Optimal make-take fees for market making regulation

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Exchanges in competition

- With the fragmentation of financial markets, exchanges are nowadays in competition.
- Traditional international exchanges are now challenged by alternative trading venues.
- Consequently, they have to find innovative ways to attract liquidity on their platforms.
- A possible solution : using a make-take fees system, that is charging in an asymmetric way liquidity provision and liquidity consumption.

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A controversial topic

- Make-take fees policies are seen as a major facilitating factor to the emergence of a new type of market makers aiming at collecting fee rebates : the high frequency traders.
- As stated by the Securities and Exchanges commission : "Highly automated exchange systems and liquidity rebates have helped establish a business model for a new type of professional liquidity provider that is distinct from the more traditional exchange specialist and over-the-counter market maker."

HFT market makers

The concern with high frequency traders becoming the new liquidity providers is two-fold.

- Their presence implies that slower traders no longer have access to the limit order book, or only in unfavorable situations when high frequency traders do not wish to support liquidity.
- They tend to leave the market in time of stress.

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Our aim

- Providing a quantitative and operational answer to the question of relevant make-take fees.
- We take the position of an exchange (or of the regulator) wishing to attract liquidity. The exchange is looking for the best make-take fees policy to offer to market makers in order to maximize its utility.
- In other words, it aims at designing an optimal contract with the (unique) market marker to create an incentive to increase liquidity.
- Principal/agent type approach : the wealth of the principal (exchange) depends on the agent's (market maker) effort (essentially his spread), but the principal cannot directly control the effort.

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Market maker's controls

• The market maker has a view on the efficient price (midprice) of the asset

$$S_t = S_0 + \sigma W_t,$$

where σ is the price volatility.

• He fixes the ask and bid prices

$$P_t^a = S_t + \delta_t^a, \quad P_t^b = S_t - \delta_t^b.$$

Arrival of market orders

- We model the arrival of buy (resp. sell) market orders by a point process $(N_t^a)_{t\geq 0}$ (resp. $(N_t^b)_{t\geq 0}$) with intensity $(\lambda_t^a)_{t\geq 0}$ (resp. $(\lambda_t^b)_{t\geq 0}$).
- The inventory of the market maker $Q_t = N_t^b N_t^a$.
- We consider a threshold inventory \bar{q} above which the market maker stops quoting on the ask or bid side.
- From financial economics arguments :

$$\lambda_t^a = \lambda(\delta_t^a) \mathbf{1}_{\{Q_t > -\bar{q}\}}, \quad \lambda_t^b = \lambda(\delta_t^b) \mathbf{1}_{\{Q_t < \bar{q}\}}.$$

where $\lambda(x) = Ae^{-k(x+c)/\sigma}$.

Equivalent probabilities

The market maker controls the spread $\delta = (\delta^a, \delta^b)$. We define the associated probability \mathbb{P}^{δ} such that

$$\widetilde{\mathsf{N}}_t^{\mathsf{a},\delta} = \mathsf{N}_t^{\mathsf{a}} - \int_0^t \lambda(\delta_s^{\mathsf{a}}) \mathbb{1}_{\{Q_s > -ar{q}\}} ds$$

and

$$\widetilde{\mathsf{N}}_t^{b,\delta} = \mathsf{N}_t^b - \int_0^t \lambda(\delta^b_s) \mathbb{1}_{\{\mathcal{Q}_s < ar{q}\}} ds$$

are martingales.

The profit and loss of the market maker

- We consider a final time horizon T > 0.
- The cash flow of the market maker

$$X_t^{\delta} = \int_0^t P_u^a dN_u^a - \int_0^t P_u^b dN_u^b.$$

- The inventory risk of the market maker is $Q_t S_t$.
- For a given contract ξ given by the exchange, seen as an F_T measurable random variable, the market maker chooses his spread δ by maximizing his utility.

The market maker problem

Under the exchange incentive policy ξ , the market maker solves now

$$V_{MM}(\xi) = \sup_{\delta} \mathbb{E}^{\delta} \left[-\exp\left(-\gamma (X_{T}^{\delta} + Q_{T}S_{T} + \xi)\right) \right].$$

- We obtain an optimal response given by $\hat{\delta}_t(\xi) = (\hat{\delta}_t^a(\xi), \hat{\delta}_t^b(\xi)).$
- We will only consider contracts such that V_{MM}(ξ) is above a threshold utility value R :

$$C = \{\xi \ \mathcal{F}_T \text{-measurable such that } V_{MM}(\xi) > R\}$$

+ integrability conditions.

