\big(X_t^{\overline{u}}\big)\big|\mathscr{F}_{t-}^\alpha\big), \overline{u}_t, \alpha_{t-}\Big),\\
\overline{h}(t) &= h\Big(t, X_t^{\overline{u}}, \mathbb{E}\big(\phi\big(X_t^{\overline{u}}\big)\big|\mathscr{F}_{t-}^\alpha\big), \overline{u}_t, \alpha_{t-}\Big),\\
\overline{g}(t) &= g\Big(X_t^{\overline{u}}, \mathbb{E}\big(\eta\big(X_t^{\overline{u}}\big)\big|\mathscr{F}_{t-}^\alpha\big), \alpha_t\Big),\\
\overline{\phi}(t) &= \phi\left(X_t^{\overline{u}}\right),\ \overline{\varphi}(t) = \varphi\left(X_t^{\overline{u}}\right),\ \overline{\psi}(t) = \psi\left(X_t^{\overline{u}}\right),\ \overline{\eta}(t) = \eta\left(X_t^{\overline{u}}\right).
\end{aligned}$$

Remark

Under suitable assumptions, for each $u \in \mathscr{U}$, (9) has a unique solution.

X^θ: the trajectory corresponding to u_t^θ
$X^{\overline{u}}$: the associated optimal trajectory

The following lemma implies $X_t^{\overline{u}}$ is the uniform mean square limit of X_t^θ as $\theta \to 0$.

Lemma

$\exists K > 0$, *for all* θ,

$$\lim_{\theta \to 0_+} \mathbb{E}\left[\sup_{0 \le t \le T} \left|X_t^\theta - X_t^{\overline{u}}\right|^2\right] \le K\theta^2.$$

"Higher Order" Sensitivity

The following lemma shows that z_t, the solution given by (11), is nothing but the mean square derivative of X_t^θ with respect to θ at $\theta = 0$.

Lemma

Let z_t be the solution to the following linear equation

$$\begin{aligned} dz_t &= \Big[\overline{b}_x(t)z_t + \overline{b}_y(t)\mathbb{E}\Big(\overline{\psi}_x(t)z_t\Big|\mathscr{F}_{t-}^{\alpha}\Big) + \overline{b}_u(t)\big(v_t - \overline{u}_t\big)\Big]dt \\ &\quad + \Big[\overline{\sigma}_x(t)z_t + \overline{\sigma}_y(t)\mathbb{E}\Big(\overline{\varphi}_x(t)z_t\Big|\mathscr{F}_{t-}^{\alpha}\Big) + \overline{\sigma}_u(t)\big(v_t - \overline{u}_t\big)\Big]dW_t \\ z_0 &= 0. \end{aligned} \tag{11}$$

Then we have

$$\lim_{\theta\to 0_+} \mathbb{E}\left[\sup_{0\le s\le T}\left|\frac{X_s^\theta - X_s^{\overline{u}}}{\theta} - z_s\right|^2\right] = 0.$$

Gateaux Derivative of J

The next lemma is concerned with the sensitivity of the cost functional J with respect to the parameter θ. It gives the Gateaux derivative of the cost functional.

Lemma

The Gateaux derivative of the cost functional J is given by

$$\begin{aligned}
&\frac{d}{d\theta} J(\bar{u}+\theta(v-\bar{u}))\Big|_{\theta=0} \\
&= \mathbb{E}\left\{ \int_0^T \left[\bar{h}_x(t) z_t + \bar{h}_y(t) \mathbb{E}(\bar{\phi}_x(t) z_t | \mathscr{F}_{t-}^{\alpha}) + \bar{h}_u(t)(v_t - \bar{u}_t) \right] dt \right\} \\
&\quad + \mathbb{E}\left[\bar{g}_x(T) z_T + \bar{g}_y(T) \mathbb{E}(\bar{\eta}_x(T) z_T | \mathscr{F}_{T-}^{\alpha}) \right].
\end{aligned}$$

Next, we will show that the Gateaux derivative of J can be expressed in terms Hamiltonian by using duality.

