

# Dynamic Investment and Financing with Internal and External Liquidity Management

Nan Chen

Systems Engineering & Engineering Management  
The Chinese University of Hong Kong

Workshop on Stochastic Control in Finance  
National University of Singapore  
July 23, 2019

This talk is based on a joint work with Yuan Tian (Ryukoku University) and Jiahui Ji (CUHK)

# Liquidity and Liquidity Management

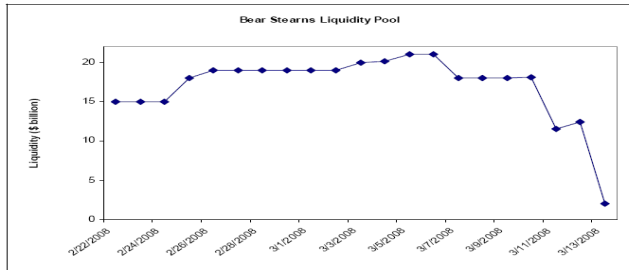
- ▶ Liquidity, which we define as the availability of cash or equivalent resources, allows firms to meet expected and unexpected obligations when needed so that their daily business operations can proceed uninterrupted.
- ▶ For a “financially constrained” firm, the decision of liquidity management is intertwined with investment, external financing, payout, debt borrowing, and liquidation decisions.

A Unified Framework for Liquidity Management



# Liquidity and Liquidity Management (Continued)

- Insufficient liquidity buffers, together with high debt levels, were the primary cause of the collapse of major players of the Wall Street in the global financial crisis of 2007-2009.



**Figure 1:** Bear Stearns' liquidity pool just before its near failure (Source: SEC Chairman Christopher Cox's letter to Basel Committee in support of New Guidance on Liquidity Management, available at: <http://www.sec.gov/news/press/2008/2008-48.htm>).

# Liquidity and Liquidity Management (Continued)

- ▶ In response to the deficiencies in the financial regulation revealed by the crisis, one major revision introduced to the Basel Accord, the global banking regulatory framework, focus on increasing liquid asset holdings.
  - ▶ **Liquidity coverage ratio (LCR):** Under Basel III, a bank should have an adequate stock of unencumbered high-quality liquid assets (HQLA) that can be converted into cash easily and immediately in private markets to meet its liquidity needs for a 30 calendar day liquidity stress scenario.

# The Paper in a Nutshell

- ▶ We propose a stochastic-control based dynamic model for firms' choices of investment, financing, liquidity management, and default policies in the presence of financial constraints.
- ▶ The model highlights the central importance of liquidity management in corporate decisions
  - ▶ The firm's optimal liquidity management policy is characterized by a **double-barrier policy**: rather than targeting on a single level of cash inventory, the firm should manage its cash reserve within a range (Miller and Orr (1966, QJE)).
  - ▶ For a financially constrained firm, the **marginal value of liquidity** to its equity highly depends on the firm's investment opportunities, cash holding level, leverage, and external financing costs. This marginal value plays a decisive role in driving the investment, debt, payout, and default policies.
  - ▶ The interaction between **debt overhang** and liquidity distorts the firm's investment incentive.
  - ▶ Imposing liquidity requirements alone is **not sufficient** to lead to a reduction in the likelihood of default of the regulated firm.

# Literature Review

- ▶ Our paper is related to three threads of literatures:
  - ▶ Neoclassical investment theory: Tobin (1969, JMCB), Uzawa (1969, JPE), Hayashi (1982, *Econometrica*), ...

Assets	Liabilities and Equity
Physical Capital $K$	Equity $E$

- ▶ Credit risk: Merton (1974, JF), Leland (1994, JF), ...

Assets	Liabilities and Equity
Assets	Debt $D$ Equity $E$

- ▶ Corporate investment/risk management in the presence of financing costs: Bolton et al. (2011, JF), Bolton et al. (2013, JFE)

Assets	Liabilities and Equity
Cash $W$ Physical Capital $K$	Equity $E$

# Model Setup

- ▶ Extending the previous literatures, our model provides a **unified framework** by examining both sides of the balance sheet of the firm.

