# Implementing Portfolio Liquidation Models

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March 18, 2019

# Outline

- portfolio liquidation models
  - general stochastic models
  - deterministic models
- empirical implementation
  - linking market impact to market microstructure
  - empirical liquidation startegies

# Portfolio liquidation

# Portfolio Liquidation

- almost all trading nowadays takes place in limit order markets
  - limit order book: list of prices and available liquidity
  - limited liquidity available at each price level
- models of optimal portfolio liquidation:
  - unaffected benchmark price
  - execution price: benchmark price + impact from trading
  - cost of trading: book value - revenues (+risk)
  - liquidation constraint: singular control problem

# Forms of market impact

- instantaneous impact
  - current trade does not affect future trades
  - pure liquidity cost; immediate recovery
- permanent impact
  - current trade affects all future trades
  - generates a drift of the benchmark price/midquote
- persistent impact
  - impact of current trade on future trades decays over time
  - generates a mean-reverting drift of the benchmark price/midquote

# General stochastic models

# Stochastic models

Consider an order to sell $X$ shares by time $T$. The portfolio process is

$$X_t = X - \int_0^t \xi_s \, ds;$$

the liquidation constraint is $X_T = 0$. The transaction price process is

$$\widetilde{S}_t = \underbrace{S_t}_{\text{unaffected price}} - \underbrace{\eta_t \xi_t}_{\text{instantaneous impact}} - \underbrace{Y_t}_{\text{persistent impact}} - \underbrace{\int_0^t \lambda_s \xi_s ds}_{\text{permanent impact}}$$

where

$$Y_t = \int_0^t \{-\rho Y_s + \gamma_t \xi_s\} \, ds.$$

denotes a mean-reverting "spread" or "midquote" or ...

# Stochastic models

The *liquidation cost* is defined as

$$
\begin{aligned}
\mathcal{C} &= \text{book value} - \text{revenue} \\
&= S_0 X - \int_0^T \widetilde{S}_t \xi_t \, dt = S_0 X - \int_0^T \widetilde{S}_t \, dX_t \\
&= S_0 X - \int_0^T \xi_t dS_t + \int_0^T \lambda_t \xi_t X_t dt + \int_0^T \eta_t \xi_t^2 \, dt + \int_0^T Y_t \xi_t \, dt
\end{aligned}
$$

Taking expectations, doing partial integration, adding a risk term:

$$
E\left[\int_0^T \left(\lambda_t \xi_t X_t + \eta_t \xi_t^2 + Y_t \xi_t + \kappa_t X_t^2\right) dt\right] \longrightarrow \min_{\xi} \quad \text{s.t. } X_T = 0.
$$

The impact terms award, the risk term penalises slow liquidation.

## Theorem (Graewe, H. & Sere (2018), H. & Xia (2018), ...)

*Suppose there is only instantaneous market impact and that*

$$\eta_t \equiv \eta(Z_t); \quad \kappa_t \equiv \kappa(Z_t)$$

*for some Itô diffusion $Z$. Under standard assumptions,*

$$V(t,z,x) = v(t,z)x^2, \quad \xi^*(t,z,x) = 2v(t,z)x$$

*where $v$ is the unique continuous viscosity/classical/$\pi$-strong solution in*

$$C_{poly}([0, T^-] \times \mathbb{R}^d)$$

*to a singular terminal value problem of the form*

$$\begin{cases} -\partial_t v - \mathcal{L}v - F = 0, & \textit{on } [0, T) \times \mathbb{R}^d, \\ \lim\limits_{t \to T} v(t,z) = +\infty & \textit{locally uniformly on } \mathbb{R}^d. \end{cases}$$

## Theorem (Graewe & H. (2017), H.& Xia (2018))

*Suppose there is only instantaneous and persistent impact*

$$\eta_t \equiv \eta; \quad \gamma_t \equiv \gamma; \quad (\rho_t), (\kappa_t) \text{ adapted processes.}$$

*Then,*

$$\xi_t^* = \frac{A_t - \gamma B_t}{\eta} X_t - \frac{\gamma C_t - B_t + 1}{\eta} Y_t$$

*where $(A, B, C)$ is the unique solution to a coupled (matrix-valued) BS(R)DE system with singular terminal condition*

$$(A_t, B_t, C_t) \longrightarrow (\infty, 1, 0) \quad \text{as } t \to T.$$

# Deterministic models

# Constant coefficients and risk neutrality

Consider the state dynamics

$$X_t = x - \int_0^t \xi_s ds$$
$$Y_t = -\rho \int_0^t Y_s ds + \gamma \int_0^t \xi_s ds$$

as well as the following cost terms:

- instantaneous impact: $H_t = \eta \int_0^t \xi_s^2 ds$
- permanent impact: $G_t = \lambda \int_0^t \xi_s ds$
- persistent impact: $Y_t = \gamma \int_0^t \xi_s e^{-\rho(t-s)} ds$
- total cost:

$$C = H_T + \int_0^T \xi_s Y_s ds + \int_0^T \xi_s G_s ds$$

# Constant coefficients and risk neutrality

- the cost from *permanent* impact is independent of the strategy
- the Euler-Lagrange ansatz yields:

$$\frac{\gamma}{2\eta}\int_0^T y' e^{-\rho|t-s|} ds + y' = C$$

- this is a Wiener-Hopf integral equation of the second kind

# Constant coefficients and risk neutrality

- only *instantaneous* impact (Almgren-Chris model)

$$X_t^* = x - \frac{t}{T}x$$

- only *persistent* impact (Obizhaeva-Wang model)

$$X_t^* = x - \frac{x}{\rho T + 2}\left(H_0(t) + \rho t + H_T(t)\right)$$

- *instantaneous and persistent* impact (Graewe-H model)

$$X_t^* = x - x\left(\frac{a + bt + c\sinh(k(t - \frac{T}{2}))}{2a + bT}\right)$$

for constants $a, b, c, k$ depending on the impact parameters

# Interpolating between AC and OW

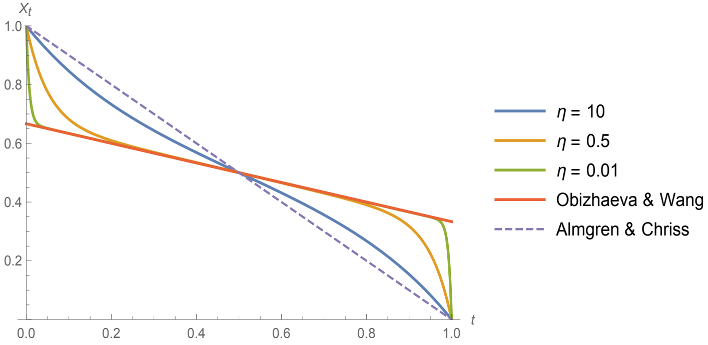



Figure: Optimal portfolio processes in the GH model

# Microstructure and market impact

# Microstructure and market impact

- *permanent impact*: drift added to the fundamental price process after each trade
  - information asymmetries
  - order imbalances
- *temporary impact*: expectation of future permanent impact, due to the persistence of trade flows
  - herding effects
  - splitting effects
  - mathematical model: Hawkes processes
- *instantaneous impact*: market makers' demand for carrying additional inventory (offer curve)

# Permanent impact (added drift)

- let

$$\delta_t = \#\text{buy} - \#\text{sell MOs since last price change}$$

- let the probability $p_t$ of a mid-price up movement be

$$p_t = f(g(\delta_t)), \quad g(\delta) = B_0 + B_1\delta, \quad f(x) = \frac{1}{1+e^{-x}}$$

- mid-price process is a martingale if

$$\delta_t = \bar{\delta} = \frac{f^{-1}(\frac{1}{2}) - B_0}{B_1}$$

- we define the permanent impact and permanent impact factor as

$$\Lambda := f(g(\bar{\delta}+1))\bar{Z}; \quad \lambda := f(g(\bar{\delta}+1))\frac{\bar{Z}}{\bar{L}}$$

# Temporary impact (expected future perm. impact)

- we assume that MO arrivals follow a Hawkes process $N$ with intensity

$$I_t^m = \mu^m + A \int_0^t e^{-B(t-t_i)} dN_s^m$$

- adding our market order placement dynamics, the intensity becomes

$$I_t^m = \mu^m + \frac{x_0}{T} + A \int_0^t e^{-B(t-t_i)} dN_s^m$$

- the expected number of additional orders is

$$x_0 \frac{P}{1-P}, \quad P = \frac{A}{B}$$

# Temporary impact (expected future perm. impact)

- equating this with the total impact in a continuous time model:

$$\frac{x_0\gamma}{T}\int_0^T e^{-\rho(T-s)}ds = \lambda x_0 \frac{P}{1-P}$$

- assuming $\gamma = \lambda$, using a Taylor approximation of order two,

$$\rho \approx \frac{2}{T}\frac{1-2P}{1-P}, \quad P < 0.5$$

- if we only consider first generation offsprings,

$$\rho \approx \frac{2}{T}(P^{-1}-1)$$

# Instantaneous impact (limit order arrivals)

- orders are added/cancelled at Poison rates
- order sizes are random
- mid-price shift implies a shift in the queues