Spaces of Functions

To define BSDEs with conditional mean-field and Markovian switching we need the following spaces of functions

$$\mathscr{S}^2_{\mathscr{F}}(0,T;\mathbb{R}) = \Big\{\varphi : [0,T]\times\Omega \to \mathbb{R},\ \mathscr{F}_t\text{-adapted cadlag process,}\ \mathbb{E}\Big[\sup_{0\le t\le T}|\varphi_t|^2\Big] < \infty\Big\},$$

$$\mathscr{L}^0_{\mathscr{F}}(0,T;\mathbb{R}) = \Big\{\psi : [0,T]\times\Omega \to \mathbb{R},\ \mathscr{F}_t\text{-progressively measurable}\Big\},$$

$$\mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}) = \Big\{\psi \in \mathscr{L}^0_{\mathscr{F}}(0,T;\mathbb{R}) : \|\psi\|_2^2 = \mathbb{E}\Big[\int_0^T |\psi_t|^2 dt\Big] < \infty\Big\},$$

$$\mathscr{M}^2_{\mathscr{F}}(0,T;\mathbb{R}) = \Big\{\lambda = (\lambda_{i_0 j_0} : i_0, j_0 \in \mathscr{M}), \text{s.t. } \lambda_{i_0 j_0} \in \mathscr{L}^0_{\mathscr{F}}(0,T;\mathbb{R}),\ \lambda_{i_0 i_0} = 0 \text{ for } i_0, j_0 \in \mathscr{M},\ \text{and} \sum_{i_0, j_0 \in \mathscr{M}} \mathbb{E}\Big[\int_0^T |\lambda_{i_0 j_0}(t)|^2 d[M_{i_0 j_0}](t)\Big] < \infty\Big\}.$$

Stochastic Integrals Driven by Martingales

Note that $M_{i_0 j_0}(t)$, $i_0, j_0 \in \mathscr{M}$, are martingales.
For $\mathscr{F}$-progressively measurable $\Lambda_t = \big(\Lambda_{i_0 j_0}(t)\big)_{i_0, j_0 \in \mathscr{M}}$, $t \geq 0$, denote

$$\int_0^t \Lambda_s \bullet dM_s = \sum_{i_0, j_0 \in \mathscr{M}} \int_0^t \Lambda_{i_0 j_0}(s) dM_{i_0 j_0}(s),$$

$$\Lambda_t \bullet dM_t = \sum_{i_0, j_0 \in \mathscr{M}} \Lambda_{i_0 j_0}(t) dM_{i_0 j_0}(t).$$

$F(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot) : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $\Psi(\cdot), \Phi(\cdot) : \mathbb{R} \to \mathbb{R}$.
ξ: an $\mathscr{F}_T$-measurable random variable.

BSDE w/ Conditional Mean Field and Markovian Switching

A triple $(Y_t, Z_t, \Lambda_t) \in \mathscr{S}^2_{\mathscr{F}}(0,T;\mathbb{R}) \times \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}) \times \mathscr{M}^2_{\mathscr{F}}(0,T;\mathbb{R})$ is a soln of the BSDE

$$\begin{aligned} dY_t &= F\Big(t, Y_t, Z_t, \mathbb{E}\big(\Psi(Y_t)\big|\mathscr{F}^{\alpha}_{t-}\big), \mathbb{E}\big(\Phi(Z_t)\big|\mathscr{F}^{\alpha}_{t-}\big)\Big)dt + Z_t dW_t + \Lambda_t \bullet dM_t, \\ Y_T &= \xi, \end{aligned} \tag{12}$$

if for $0 \le t \le T$,

$$\begin{aligned} Y_t = \xi - \int_t^T F\Big(s, Y_s, Z_s, \mathbb{E}\big(\Psi(Y_s)\big|\mathscr{F}^{\alpha}_{s-}\big), \mathbb{E}\big(\Phi(Z_s)\big|\mathscr{F}^{\alpha}_{s-}\big)\Big)ds \\ - \int_t^T Z_s dW_s - \int_t^T \Lambda_s \bullet dM_s. \end{aligned}$$

Theorem

Under suitable conditions, (12) *has a unique solution* $(Y_t, Z_t, \Lambda_t) \in \mathscr{S}^2_{\mathscr{F}}(0,T;\mathbb{R}) \times \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R}) \times \mathscr{M}^2_{\mathscr{F}}(0,T;\mathbb{R})$.