Assets	Liabilities and Equity
Cash $W$	Debt $D$
Physical Capital $K$	Equity $E$

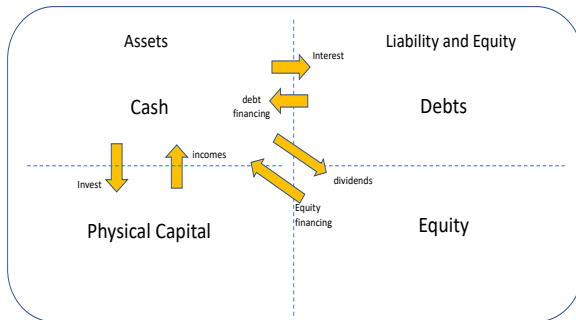
Table: The balance sheet of the firm in our paper

- ▶ On one hand, we explicitly model debts on the liabilities side and assume that the firm can adjust debt levels by issuing new debt and repurchasing existing debt.  
⇒ **Capital requirement/credit risk** (large literature)
- ▶ On the other hand, we incorporate liquid cash in addition to illiquid, productive physical capital on the assets side.  
⇒ **Liquidity requirement** (small literature)
- ▶ A even smaller literature on the interaction between the two sides of balance sheet.

# Model Setup: Cash Flow



## Cash Flows in the Model





# Model Setup: Production Technology/Physical Capital

- ▶ The firm owns physical capital and puts it for producing revenue. Suppose that the (cumulative) productivity  $A_t$  is subject to stochastic shocks:

$$dA_t = \mu dt + \sigma dZ_t.$$

where  $Z_t$  is a standard BM. At time  $t$ , when the amount of physical capital owned by the firm is  $K_t$ , the operating revenue of the firm is given by  $K_t dA_t$ .

- ▶ The firm also invests to create new physical capital. The firm's capital stock  $K$  evolves according to

$$dK_t = (I_t - \delta K_t)dt, \quad t \geq 0. \quad (1)$$

where

- ▶  $I_t$ : gross investment rate (to be determined endogenously in the equilibrium)
- ▶  $\delta$ : capital depreciation rate

# Model Setup: Operating Profits

- ▶ Assume that the firm's investment is costly. It incurs adjustment costs denoted by  $G(I, K)$ .
  - ▶ Following the standard modelling device in the literature of neoclassical investment theory, we assume that  $G$  is homogeneous of degree one in  $I$  and  $K$ ; that is,  $G(I, K) = g(i)K$ , where  $i = I/K$  and  $g(i)$  is an increasing and convex function. A special case considered in the paper is that  $g(i) = \theta i^2/2$ , with the parameter  $\theta$  measures the degree of the adjustment cost (Uzawa(1969, Econometrica), Hayashi (1982, Econometrica)).
- ▶ The firm's operating profit rate is

$$dY_t = K_t dA_t - I_t dt - G(I_t, K_t) dt, \quad t \geq 0. \quad (2)$$

- ▶ The pecking order in the firm's financing choices: it prioritizes usage of the internal funding over the external ones.

# Model Setup: Debt Structure

- ▶ We assume that the firm finances its business partially through debts. It continuously adjusts the outstanding debt level  $P_t$  through issuances of new debts and repurchase of old ones. All the adjustments are made at the market price and

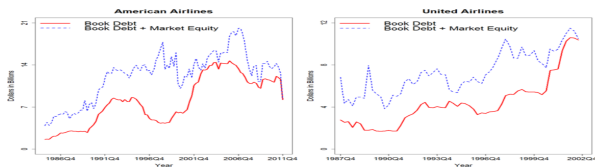
$$dP_t = Q_t dt, \quad t \geq 0, \quad (3)$$

where  $Q_t$  is the adjustment rate of the debt level.

