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Systemic risk & supercooled Stefan problem on networks

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March 18, 2019

Outline

Motivation

Goal: understand how default cascades arise in banking networks.

More generally: how do microscopic effects in a large system lead to macroscopic phenomena?

Example 1: systemic risk

- Banking system with banks borrowing from each other.
- Banks default $\rightarrow$ losses to other banks.
- Losses $\rightarrow$ new defaults.
- New defaults $\rightarrow$ new losses $\rightarrow$ default cascade.

Other examples

Example 2: neural networks

- Neurons in a part of the brain, e.g. 10^6 in the human hippocampus.
- Membrane potential of a neuron reaches a critical level ("spike") $\rightarrow$ the neuron fires.
- Neuron firing $\rightarrow$ spike in surrounding neurons.
- Surrounding neurons may fire $\rightarrow$ synchronization.

Example 3: supercooling

- Liquid, e.g. water, cooled below its freezing temperature, but remaining liquid. Freezes when poured onto a warmer surface.
- Local freezing $\rightarrow$ global freezing.

Basic structural model for default cascades, in words

- Log-assets of N banks: $Y^1(0), Y^2(0), \dots, Y^N(0) \in [0, \infty)$.
- Evolve according to **indepedent Brownian motions**, each with **drift** α, **dispersion** σ.
- When a process hits 0, it is **absorbed**. $\leftrightarrow$ default
- This leads to **immediate downward jumps** by other processes, tuned by a parameter $C > 0$. $\leftrightarrow$ losses
- If some processes cross 0 due to jumps, these processes are **removed**, jump sizes of remaining particles are **adjusted**, etc. $\leftrightarrow$ new losses
- When **cascade resolved**: remaining processes **continue as BMs**, etc.

Our model, in formulas

- **Process locations**: Y^1, Y^2, $\ldots$, Y^N.
- As long as processes on $(0,\infty)$:

$$\mathrm{d}Y^i_t = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}B^i_t, \quad i = 1, 2, \ldots, N,$$

  B^1, B^2, $\ldots$, B^N **independent standard BMs**.
- **Hitting times**:

$$\tau^i = \inf\{t > 0 :\ Y^i_t \leq 0\}, \quad i = 1, 2, \ldots, N.$$

- Suppose Y^i hits 0 at time t and is **removed**.

Our model: cascades, in words

- **Shift** the remaining processes by
$$C \log\left(1 - \frac{1}{S_{t-}}\right),$$
where S_{t-} is the **pre-cascade size of the system**.
- **Note**: factor $\downarrow$ in size S_{t-}, $\uparrow$ in parameter C.
- Update may lead to processes $i_1, i_2, \ldots, i_k$ crossing 0, these are removed, and we **adjust the shift** to
$$C \log\left(1 - \frac{k+1}{S_{t-}}\right).$$
- May cause **more immediate absorptions**, in which case **repeat** procedure etc., until determine all processes to remove at time t.

Our model: cascades, in formulas

- **System size**: $S_t := \sum_{i=1}^{N} \mathbf{1}_{\{\tau^i > t\}}$.
- **Order statistics**: $Y_{t-}^{(1)} \leq Y_{t-}^{(2)} \leq \cdots \leq Y_{t-}^{(S_{t-})}$ of $(Y_{t-}^i : \tau^i \geq t)$.
- **# of processes removed at time** t:

  $D_t := \inf \left\{ k : \; Y_{t-}^{(k)} + C \log \left(1 - \frac{k-1}{S_{t-}} \right) > 0 \right\} - 1.$
- **Log-asset dynamics**:

  $Y_t^i := Y_0^i + \alpha t + \sigma B_t^i + \sum_{u \leq t} C \log \left(1 - \frac{D_u}{S_{u-}} \right).$

Large system limit: starting point

To study macroscopic default cascades:

- take $N \to \infty$;
- macroscopic default cascades $\leftrightarrow$ blow-ups in a limiting process.

Crucial observation: sum of jumps

$\sum_{u \le t} C \log\left(1 - \frac{D_u}{S_{u-}}\right) = \sum_{u \le t} C \log\left(\frac{S_u}{S_{u-}}\right) = C \log\left(\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{\{\tau^j > t\}}\right)$.