Adjoint Equation

$$
\begin{aligned}
d\overline{p}_t &= -\Big[\overline{b}_x(t)\overline{p}_t + \overline{\sigma}_x(t)\overline{q}_t + \overline{h}_x(t)\Big]dt \\
&\quad - \Big[\mathbb{E}\big(\overline{b}_y(t)\overline{p}_t \big| \mathscr{F}^{\alpha}_{t-}\big)\overline{\psi}_x(t) + \mathbb{E}\big(\overline{\sigma}_y(t)\overline{q}_t \big| \mathscr{F}^{\alpha}_{t-}\big)\overline{\varphi}_x(t) \\
&\qquad + \mathbb{E}\big(\overline{h}_y(t)\big|\mathscr{F}^{\alpha}_{t-}\big)\overline{\phi}_x(t)\Big]dt + \overline{q}_t dW_t + \overline{\lambda}_t \bullet dM_t, \qquad (13) \\
\overline{p}_T &= \overline{g}_x(T) + \mathbb{E}\big(\overline{g}_y(T)\big|\mathscr{F}^{\alpha}_{T-}\big)\overline{\eta}_x(T). \qquad (14)
\end{aligned}
$$

Lemma

The following identity holds

$$
\begin{aligned}
\mathbb{E}\big(\overline{p}_T z_T\big) = \mathbb{E}\Bigg[\int_0^T \Big(&\overline{p}_t \overline{b}_u(t)(v_t - \overline{u}_t) \\
&-\overline{h}_x(t) z_t - \mathbb{E}\big(\overline{h}_y(t)\big|\mathscr{F}^{\alpha}_{t-}\big)\overline{\phi}_x(t) z_t + \overline{q}_t \overline{\sigma}_u(t)(v_t - \overline{u}_t)\Big)dt\Bigg].
\end{aligned}
$$

Hamiltonian

For $(t, x, \mu, u, p, q, i_0) \in [0, T] \times \mathbb{R} \times \mathscr{P} \times \mathbb{R} \times \mathscr{M}$, define the Hamiltonian

$$\begin{aligned} & H(t, x, u, p, q, i_0) \\ & \quad = h\big(t, x, \mathbb{E}(\phi(x)|\mathscr{F}_{t-}^{\alpha}), u, i_0\big) \\ & \qquad + b\big(t, x, \mathbb{E}(\psi(x)|\mathscr{F}_{t-}^{\alpha}), u, i_0\big) p \\ & \qquad + \sigma\big(t, x, \mathbb{E}(\varphi(x)|\mathscr{F}_{t-}^{\alpha}), u, i_0\big) q. \end{aligned} \tag{15}$$

Representation of the Gateaux Derivative of J.

We can represent the Gateaux derivative of the cost functional in terms of the Hamiltonian H.

Lemma

The Gateaux derivative of the cost functional can be expresses in terms of the Hamiltonian H in the following way

$$\begin{aligned}\frac{d}{d\theta}J\big(\overline{u}+\theta(v-\overline{u})\big)\Big|_{\theta=0} &= \mathbb{E}\bigg(\int_0^T\Big(\overline{h}_u(t)(v_t-\overline{u}_t)+\overline{p}_t\overline{b}_u(t)(v_t-\overline{u}_t)\\ &\qquad +\overline{q}_t\overline{\sigma}_u(t)(v_t-\overline{u}_t)\Big)dt\bigg)\\ &= \mathbb{E}\bigg(\int_0^T\frac{d}{du}H(t,\overline{X}_t,\overline{u}_t,\overline{p}_t,\overline{q}_t,\alpha_{t-})(v_t-\overline{u}_t)dt\bigg).\end{aligned} \tag{16}$$

Necessary Conditions

Theorem

(Necessary Conditions). *Under suitable assumptions, if* $\overline{u}_t$ *is an optimal control with state trajectory* $\overline{X}_t$, *then* $\exists\ (\overline{p}_t, \overline{q}_t, \overline{\lambda}_t)$ *of adapted processes which satisfies the BSDE* (13) *and* (14) *such that*

$$\frac{d}{du}H(t, \overline{X}_t, \overline{u}_t, \overline{p}_t, \overline{q}_t, \alpha_{t-})(v - \overline{u}_t) \geq 0, \quad \mathbb{P}\text{-a.s., for all } t \in [0, T] \text{ and } v \in U.$$