- ▶ if  $Q_t > 0$ , the firm issues new debts of face value  $Q_t dt$  over  $(t, t + dt)$
  - ▶ if  $Q_t < 0$ , the firm repurchases debts amount of  $Q_t dt$  back.
- ▶ For mathematical tractability, we assume that  $Q_t dt = p dK_t$ , i.e., the total outstanding amount of debt is maintained to be proportional to the size of the firm's assets.

# Model Setup: Debt Structure (Continued)

- ▶ Many existing models in the literature of credit risk (e.g. Merton (1974), Leland (1994, 1998)) assume that the firm is committed to a fixed amount of debts, irrespective of the evolution of the firm's fundamentals. However, that static capital structure is not consistent with practice



**Figure:** Debt and enterprise values for American and United Airlines.  
Data source: DeMarzo and He (2016)

- ▶ Dynamic capital structure:
  - ▶ Fischer, Heinkel, Zechner (1989, JF), Goldstein, Ju, and Leland (2001, JB), Titman and Tsyplakov (2007, RF), Strebulaev (2007, JF), Dangl and Zechner (2016, JFI), DeMarzo and He (2016)

# Model Setup: External Equity Financing and Its Costs

- ▶ Compared with debt financing, firms face significant costs when they attempt to raise funds through external equity financing, due to both transaction costs and the influence of asymmetric information and managerial incentive problems (Jensen and Meckling (1976, JFE), Leland and Pyle (1977, JF), Myers and Majluf (1984, JFE), Calomiris and Himmelberg (1997)).
- ▶ We take a reduced-form assumption to model this financing cost that, whenever the firm chooses to raise external equity, it will incur a fixed cost  $\phi$  and a marginal cost  $\gamma$  (i.e., if it raises an amount of  $A$ , it needs to pay  $\phi + A\gamma$ ). In the presence of fixed costs, the firm can tap equity market for financing only intermittently.

# Model Setup: Cash Reserve

- ▶ The inventory of cash reserve is a central state variable in our model. Let  $W_t$  be the cash inventory. It evolves according to

$$\begin{aligned} dW_t = & (r - \lambda)W_t dt + \underbrace{dY_t}_{\text{Operating profit}} + \underbrace{v_t dP_t - (1 - \pi)cP_t dt}_{\text{debt issuance - interest service}} \\ & + \underbrace{dH_t}_{\text{external equity financing}} - \underbrace{dU_t}_{\text{dividend payouts}}. \end{aligned}$$

where

- ▶  $\lambda$  the cash-carrying cost and  $r$  the risk free rate.
  - ▶  $v_t$  market price of debt of unit face value
  - ▶  $c$  coupon rate and  $\pi$  tax rate
  - ▶  $dH_t$  cash inflow from external equity financing
  - ▶  $dU_t$  cash outflow from dividend payout
- ▶ The presence of debt introduce a tradeoff

# Equity and Debt Evaluation

- Equity value:

$$\begin{aligned} & E(K_t, W_t) \\ &= \max_{l, U, H, \tau_b} E \left[ \int_t^{\tau_b} e^{-r(s-t)} (dU_s - dH_s - dX_s) + e^{-r(\tau_b-t)} l(K_{\tau_b}, W_{\tau_b}) \middle| K_t, W_t \right] \end{aligned} \quad (4)$$

subject to

$$\begin{aligned} dW_t &= (r - \lambda)W_t dt + dY_t + v_t dP_t - (1 - \pi)cP_t dt + dH_t - dU_t. \\ dK_t &= (I_t - \delta K_t) dt \end{aligned}$$

where

- $\tau_b$  default time
- $X_t$  accumulative financing cost
- $dH_t$  cash inflow from external equity financing
- $dU_t$  cash payout rate
- $\alpha$ : liquidation value and the after-liquidation equity value

$$l(K_t, W_t) = (\alpha K_t + W_t - pK_t)^+.$$

# Equity and Debt Evaluation (Continued)

- ▶ Debt value:

$$v(K_t, W_t) = E \left[ \int_t^{\tau_b} e^{-r(s-t)} c dt + e^{-r(\tau_b-t)} v(K_{\tau_b}, W_{\tau_b}) \middle| K_t, W_t \right] \quad (5)$$

with the default value of

$$v(K_{\tau_b}, W_{\tau_b}) = \min \left\{ \frac{\alpha K_{\tau_b} + W_{\tau_b}}{p K_{\tau_b}}, 1 \right\}$$