• For $\xi = 0$, well studied problem since Avellaneda and Stoikov.

We assume that the exchange

- Earns c > 0 for each market order occurring in its platform.
- Pays the incentive policy ξ to the market maker.

The profit and loss of the exchange is

$$c(N_T^a+N_T^b)-\xi.$$

The exchange problem

The exchange designs the contract ξ by solving

$$V_{E} = \sup_{\xi \in \mathcal{C}} \mathbb{E}^{\hat{\delta}(\xi)} \left[-\exp\left(-\eta (c(N_{T}^{a} + N_{T}^{b}) - \xi)\right) \right],$$

where η is the risk aversion of the exchange.

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Dynamic programming principle

- We fix ξ and compute the best response of the market maker.
- Let τ be a stopping time with values in [t, T] and μ ∈ A_τ, where A_τ denotes the restriction of the set of admissible controls A to controls on [τ, T].

• Let
$$J_T(\tau,\mu) = \mathbb{E}_{\tau}^{\mu} \left[-e^{-\gamma \int_{\tau}^{T} (\mu_u^a dN_u^a + \mu_u^b dN_u^b + Q_u dS_u)} e^{-\gamma \xi} \right]$$
 and
 $V_{\tau} = \operatorname{ess\,sup}_{\mu \in \mathcal{A}_{\tau}} J_T(\tau,\mu).$

• Dynamic programming principle :

$$V_t = \operatorname{ess\,sup}_{\delta \in \mathcal{A}} \mathbb{E}_t^{\delta} \Big[- e^{-\gamma \int_t^{\tau} (\delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u)} V_{\tau} \Big].$$

A convenient super-martingale

Let

$$U_t^{\delta} = V_t e^{-\gamma \int_0^t \delta_u^a dN_u^a + \delta_u^b dN_u^b + Q_u dS_u}.$$

• $U_0^\delta = V_0$ and

$$U_T^{\delta} = -e^{-\gamma \left(\int_0^T \delta^a_u dN^a_u + \delta^b_u dN^b_u + Q_u dS_u + \xi
ight)}$$

- From the DPP, we get that U_t^{δ} is a \mathbb{P}^{δ} -super-martingale. We want to find the optimal controls (δ^a, δ^b) turning it into a martingale.
- To do so we find a suitable representation of U_t^{δ} .

Doob-Meyer and martingale representation

- Doob-Meyer : U^δ_t = M^δ_t A^δ_t, where M^δ is a ℙ^δ-martingale and A^δ is an integrable non-decreasing predictable process starting at zero.
- Martingale representation theorem : There exists a predictable process $\widetilde{Z}^{\delta} = (\widetilde{Z}^{\delta,S}, \widetilde{Z}^{\delta,a}, \widetilde{Z}^{\delta,b})$ such that M_t^{δ} can be represented as

$$V_0 + \int_0^t \widetilde{Z}_r^{\delta} d\chi_r - \int_0^t \widetilde{Z}_r^{\delta,a} \lambda(\delta_r^a) \mathbb{1}_{\{Q_r > -\bar{q}\}} dr - \int_0^t \widetilde{Z}_r^{\delta,b} \lambda(\delta_r^b) \mathbb{1}_{\{Q_r < \bar{q}\}} dr,$$

with $\chi = (S, N^a, N^b)$.

Solving the market maker problem for a given contract

Reducing the class of contracts

- Let Y be the process defined by $V_t = -e^{-\gamma Y_t}$.
- $Y_T = \xi$ and using Ito's formula together with the previous result and the martingale property of U_t for the optimal controls we get

$$dY_t = Z_t^a dN_t^a + Z_t^b dN_t^b + Z_t^S dS_t - H(Z_t, Q_t) dt,$$

for an explicit function H and where the Z^i do not depend on δ .

- Any contract ξ can be (uniquely) represented under the preceding form ! We can restrict ourselves to such contracts.
- Natural financial interpretation of the contracts :
 - The exchange rewards the market maker by Z^a (resp. Z^b) for each buy (resp.sell) market order.
 - The exchange participates to the market/inventory risk of the market maker by taking $-Z^S$ of his share.
 - The market maker pays a continuous coupon $H(Z_t, Q_t)dt$.