Additional Assumptions

(C3) The functions $\psi(\cdot)$, $\phi(\cdot)$, $\varphi(\cdot)$, and $\eta(\cdot)$ are convex, the function $g(\cdot,\cdot,\cdot)$ is convex in (x,y), and the Hamiltonian $H(\cdot,\cdot,\cdot,\cdot,\cdot,\cdot)$ is convex in $(\widehat{x},u)$.

(C4) The functions $b_y(\cdot,\cdot,\cdot,\cdot,\cdot)$, $\sigma_y(\cdot,\cdot,\cdot,\cdot,\cdot)$, $h_y(\cdot,\cdot,\cdot,\cdot,\cdot)$, and $g_y(\cdot,\cdot,\cdot)$ are nonnegative.

Sufficient Conditions

Assume additionally

- The functions ψ, ϕ, φ, and η are convex, the function g is convex in (x, y), and the Hamiltonian H is convex in (x, y, u).
- The functions b_y, σ_y, h_y, and g_y are nonnegative.

Theorem

(Sufficient Conditions). *Assume that conditions in the result of necessary conditions and above hold. Let $\bar{u}$ be a control in $\mathscr{U}$ with the corresponding state trajectory $\bar{X}_t$. Let $(\bar{p}_t, \bar{q}_t, \bar{\lambda}_t)$ be the solution to the adjoint equation. If*

$$H(t, \bar{x}_t, \bar{u}_t, \bar{p}_t, \bar{q}_t) = \inf_{v \in \mathscr{U}} H(t, \bar{x}_t, v, \bar{p}_t, \bar{q}_t), \quad \forall t \in [0, T] \tag{17}$$

then $\bar{u}_t$ is an optimal control.

Application to LQG Contol

Consider the following LQG problem with Markovian switching and mean-field interaction

System:

$$
\begin{aligned}
dX_t = & \Big[A(\alpha_{t-})X_t + \widehat{A}(\alpha_{t-})\mathbb{E}\big(X_t \big| \mathscr{F}^{\alpha}_{t-}\big) + B(\alpha_{t-})u_t\Big]dt \\
& + \Big[C(\alpha_{t-})X_t + \widehat{C}(\alpha_{t-})\mathbb{E}\big(X_t \big| \mathscr{F}^{\alpha}_{t-}\big) + D(\alpha_{t-})u_t\Big]dW_t \\
X_0 = & \, x_0.
\end{aligned} \tag{18}
$$

Cost:

$$
J(u) = \frac{1}{2}\mathbb{E}\Big[\int_0^T \Big(R(\alpha_{t-})X_t^2 + N(\alpha_{t-})u_t^2\Big)dt + S(\alpha_T)X_T^2\Big]. \tag{19}
$$

We will use maximum principle to solve the optimal control problem.

Adjoint equation:

$$
\begin{aligned}
dp_t = &-\Big[A(\alpha_{t-})p_t + C(\alpha_{t-})q_t + R(\alpha_{t-})X_t + \widehat{A}(\alpha_{t-})\mathbb{E}\big(p_t \big| \mathscr{F}^{\alpha}_{t-}\big) + \widehat{C}(\alpha_{t-})\mathbb{E}\big(q_t \big| \mathscr{F}^{\alpha}_{t-}\big)\Big] dt \\
&+ q_t dW_t + \Lambda_t \bullet dM_t, \\
p_T = &\, S(\alpha_{T-})X_T.
\end{aligned}
$$

Hamiltonian:

$$
\begin{aligned}
H(t,x,u,p,q,i_0) = &\frac{1}{2}\Big[R(i_0)x^2 + N(i_0)u^2\Big] + \Big[A(i_0)x + \widehat{A}(i_0)\mathbb{E}\big(x \big| \mathscr{F}^{\alpha}_{t-}\big) + B(i_0)u\Big]p \\
&+ \Big[C(i_0)x + \widehat{C}(i_0)\mathbb{E}\big(x \big| \mathscr{F}^{\alpha}_{t-}\big) + D(i_0)u\Big]q.
\end{aligned}
$$