# Instantaneous impact (limit order arrivals)

- liquidity is scattered throughout the book ("many holes")
- regression equation

$$O_i = p_0 + \eta E_i + \epsilon_i$$

where

- $O_i$ are the price offsets (differences between price levels with liquidity)
- $p_0$ is the minimum spread
- $E_i$ is the aggregated average liquidity

| level | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| arrival | 0.875 | 0.254 | 0.156 | 0.098 |
| cancellation | 0.507 | 0.130 | 0.079 | 0.084 |
| aggregate shares | 95 | 190 | 290 | 356 |
| offests | 42.8 | 61.6 | 76.3 | 89.7 |

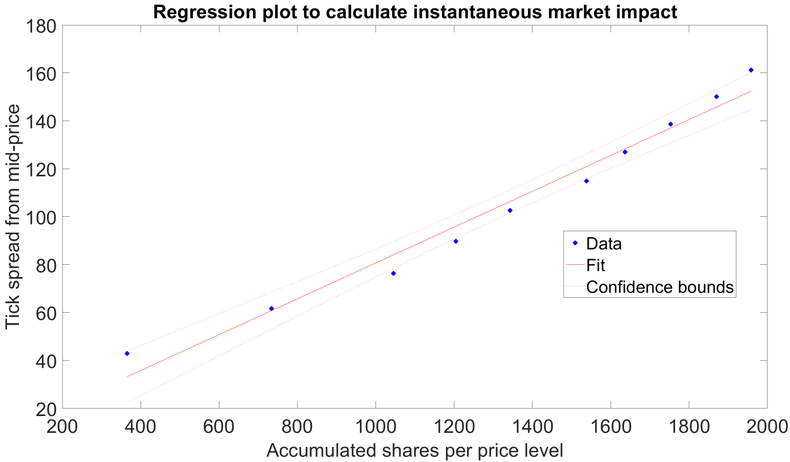



Figure: Instantaneous impact factor for AMZN.

# A two-layer order book model

- order layer:
  - limit orders + cancellation (Poisson arrivals)
  - market orders (Hawkes arrivals)
    - originating from the market
    - originating from us
- price layer (Poisson arrivals with rate $\mu^P$)

# A two-layer order book model

- we calibrate the model
- simulate the LOB with and without our strategy
- compute the cost of liquidation for different models

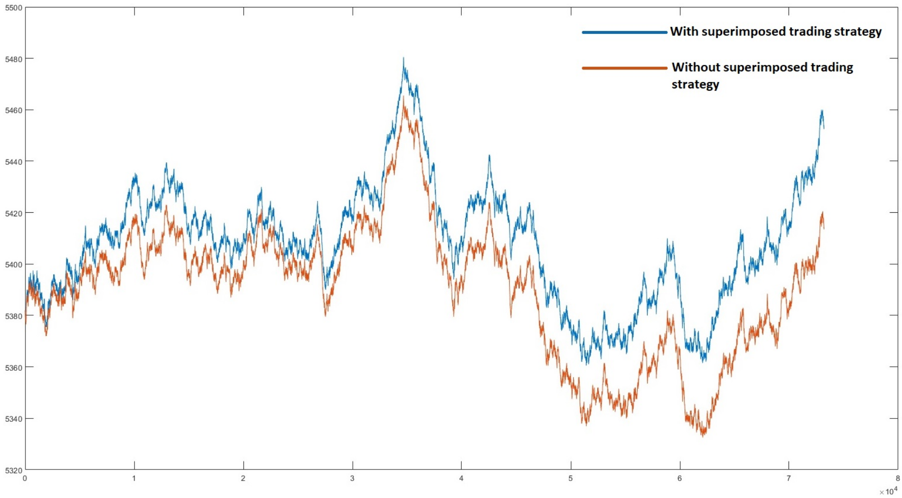



Figure: A sample path of midprice shift due to our trading activity.