New super-martingale representation

• In term of this new representation, we obtain

$$U_t^{\delta} = M_t^{\delta} + \gamma \int_0^t U_u^{\delta} (H(Z_u, q_u) - h(\delta_u, Z_u, q_u)) du,$$

where h is explicit and

$$H(z,q) = \sup_{|\delta^a| \vee |\delta^b| \leq \delta_\infty} h(\delta, z, q).$$

 The process U^δ_t becomes a martingale if and only if δ is chosen as the maximizer of h.

Optimal quotes

Let $\boldsymbol{\xi}$ be an admissible contract. The unique optimal spread of the market maker is given by

$$\hat{\delta}^{a}_{t}(\xi) = -Z^{a}_{t} + rac{1}{\gamma}\log(1+rac{\sigma\gamma}{k}), \quad \hat{\delta}^{b}_{t}(\xi) = -Z^{b}_{t} + rac{1}{\gamma}\log(1+rac{\sigma\gamma}{k}).$$

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Solving the exchange problem

By representing any contract $\xi = Y_0^{Y_0^{\xi}, Z^{\xi}}$, the exchange problem

$$V_{E} = \sup_{\xi \in \mathcal{C}} \mathbb{E}^{\hat{\delta}(\xi)} \left[-\exp\left(-\eta (c(N_{T}^{a} + N_{T}^{b}) - \xi)\right) \right]$$

is equivalent to

$$V_{E} = \sup_{Z} \mathbb{E}^{\hat{\delta}(Y_{T}^{Y_{0},Z})} \left[-\exp\left(-\eta (c(N_{T}^{a} + N_{T}^{b}) - Y_{T}^{Y_{0},Z})\right) \right]$$

Reduction to a classical control problem

$$V_{E} = \sup_{Z} \mathbb{E}^{\hat{\delta}(Y_{T}^{0,Z})} \Big[-\exp\left(-\eta \int_{0}^{T} (c - Z_{t}^{a}) dN_{t}^{a} + (c - Z_{t}^{b}) dN_{t}^{b} - Z_{t}^{S} dS_{t} + H(Z_{t}, Q_{t}) dt \Big) \Big]$$

Reduction to a HJB equation

• $V_E = v(0, Q_0)$ with

$$\partial_t v(t,Q) + \sup_z h_E(Q,v(t,Q),v(t,Q+1),v(t,Q-1),z) = 0$$

and v(T,q) = -1, where the function h_E is explicit.

• The optimal control Z^{*} is obtained so that Z^{*}_t is solution of the maximization problem of

 $z\mapsto h_E(Q_t,v(t,Q_t),v(t,Q_t+1),v(t,Q_t-1),z).$

It is explicit in terms of the parameters of h_E .

Reduction to a linear equation

If we take $u(t,Q) = (-v(t,Q))^{-rac{k}{\sigma\eta}}$, we get

$$\begin{cases} \partial_t u(t,Q) + C_1 Q^2 - C_2 (u(t,Q-1) \mathbb{1}_{\{Q > -\bar{q}\}} + u(t,Q+1) \mathbb{1}_{\{Q < \bar{q}\}}) = 0, \\ u(\mathcal{T},Q) = 1, \end{cases}$$

with C_1 and C_2 are positive explicit constants.

- Guarantees the existence and uniqueness of v.
- Easy numerical computation of *u* and *v*.

Theorem

The contract ξ^{\star} that solves the exchange problem is given by

$$\xi^{\star} = Y^{\star} + \int_{0}^{T} Z_{t}^{a,\star} dN_{t}^{a} + Z_{t}^{b,\star} dN_{t}^{b} + Z_{t}^{S,\star} dS_{t} - H(Z_{t}^{\star}, Q_{t}) dt,$$

with

$$Z_t^{a\star} = -\frac{\sigma}{k} \log \left(\frac{u(t, Q_t)}{u(t, Q_t - 1)} \right) + \hat{c},$$
$$Z_t^{b\star} = -\frac{\sigma}{k} \log \left(\frac{u(t, Q_t)}{u(t, Q_t + 1)} \right) + \hat{c}, \quad Z_t^{S\star} = -\frac{\gamma}{\eta + \gamma} Q_t.$$
with $\hat{c} = c + \frac{1}{\eta} \log \left(1 - \frac{\sigma^2 \gamma \eta}{(k + \sigma \gamma)(k + \sigma \eta)} \right).$

Discussion

The quantities

$$-\log\left(rac{u(t,Q_t)}{u(t,Q_t-1)}
ight)$$
 and $-\log\left(rac{u(t,Q_t)}{u(\mathcal{T},Q_t+1)}
ight)$

are roughly proportional respectively to Q_t and $-Q_t$.