# Rational Expectation Equilibrium: Equity and Debt

- ▶ Definition of equity and debt values:
  - ▶  $E(K_t, W_t)$  and  $v(K_t, W_t)$  are the values of the firm's equity and debt in a rational expectation equilibrium if they satisfy
    - (a) Given  $v(\cdot, \cdot)$ , the equity function  $E$  solves the optimization problem (4);
    - (b) Given  $E$  and the associated actions  $(I, U, H, \tau_b)$ , the debt value  $v$  satisfies (5).
- ▶ Optimal capital structure:
  - ▶ At time 0, the firm chooses  $p^*$  to optimize the initial firm value:

$$p^* = \arg \max_p [E(K_0, W_0) + pK_0v(K_0, W_0)]$$

- ▶ The firm commits to the debt policy  $Q_t dt = p^* dK_t$  for  $t \geq 0$ .

# Model Solution

- ▶ We can use a PDE system to characterize the debt and equity value function  $(v, E)$ . It is a variational inequality embedded with a fixed-point structure.
  - ▶ Given  $v$ , the equity value  $E$  solves the following variational inequality:

$$0 = \max \left\{ \underbrace{\max_I \mathcal{L}E - rE}_{\text{internal investment region}}, \underbrace{1 - E_W}_{\text{payout region}}, \underbrace{\Delta E - E}_{\text{refinancing/liquidation region}} \right\}$$

with

$$\begin{aligned} \mathcal{L} = & (I - \delta K) \frac{\partial}{\partial K} + \left[ (r - \lambda)W + \mu K - I - G(I, K) + (I - \delta K)p v(W, K) - (1 - \pi)cpK \right] \frac{\partial}{\partial W} \\ & + \frac{1}{2} \sigma^2 K^2 \frac{\partial^2}{\partial W^2}. \end{aligned}$$

and

$$\Delta E = \max \left\{ \max_m [E(W + mK, K) - (\phi K + (1 + \gamma)mK)], (\alpha K_t + W_t - pK_t)^+ \right\}.$$

- ▶ Given  $E$ , the debt value  $v$  solves  $\mathcal{L}v + c = rv$

## Model Solution (Continued)

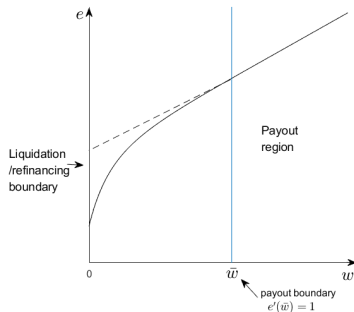
- ▶ Homogenous structure of the problem
- ▶ Let  $w = W/K$ , the cash-capital ratio. Both the firm's equity and debt value can be reduced down to functions of  $w$ :

$$e(w) = \frac{E(K, W)}{K} \quad \text{and} \quad d(w) = \frac{pKv(K, W)}{K}.$$

- ▶  $e$  and  $d$  can be determined through the above differential equation system.

# Model Solution (Continued)

## ► We can prove



- $e$  is an concave and increasing function
- $e'(w) \geq 1$  for all  $w$
- There exists a  $\bar{w}$  such that  $e'(\bar{w}) = 1$ . In  $(\bar{w}, +\infty)$ , the firm pays out excess cash as dividend to the equity holder.
- The firm chooses either liquidating or refinancing when  $w = 0$

- liquidation:  $e(0) = (\alpha - p)^+$
- refinancing: raises  $m^*$  in which

$$m^* = \arg \max_m [e(m) - (\phi + (1 + \gamma)m)]$$

# Model Solution (Continued)

- ▶ Three regions:
  - ▶ Liquidation/refinancing:  $\{0\}$
  - ▶ Payout:  $[\bar{w}, +\infty)$
  - ▶ Internal financing:  $(0, \bar{w})$
- ▶ Marginal value of cash:
  - ▶ Either liquidation or refinancing is costly from the perspective of equity holders. More cash liquidity helps keep the firm away from the boundary 0. Hence, the marginal value of one dollar of cash is worth more than 1.
  - ▶ Such high marginal value of cash highlights the importance of liquidity for the firms in distress.