# Calibration

# Data (LOBSTER, April 4, 2018)

| Level | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| Submission | 39,767 | 11,937 | 7,603 | 4,953 | 3,395 | 2,804 |
| Cancellation | 30,987 | 10,392 | 6,453 | 4,176 | 2,846 | 2,363 |
| Execution | 8,775 | 1,537 | 1,142 | 772 | 548 | 439 |

Table: Event counts per level: AMZN

# Impact factors: AMZN, 5% of ADV ($\Phi = 20$ sec.)

- price change parameters:

| Parameters | $\mu^P$ | $B_0$ | $B_1$ | $\bar{L}$ | $\bar{Z}$ |
|---|---|---|---|---|---|
| Mid price change | 2.38 | 0.033 | 0.894 | 33.807 | 10.797 |

- market order parameters:

| Parameters | $\mu^m$ | $A$ | $B$ |
|---|---|---|---|
| Sell orders | 0.167 | 8.375 | 18.53 |

- impact factors:

| $\lambda$ | $\rho$ | $\eta$ |
|---|---|---|
| 0.0162 bps | 0.0034 %/second | 0.013 bps |

# Optimal strategies

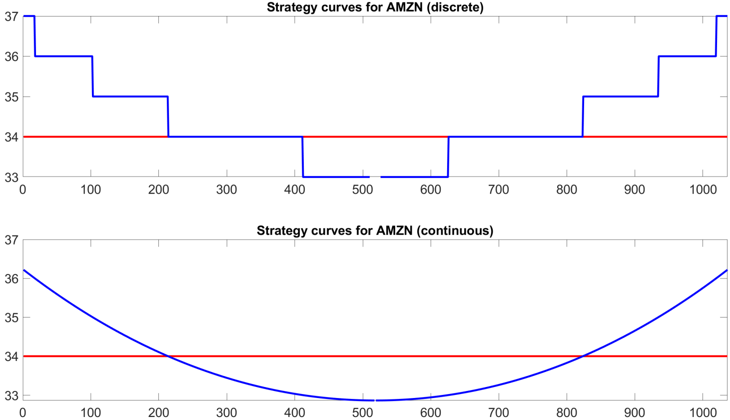



Figure: Optimal liquidation strategies for Amazon (+6%)

# Optimal strategies

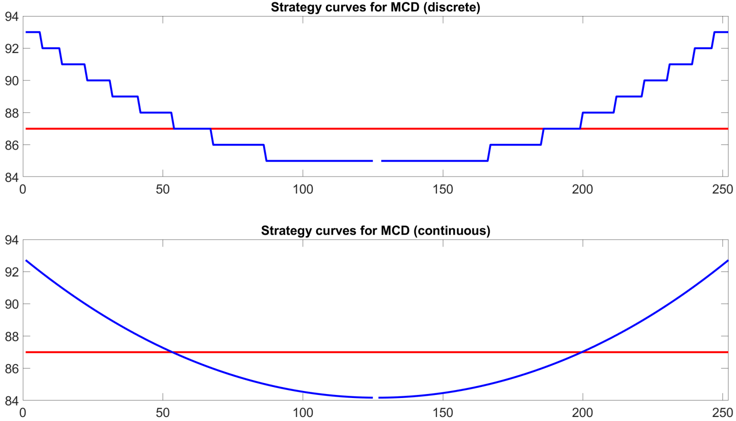



Figure: Optimal liquidation strategies for McDonald's (+6%)

# Optimal strategies

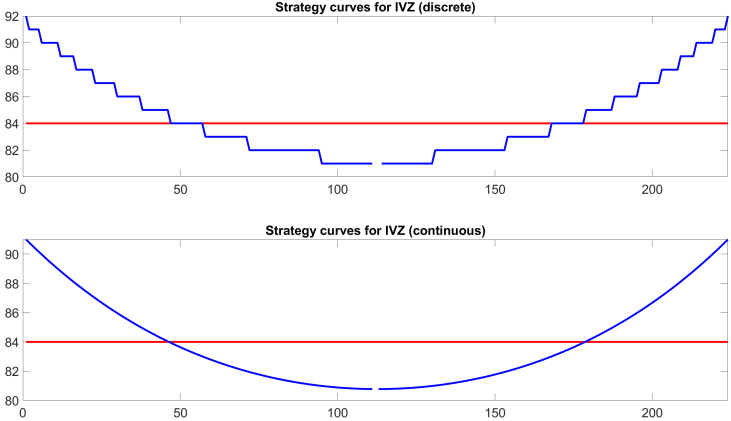



Figure: Optimal liquidation strategies for Ivesco (+10%)