• Thus, when the inventory is highly positive, the exchange provides incentives to the market-maker so that it attracts buy market orders and tries to dissuade him to accept more sell market orders, and conversely for a negative inventory.

Discussion

The integral

$$\int_0^T Z_u^{S\star} dS_u$$

can be understood as a risk sharing term.

- Indeed, \$\int_0^t Q_u dS_u\$ corresponds to the price driven component of the inventory risk \$Q_t S_t\$. Hence in the optimal contract, the exchange supports part of this risk so that the market maker maintains reasonable quotes despite some inventory.
- The proportion of risk handled by the platform is $\frac{\gamma}{\gamma+\gamma_n}$

Comments on the optimal contract

Discussion

- We see that when acting optimally, the exchange transfers the totality of the taker fee c to the market maker. It is neutral to the value of c (its optimal utility function does not depend on c).
- However, c plays an important role in the optimal spread offered by the market maker which is approximately given by

$$-2c - \frac{2}{\gamma_{\boldsymbol{\rho}}} \log \left(1 - \frac{\sigma^2 \gamma \gamma_{\boldsymbol{\rho}}}{(k + \sigma \gamma)(k + \sigma \gamma_{\boldsymbol{\rho}})}\right) + \frac{2}{\gamma} \log(1 + \frac{\sigma \gamma}{k}).$$

• The exchange may fix in practice the transaction cost *c* so that the spread is close to one tick by setting

$$c \approx -\frac{1}{2} \mathsf{Tick} - \frac{1}{\gamma_{p}} \log \left(1 - \frac{\sigma^{2} \gamma \gamma_{p}}{(k + \sigma \gamma)(k + \sigma \gamma_{p})} \right) + \frac{1}{\gamma} \log(1 + \frac{\sigma \gamma}{k}).$$

• For $\sigma \gamma / k$ small enough, $c \approx \frac{\sigma}{k} - \frac{1}{2}$ Tick.

Analyzing the effect of the exchange optimal incentive policy

The benefits of the incentive policy

- We can compute the spread, optimal contract, profit and losses of the market maker and exchange, order flows...
- We compare these quantities to the ones obtained in the case where $\xi = 0$.

$$\begin{split} T &= 600s, \quad \sigma = 0.3 \text{Tick.} s^{-1/2}, \quad A = 0.9 s^{-1}, \quad k = 0.3 s^{-1/2}, \\ \bar{q} &= 50 \text{ unities}, \quad \gamma = 0.01 \text{Tick}^{-1}, \quad \eta = 1 \text{Tick}^{-1}, \quad c = 0.5 \text{Tick.} \end{split}$$

Impact of the incentive policy on the spread

The optimal spread is given by $S_t^{\star} = \delta_t^{a\star} + \delta_t^{b\star}$ with

$$\delta_t^{i\star} = \delta_t^i(\xi^*) = -Z_t^{i\star} + \frac{1}{\gamma} \log\left(1 + \frac{\sigma\gamma}{k}\right), \quad i = a, b.$$



FIGURE – Optimal initial spread with/without the exchange incentive policy as a function of the initial inventory Q_0 .

Impact of the incentive policy on the spread



FIGURE – Optimal initial ask (left) and bid (right) spread component with/without the exchange incentive policy as a function of the initial inventory Q_0 .

Impact of the volatility on the incentive policy



FIGURE – The initial optimal spread difference between both situations with/without incentive policy from the exchange toward the market maker as a decreasing function of the volatility σ .

Impact of the incentive policy on the market liquidity



FIGURE – Average order flow on [0, T] with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).

Impact of the incentive policy on the market maker and exchange profit and loss



FIGURE – Average total P&L of the market maker and the exchange on [0, T] with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).

Impact of the incentive policy on trading costs

We consider that there is only one market taker who wants to buy a fixed quantity $Q_{final} = 200$ units. We compute the trading cost in both situations : $\int_{0}^{T} \delta_{s}^{a} dN_{s}^{a}.$



FIGURE – Average trading cost on [0, T] with 95% confidence interval, with/without incentive policy from the exchange (5000 scenarios).

Benefits of the exchange incentive policy

- Smaller spreads.
- Better market liquidity.
- Increase of the profit and loss of the market maker and the exchange.
- Lower transaction costs.