$\longrightarrow$ functional of the **empirical measure** $\varrho^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{Y^i}$

$\longrightarrow$ **interaction of mean-field type**

Large system limit: McKeav-Vlasov heuristics

McKean-Vlasov heuristics (cf. **Sznitman '89**):

- **Classical setting**:

  $Y_t^i = Y_0^i + \int_0^t b(Y_s^i, \varrho_s^N)\,\mathrm{d}s + \int_0^t \sigma(Y_s^i, \varrho_s^N)\,\mathrm{d}B_s^i, \; i = 1, 2, \ldots, N.$

- **Guess**: $\varrho^N \overset{N\to\infty}{\longrightarrow} \varrho$, **deterministic**.

- $\Longrightarrow$ for large N, particle locations **well-approximated** by

  $\overline{Y}_t^i = \overline{Y}_0^i + \int_0^t b(\overline{Y}_s^i, \varrho_s)\,\mathrm{d}s + \int_0^t \sigma(\overline{Y}_s^i, \varrho_s)\,\mathrm{d}B_s^i, \; i = 1, 2, \ldots, N.$

- $\Longrightarrow \varrho = \lim_{N\to\infty} \varrho^N = \lim_{N\to\infty} \overline{\varrho}^N = \mathcal{L}(\overline{Y}^1)$.

- **Conclusion**: in $N \to \infty$ limit, Y^i converge to **unique solution** of

  $\overline{Y}_t = \overline{Y}_0 + \int_0^t b(\overline{Y}_s, \mathcal{L}(\overline{Y}_s))\,\mathrm{d}s + \int_0^t \sigma(\overline{Y}_s, \mathcal{L}(\overline{Y}_s))\,\mathrm{d}B_s.$

Large system limit: our setting

- **McKean-Vlasov heuristics** suggests Y^i converge to **unique sol.** of

  $$\overline{Y}_t = \overline{Y}_0 + \alpha t + \sigma \overline{B}_t + \Lambda_t,$$

  where

  $$\Lambda_t := C \log \mathbb{P}(\overline{\tau} > t), \quad \overline{\tau} := \inf\{t \geq 0 : \overline{Y}_t \leq 0\}.$$

- **Problems**: **non-uniqueness**, **non-existence** in $C([0,\infty),\mathbb{R})$.
- $\mathbb{P}(\overline{\tau} > t)$ or $\frac{1}{N}\sum_{j=1}^{N} \mathbf{1}_{\{\tau^j > t\}}$ do not specify **cascade mechanism**.
- $\rightsquigarrow D_t := \inf\{y > 0 : y - F_t(y) > 0\}$

  $$:= \inf\left\{y > 0 : y + C \log\left(1 - \frac{\mathbb{P}(\overline{\tau} \geq t, \overline{Y}_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau} \geq t)}\right) > 0\right\}.$$

- **Specify** $\Lambda_t = \Lambda_{t-} + F_t(D_t)$, rcll.
- Call solutions with **correct cascade mechanism physical solutions**.

A first limit theorem

Theorem (Nadtochiy, S. '17) Suppose $\frac{1}{N}\sum_{i=1}^{N}\delta_{Y_i(0)} \to \nu$; ν has a bounded density f_ν on $[0,\infty)$ vanishing in a neighborhood of 0.

Then:

The sequence $\frac{1}{N}\sum_{i=1}^{N}\delta_{\widetilde{Y}^i}$, $N \in \mathbb{N}$ is tight and any limit point is supported on physical solutions $\overline{Y}$ with $\overline{Y}_0 \stackrel{d}{=} \nu$.

Technical points: theorem for auxiliary log capitalization processes; Skorokhod M1 topology on rcll paths (key observation of **Delarue, Inglis, Rubenthaler, Tanré '15**).

Connection to Stefan problem

- Consider a variant of the **limiting problem**:

$$\overline{Y}_t = \overline{Y}_0 + B_t - \Lambda_t, \;\; t \leq \overline{\tau}, \quad \overline{Y}_t = \overline{Y}_{\overline{\tau}}, \;\; t > \overline{\tau},$$
$$\Lambda_t = C\,\mathbb{P}(\overline{\tau} \leq t).$$

- If Λ' exists on $[0, T)$, densities $p(t,\cdot)$ of $\overline{Y}_t$ solve

$$\partial_t p = \frac{1}{2}\partial_{xx} p + \Lambda'_t \partial_x p, \quad p(0,\cdot) = f, \quad p(\cdot, 0) = 0,$$
$$\Lambda'_t = \frac{C}{2}\partial_x p(t,0), \quad t \in [0, T).$$

- Change of variables: $u(t,x) := p(t, x - \Lambda_t), \;\; x \geq \Lambda_t, \;\; t \in [0, T)$.