# Optimal strategies

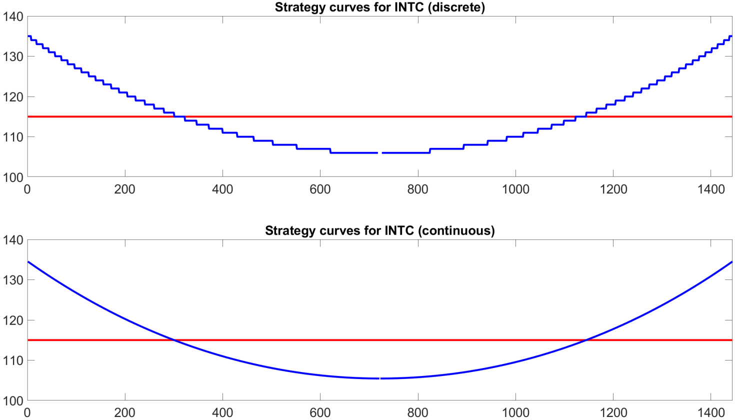



Figure: Optimal liquidation strategies for Intel (+17%)

# Optimal strategies

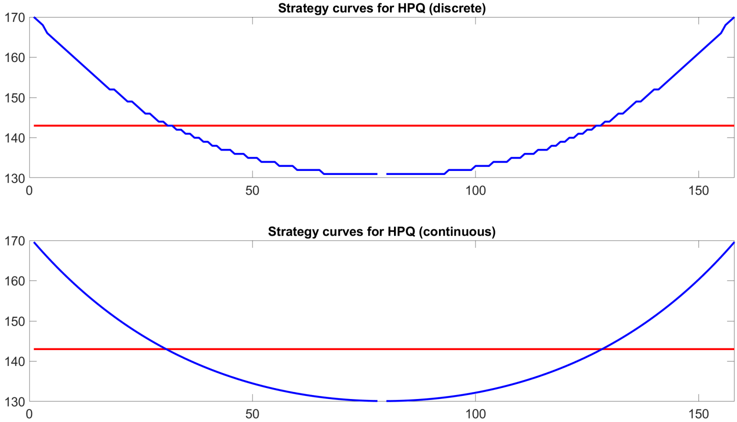
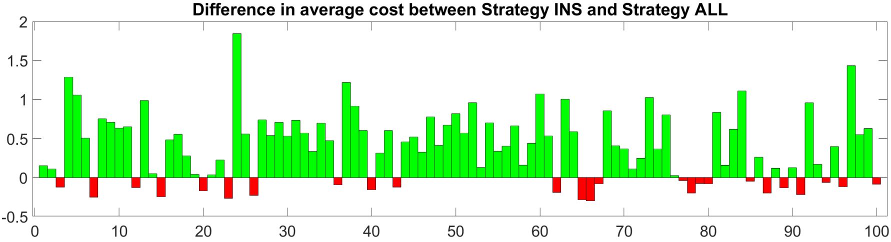



Figure: Optimal liquidation strategies for HP (+20%)

# Cost comparison



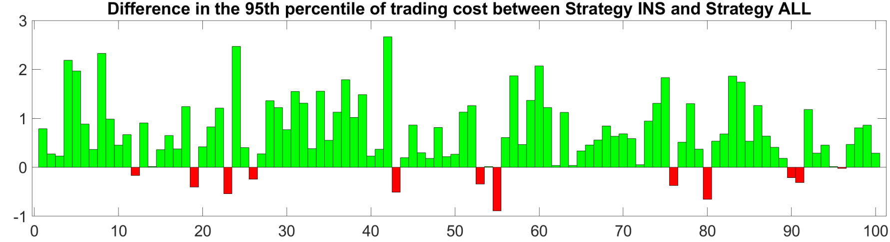




Figure: Model performance: AC vs. GH; for AMZN we save 1/3 spread

# Conclusion

- models of optimal portfolio liquidation with continuous trading
  - abstract existence and uniqueness of solutions results
  - closed form solutions for models with constant coefficients
- we compared the performance of models:
  - only instantaneous impact (AC model)
  - instantaneous and persistent impact (GH model)
- GH outperforms AC in most cases

# Thank you!