Mean-Risk Portfolio Selection with Law-Invariant Coherent Risk Measure

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A Joint work with Xuedong He and Xun Yu Zhou

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    - the preference is not a *total order*.
- We study the explicit way, which is more intuitive.

Mean-risk portfolio selection in continuous-time market

- We consider our mean-risk problem in a continuous-time, *arbitrage-free*, and *complete* financial market, with interest rate $r \equiv 0$.
  - A standard Black-Scholes market is an often-used example.
  - Here we do not need many details of the market other than the unique *pricing kernel*, denoted as ξ, which satisfies $\mathbb{E}[\xi] = 1$.
  - For any terminal wealth X_T, we have $X_0 = \mathbb{E}[\xi X_T]$.
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- Assumption: $\xi > 0$ admits *no atom*, i.e., $P(\xi = x) = 0$ for $\forall\, x \in \mathbb{R}$.
- For an investor with initial wealth $x > 0$, the mean-risk portfolio selection in the time period $[0, T]$ can be formulated as

$$\begin{aligned} \text{Min} \quad & \rho(X_T) \\ s.t. \quad & (X_\cdot, \pi_\cdot) \text{ is a wealth-portfolio process with } X_0 = x, \\ & \mathbb{E} X_T \geq l. \end{aligned} \tag{1}$$

where ρ is the investor's sense on the risk, $l > x$ is a target level.

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- *Law-invariant* risk measure: risk is fully described by distribution.

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- Any *convex combination* of $CV@R$ is law-invariant, coherent, and comonotonic.
- We aim at the mean-risk portfolio selection with law-invariant coherent risk measure.

Representation of law-invariant coherent risk measures

Theorem 1: Denote $\mathcal{P}([0,1])$ as the set of probability measure on $[0,1]$. With some regularity condition, ρ is a law-invariant convex risk measure if and only if

$$\rho(X) = \sup_{\mu \in \mathcal{A}} \left\{ \int_{[0,1]} CV@R_z(X)\mu(dz) \right\}$$

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ρ is furthermore comonotonic if and only if

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- For $m \in \mathcal{P}([0,1])$, define the weighted V@R by

$$WV@R_m(X) = \int_0^1 V@R_z(X)m(dz),$$

then a law-invariant coherent risk measure can be written as $\sup_{\mu\in\mathcal{B}} WV@R_{\varphi(\mu)}(X)$, where $\varphi(\mu)$ is defined as the probability measure $\varphi(\mu)(dz) = \int_z^1 \frac{1}{\beta}\mu(d\beta)$.

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- $WV@R$ is the building block for law-invariant coherent risk measure.

Martingale approach

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- For the portfolio selection problem (1), we can firstly solve the optimal terminal wealth X^* by

$$\begin{aligned} \text{Min} \quad & \rho(X) \\ s.t. \quad & \mathbb{E}[\xi X] = x_0, \\ & \mathbb{E}[X] \geq l. \end{aligned} \tag{2}$$

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  and then replicate the optimal terminal wealth X^*.
- Since replication in a complete market is theoretically easy by, e.g., martingale representation, we focus the first step for optimal terminal wealth X^*.

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- Redefine $\rho(G) := \rho(X)$ with $G = F_X^{-1}$, and denote $Z = F_\xi(\xi)$.
- Then (2) can be reformulated into

$$
\begin{aligned}
\mathrm{Min} \quad & \rho(G) \\
s.t. \quad & \mathbb{E}[\xi G(1 - Z)] = x_0, \\
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Theorem 2: Denote V as the optimal value for problem (3), and suppose $\operatorname{essinf} \xi = 0$. Then

(i) If $\gamma^* > 1$, then $V = -\infty$; If $\gamma^* \leq 1$, then $V = -x$.

(ii) If $\gamma^* < 1$, there exists a sequence of $X_n = a_n + b_n \mathbf{1}_{\xi \leq c_n}$ asymptotically optimal, where $c_n \downarrow 0$, $b_n \uparrow +\infty$ and $a_n \to x$.

(iii) If $\gamma^* = 1$ and achieved by c^*, then $X^* = a + b\mathbf{1}_{\xi \leq F_\xi^{-1}(1-c^*)}$ for some $a \in \mathbb{R}$, $b > 0$.

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Theorem 3: Suppose $\mathrm{essinf}\xi = 0$. Then

(i') If $\gamma^* \leq 1$, then $\hat{V} = -x$. If $\gamma^* \in (1, +\infty)$, then $\hat{V} = -\gamma^* x$.

(ii') There exists an optimal solution X^* iff $V > -\infty$ and γ^* is obtained by some $c^* \in (0, 1)$, in which case $X^* = b\mathbf{1}_{\xi \leq F_\xi^{-1}(1-c^*)}$ for some b.

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- The optimal value $\hat{V}$ does not depend on l, but does depend on μ.
- $\hat{V}$ may not be asymptotically approached by $X_n = a_n + b_n\mathbf{1}_{\xi \leq c_n}$.

When ρ is coherent and law-invariant

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- If we can *swap* $\min$ and $\sup$, then the minimization over G is the same as that for WV@R.

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- For problem (4) w/o no-bankruptcy constraint, we have another critical quantity

$$\gamma_{\mathcal{A}} := \inf_{\mu \in \mathcal{A}} \sup_{0<c<1} \frac{\varphi(\mu)((c,1])}{\int_c^1 F_\xi^{-1}(1-z)dz}.$$

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Theorem 5: Suppose $\text{essinf}\xi = 0$. Denote V_c and $\hat{V}_c$ as the optimal value for problem (4) without and with the no-bankruptcy constraint.

- For problem (4) without no-bankruptcy constraint, $V_c > -\infty$ iff $\gamma_{\mathcal{A}} \le 1$. When $V_c > -\infty$, we have $V_c = -x$.
- For problem (4 with no-bankruptcy constraint, $\hat{V}_c > -\infty$ iff $\gamma_{\mathcal{A}} < +\infty$. When $V_c > -\infty$, we have $V_c = -x \max(\gamma_{\mathcal{A}}, 1)$.

Questions and Comments