Connection to Stefan problem cont.

- **Supercooled Stefan problem (1D, one phase)**:

$$\partial_t u = \frac{1}{2}\partial_{xx} u \quad \text{on} \quad \{(t,x) \in [0,\infty)^2 : x \geq \Lambda_t\},$$

$$\Lambda'_t = C\partial_x u(t,\Lambda_t), \quad t \geq 0,$$

$$u(0,x) = f(x), \quad x \geq 0 \quad \text{and} \quad u(t,\Lambda_t) = 0, \quad t \geq 0,$$

where $f \geq 0$, $C \geq 0$.

- **Blow-up**: for some $T < \infty$, $\lim_{t\uparrow T} \Lambda'_t = \infty$.
- **Classical solution** on $[0, T)$.

Brief history of Stefan problems

- **Stefan 1889–1891**: **free boundary problems** for the **heat equation**.
- Physical models of ice formation; evaporation & condensation.
- Dormant until **Brillouin '31**, **Rubinshtein**: ≈**2500 papers** by **'67**.
- **Kamenomostkaja '61**: definitive solution.
- **'70-today**: **supercooled Stefan problem**.
- **Sherman '70**: presence of **blow-ups**.
- For some $T < \infty$: **boundary speed** $\to \infty$.

Back to our physical solutions: questions

- By the theorem, a **physical solution** $\overline{Y}$ with **rcll paths exists**.
- How do the **jumps** in $\overline{Y}$ arise? $\longleftrightarrow$ systemic crises, synchronization of neurons, leaps of the solid/liquid frontier.
- E.g., what can one say about

$$t_\Delta := \inf\{t \geq 0 : \Delta \overline{Y}_t \neq 0\}$$

and the empirical log-asset distribution $\mathcal{L}(\overline{Y}_{t_\Delta -})$ right before t_Δ?

Main theorem I: regular interval

Theorem (Nadtochiy, S. '17) Suppose $\overline{Y}_0 \stackrel{d}{=} \nu$ has a density $f_\nu \in W_2^1([0,\infty))$ and $f_\nu(0) = 0$.

Then: there exists $t_{reg} > 0$ such that on $[0, t_{reg})$ all physical solutions are indistinguishable and satisfy

$$\overline{Y}_t = \overline{Y}_0 + \alpha t + \sigma B_t + \int_0^t \lambda_s \,\mathrm{d}s, \quad t \in [0, \overline{\tau} \wedge t_{reg}),$$
$$\lambda_t = C\,\partial_t \log \mathbb{P}(\overline{\tau} > t), \quad t \in [0, t_{reg}).$$

Moreover, $t_{reg} = \inf\{t > 0 : \|\lambda\|_{L^2([0,t])} = \infty\}$.

Main theorem II: description of jumps

Theorem (Nadtochiy, S. '17) Consider a physical solution $\overline{Y}$.

Then:

(a) the time of the first jump $t_\Delta := \inf\{t \geq 0 : \; \Delta \overline{Y}_t \neq 0\}$ is given by

$$t_\Delta = \inf\left\{t \geq 0 : \; \exists\, \eta > 0 \;\text{ s.t. }\; \frac{\mathbb{P}(\overline{\tau} \geq t, \overline{Y}_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau} \geq t)} \geq \frac{y}{C},\; y \in [0,\eta]\right\},$$

and

(b) the size of the jump at t_Δ is

$$\sup\left\{\eta \geq 0 : \; \frac{\mathbb{P}(\overline{\tau} \geq t_\Delta, \overline{Y}_{t_\Delta -} \in (0,y))}{\mathbb{P}(\overline{\tau} \geq t_\Delta)} \geq \frac{y}{C},\; y \in [0,\eta]\right\}.$$

All regimes, uniqueness (Delarue, Nadtochiy, S. '19)