Necessary condition for optimality leads to

$$
N(\alpha_{t-})u_t = -\Big[B(\alpha_{t-})p_t + D(\alpha_{t-})q_t\Big].
$$

Denote $\widehat{X}_t = \mathbb{E}(X_t | \mathscr{F}^{\alpha}_{t-})$, $\widehat{p}_t = \mathbb{E}(p_t | \mathscr{F}^{\alpha}_{t-})$, $\widehat{q}_t = \mathbb{E}(q_t | \mathscr{F}^{\alpha}_{t-})$, $A_t = A(\alpha_{t-})$, and $\widehat{A}_t = \widehat{A}(\alpha_{t-})$. The functions $B_t, C_t, \widehat{C}_t, \ldots$ are defined by a similar way.

The feedback control system takes the form

$$\begin{aligned}
dX_t &= \left[A_t X_t + \widehat{A}_t \widehat{X}_t - \frac{B_t^2}{N_t} p_t - \frac{B_t D_t}{N_t} q_t\right] dt + \left[C_t X_t + \widehat{C}_t \widehat{X}_t - \frac{D_t B_t}{N_t} p_t - \frac{D_t^2}{N_t} q_t\right] dW_t, \\
dp_t &= -\left[A_t p_t + C_t q_t + R_t X_t + \widehat{A}_t \widehat{p}_t + \widehat{C}_t \widehat{q}_t\right] dt + q_t dW_t + \Lambda_t \bullet dM_t, \\
X_0 &= x, \\
p_T &= S(\alpha_T) X_T,
\end{aligned}$$

a fully coupled mean-field forward-backward SDE w/ Markovian switching.

To solve this system, we put

$$p_t = v(t, \alpha_t) X_t + \gamma(t, \alpha_t) \mathbb{E}(X_t | \mathscr{F}^{\alpha}_{t-}) \tag{20}$$

By equalizing the coefficients of X_t and then $\widehat{X}_t$ we obtain the following equations for v_t and γ_t.

$$\begin{cases} v_t' + \left(2A_t + C_t^2\right) v_t - \dfrac{\left(B_t + D_t C_t\right)^2 v_t^2}{N_t + v_t D_t^2} + R_t + Q v_t(\alpha_t) = 0, \\ v_T = S(\alpha_T), \end{cases} \tag{21}$$

and

$$\begin{cases} \gamma_t' + 2\left(A_t + \widehat{A}_t\right)\gamma_t + \left(2C_t\widehat{C}_t + \widehat{C}_t^2 + 2\widehat{A}_t\right) v_t \\ \qquad - \dfrac{B_t\gamma_t + \widehat{C}_t D_t v_t}{N_t + D_t^2 v_t}\left[\left(2B_t + 2C_t D_t + \widehat{C}_t D_t\right) v_t + B\gamma_t\right] + Q\gamma_t(\alpha_t) = 0, \\ \gamma_T = 0. \end{cases} \tag{22}$$

Note that (21) is a Riccati equation which can be rewritten as follow

$$\begin{cases} v'(t,i_0) + \left(2A(i_0) + C^2(i_0)\right) v(t,i_0) - \dfrac{\left(B(i_0) + D(i_0)C(i_0)\right)^2 v^2(t,i_0)}{N(i_0) + v(t,i_0)D(i_0)^2} \\ \qquad + R(i_0) + \sum_{j_0\in\mathscr{M}} q_{i_0 j_0}\left(v(t,j_0) - v(t,i_0)\right) = 0, \\ v(T,i_0) = S(i_0). \end{cases} \tag{23}$$