## Model Solution (Continued)

- ▶ In  $(0, \bar{w})$ , the equity and debt value functions,  $e(\cdot)$  and  $d(\cdot)$  satisfy the following PDE system

- ▶ If liquidating at  $w = 0$

$$\begin{cases} re(w) = (i - \delta)e(w) + \tilde{\mathcal{L}}e(w), & w \in (0, \bar{w}) \\ e(0) = (\alpha - p)^+, & e'(\bar{w}) = 1 \end{cases}$$

and

$$\begin{cases} rd(w) = \tilde{\mathcal{L}}d(w) + cp, & w \in (0, \bar{w}) \\ d(0) = \min\{\alpha, p\}, & d'(\bar{w}) = 0. \end{cases}$$

- ▶ If refinancing at  $w = 0$

$$\begin{cases} re(w) = (i - \delta)e(w) + \tilde{\mathcal{L}}e(w), & w \in (0, \bar{w}) \\ e(0) = \max_m [e(m) - (\phi + (1 + \gamma)m)], & e'(\bar{w}) = 1 \end{cases}$$

and

$$d(w) = cp/r$$

▶

$$\begin{aligned} \tilde{\mathcal{L}} &= -(i - \delta)w \frac{\partial}{\partial w} + \left[ (r - \lambda)w + \mu - i - \frac{\theta}{2}i^2 + (i - \delta)d(w) - (1 - \pi)cp \right] \frac{\partial}{\partial w} \\ &+ \frac{1}{2}\sigma^2 \frac{\partial^2}{\partial w^2}; \end{aligned}$$

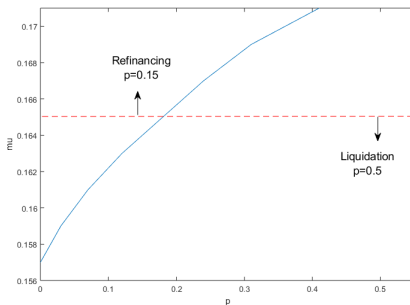
and we use  $e''(\bar{w}) = 1$  to determine  $\bar{w}$ .

# Numerical Results

- ▶ The mean and volatility of the risk-adjusted productivity shock:  $\mu = 0.165$ ,  $\sigma = 9\%$ ; rate of depreciation  $\delta = 10.07\%$
- ▶ risk-free rate  $r = 6\%$ ; cash-carrying cost  $\lambda = 1\%$ ;
- ▶ adjustment cost  $\theta = 1.5$ .
- ▶ proportional financing cost  $\gamma = 6\%$ ; fixed financing cost  $\phi = 7\%$
- ▶ liquidation cost  $1 - \alpha = 0.1$
- ▶ Debt coupon rate  $c = 8\%$ , tax rate  $\pi = 20\%$ .

# Numerical Results: Liquidation vs. Refinancing

- ▶ The debt burden plays an important role in the firm's liquidation/refinancing decision when it runs out of its liquidity reserve: a more debt-laden firm tends to be more likely to choose to liquidate.
- ▶ Parameter regions of liquidation and refinancing





# Numerical Results: Liquidation vs. Refinancing (Continued)

- ▶ Debt overhang (Merton (1974, JF), Myers (1977, JFE)) can be used to explain the decisional transition from refinancing to liquidation as the debt amount increases in the model.
- ▶ Outcomes for the firm's debt and equity values with and without external refinancing