- For **uniqueness**, need to understand **all** regimes.
- **Case 1:** $\overline{Y}_{t-}$ has a density $f \in C^1([0,\infty)) \cap C^\omega((0,\infty))$, $f(0) = 0$.
  $\Longrightarrow \dot{\Lambda} = \lambda$ is continuous on $[t, t+\varepsilon)$ for some $\varepsilon > 0$.
- **Case 2:** $\overline{Y}_{t-}$ has a density $f \in C^\omega((0,\infty))$, $f(0+) \in [0, 1/C)$.
  $\Longrightarrow \Lambda$ is $(1/2+\delta)$-Hölder on $[t, t+\varepsilon)$, back to **Case 1** on $(t, t+\varepsilon)$.
- **Case 3:** $\overline{Y}_{t-}$ has a density $f \in C^\omega((0,\infty))$, $f(0+) \geq 1/C$.
  $\Longrightarrow \Lambda_t - \Lambda_{t-} = -\inf\left\{y \geq 0 : \mathbb{P}(\overline{Y}_{t-} \in (0,y] < y/C)\right\}$,
  back to **Case 1** on $(t, t+\varepsilon)$.
- **Uniqueness** follows from this and sandwiching between two **maximal** physical solutions.

General networks (Nadtochiy, S. '18)

- More generally, may replace

$$\overline{Y}_t = \overline{Y}_0 + \alpha t + \sigma \overline{B}_t + \Lambda_t,$$

$$\Lambda_t := C \log \mathbb{P}(\overline{\tau} > t), \quad \overline{\tau} := \inf\{t \geq 0 : \ \overline{Y}_t \leq 0\}$$

by its **network** version:

$$\overline{Y}_t^x = \overline{Y}_0^x + \int_0^t \alpha_s^x \, \mathrm{d}s + \int_0^t \sigma_s^x \, \mathrm{d}\overline{B}_s + \Lambda_t^x,$$

$$\Lambda_t^x := C^x \int_{\mathcal{X}} \log \mathbb{P}(\overline{\tau}^{x'} > t) \, \kappa(x, \mathrm{d}x'), \quad \overline{\tau}^{x'} := \inf\{t \geq 0 : \ \overline{Y}_t^{x'} \leq 0\}.$$

- Hereby, $x \in \mathcal{X}$ are different types of banks, stochastic kernel κ defines a **weighted directed network** on $\mathcal{X}$.

General networks: questions

- Does a solution $\{\overline{Y}^x\}_{x\in\mathcal{X}}$ with rcll paths still **exist**?

  Yes, can prove this directly using a **new Schauder's theorem** for the **Skorokhod M1 topology**.

- How do the **jumps** in $\{\overline{Y}^x\}_{x\in\mathcal{X}}$ arise?

  $\longleftrightarrow$ systemic crises/synchronization of neurons.

- E.g., what can one say about

$$t_\Delta := \inf\{t \geq 0 : \Delta\overline{Y}^x_t \neq 0 \text{ for some } x \in \mathcal{X}\}$$

and empir. log-asset distributions $\{\mathcal{L}(\overline{Y}^x_{t_\Delta-})\}_{x\in\mathcal{X}}$ **right before** t_Δ?

Preparation: logarithmic Perron-Frobenius eigenvalue

- Let $\mathcal{X}$ is **finite**.
- Suppose that the time $t-$ configuration satisfies

$$\frac{\mathbb{P}(\overline{\tau}^x \geq t,\ \overline{Y}^x_{t-} \in (0,y))}{\mathbb{P}(\overline{\tau}^x \geq t)} \geq c^x y, \quad y \in [0,\eta^x], \quad x \in \mathcal{X}.$$

- If the matrix $(C^x \kappa(x,\{x'\}) c^{x'})_{x,x' \in \mathcal{X}}$ is **irreducible**, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \int_{\mathcal{X}^n} \prod_{m=0}^{n-1} \left(C^{x_m} c^{x_{m+1}}\right) \kappa(x_0, \mathrm{d}x_1) \ldots \kappa(x_{n-1}, \mathrm{d}x_n)$$

is referred to as its **logarithmic Perron-Frobenius eigenvalue**.

General networks: jump criterion

<u>Theorem</u> (Nadtochiy, S. '18) Suppose that $\alpha^x \leq \overline{\alpha}$, $x \in \mathcal{X}$ and $\underline{\sigma} \leq \sigma^x \leq \overline{\sigma}$, $x \in \mathcal{X}$ with suitable $\overline{\alpha} \in \mathbb{R}$ and $\underline{\sigma}, \overline{\sigma} \in (0, \infty)$.

If $(C^x \kappa(x, \{x'\}) c^{x'})_{x,x' \in \mathcal{X}}$ is irreducible with a strictly positive logarithmic Perron-Frobenius eigenvalue, then

$$\Delta \overline{Y}_t^x \neq 0$$

for at least one $x \in \mathcal{X}$.

THANK YOU
FOR YOUR ATTENTION!