Theorem

The optimal control $u_t \in \mathscr{U}$ for the linear quadratic control problem is given in feedback form by

$$u_t = -\frac{\Big[B(\alpha_{t-}) + C(\alpha_{t-})D(\alpha_{t-})\Big]\nu_t X_t + \Big[B(\alpha_{t-})\gamma_t + \widehat{C}(\alpha_{t-})D(\alpha_{t-})\nu_t\Big]\widehat{X}_t}{N(\alpha_{t-}) + D^2(\alpha_{t-})\nu_t}$$

where ν_t and γ_t are solutions to (21) *and* (22), *respectively.*

Numerical Simulation

Let us consider the linear quadratic equation in which the Markovian process takes two possible values 1 and 2 (i.e., $\mathscr{M} = \{1,2\}$) with the generator

$$Q = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$$

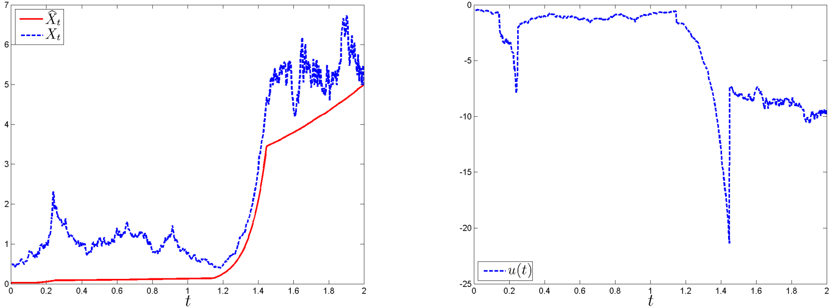
and the initial condition $x_0 = 5$. For illustration purpose, assume that $T = 2$ and that the coefficients of the dynamic equation are given by

$$\begin{array}{llllll} A(1) = 2, & \widehat{A}(1) = 1, & B(1) = 2, & C(1) = 1, & \widehat{C}(1) = 2, & D(1) = 2, \\ A(2) = 5, & \widehat{A}(2) = 5, & B(2) = 4, & C(2) = 2, & \widehat{C}(2) = 3, & D(2) = 1. \end{array}$$

We also consider the cost function defined by equation (19) with

$$\begin{array}{lll} N(1) = 1, & R(1) = 2, & S(1) = 4, \\ N(2) = 4, & R(2) = 5, & S(2) = 2. \end{array}$$

To study the behavior of the solution to equation (18) and the corresponding mean-field terms, we first generate 10000 independent Brownian motions and then use Euler's method to achieve approximations of X_t and $\widehat{X}_t = \mathbb{E}(X_t | \mathscr{F}^{\alpha}_{t-})$. Their graphs are shown in Figure 2 in which the more fluctuating function is X_t and the smoother function is $\widehat{X}_t$.



(a) A trajectory of solution X_t and $\widehat{X}_t$

(b) The optimal control u_t

Figure 2: Simulations for X_t, $\widehat{X}_t$, and u_t

Further Remarks

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- Regarding the Limit Theory
  - The McKean-Vlasov involves random measure
  - The rate of convergence is a future research topic

Remarks

- Regarding the Limit Theory
  - The McKean-Vlasov involves random measure
  - The rate of convergence is a future research topic
- Regarding maximum principle
  - Consideration of the models is largely due to the current needs of handling networked control systems.
  - The results cannot be obtained using known results in the literature.
  - Main idea and insight of the paper are the use of conditional mean field and the use of a law of large numbers for such processes.
  - Future work: mean-field games, more general maximum principles.

Thank you

Assumptions for BSDE

(B1) $\mathbb{E}|\xi|^2 < \infty$.

(B2) $F(t,Y,Z,y,z)$ is $\mathscr{F}$-progressively measurable for each (Y,Z,y,z) and $F(t,0,0,0,0) \in \mathscr{L}^2_{\mathscr{F}}(0,T;\mathbb{R})$.

(B3) There exists a constant $K > 0$ such that for $t \in [0,T]$ and $Y,Z,Y',Z',y,z,y',z' \in \mathbb{R}$,

$$\begin{aligned}&\Big|F\big(t,Y,Z,y,z\big) - F\big(t,Y',Z',y',z'\big)\Big| \\ &\quad \le K\Big(|Y-Y'| + |Z-Z'| + |y-y'| + |z-z'|\Big) \quad \text{a.s.}\end{aligned}$$

and

$$\Big|\Psi(y) - \Psi(y')\Big| \vee \Big|\Phi(z) - \Phi(z')\Big| \le K\Big(|y-y'| + |z-z'|\Big).$$