	Liquidation	Refinancing
New Investment from equity holders		$\phi K + (1 + \gamma m^* K)$
Total firm value	$\alpha K$	$E(K, m^* K) + pKc/r$
Debt	$\min\{pK, \alpha K\}$	$pKc/r$
Equity	$(\alpha K - pK)^+$	$E(K, m^* K)$

- ▶ Net benefit from the firm's perspective:

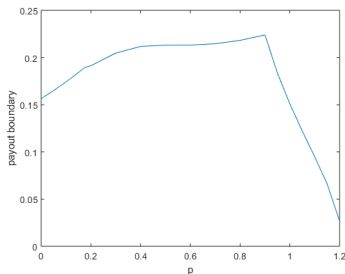
$$E(K, m^* K) + \frac{pKc}{r} - \alpha K - (\phi K + (1 + \gamma)m^* K) = 0.0874K$$

- ▶ Net benefit from the equity's perspective:

$$E(K, m^* K) - (\alpha K - pK)^+ - (\phi K + (1 + \gamma)m^* K) = -0.0792K$$

## Numerical Results: Payout Boundary

- We find that the change of  $\bar{w}$ , the boundary of payout region, is not monotone as the debt amount increases.



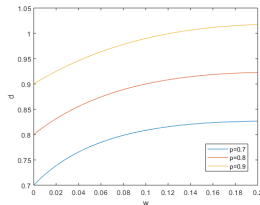
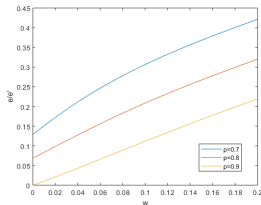
- Fix  $\mu = 0.165$ . Calculate  $\bar{w}$  for different  $p$ .
- For small and medium size of debts, the firm will defer dividend payouts as the debt amount increases because it needs to preserve more liquidity; however, a firm with large debt burden will keep less cash.
- Cashing out in financial distress: firms have a propensity to pay more dividends in the recent financial crisis (Floyd et al. (JFE, 2015))

# Numerical Results: Investment

- ▶ Let  $I^*$  be the optimal investment and  $i^* = I/K$ . Then,  $i^*$  is determined by

$$1 + \theta i^*(w) = \frac{e(w)}{e'(w)} + d(w) - w.$$

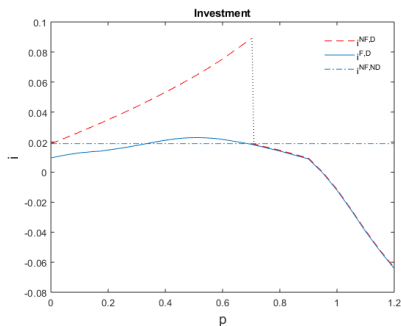
- ▶ Debts has two opposing effects on the firm's investment.
  - ▶ The proceeds of debt issuance may be used for investment.
  - ▶ On the other hand, too high debt will dis-incentivize the investment decisions of equity holder.



# Numerical Results: Investment (Continued)

To better understand the effects of debt and refinancing costs on the firm's investment, we compare the following three setups:

- ▶ No frictions, no debt ( $\phi = \gamma = 0, p = 0$ ) (Uzawa (1969, JPE))
- ▶ No frictions, but with debt ( $\phi = \gamma = 0, p \neq 0$ )
- ▶ Frictions and debt ( $\phi, \gamma, p \neq 0$ )



## More Discussions: Tobin's Average Q and Marginal Q

- ▶ Tobin (1969), Hayashi (1982), Hennessy(JF2004): In dynamic investment models, the shadow price of productive capital, or marginal q, is a sufficient statistic for investment. Since marginal q is unobservable, Tobin's average q, the market price of one unit of capital, is commonly used as an empirical proxy.
- ▶ Average q:

$$q_a(w) = \frac{E(K, W) + D(K, W) - W}{K} = e(w) + d(w) - w.$$

- ▶ Marginal q:

$$\begin{aligned} q_m(w) &= \frac{d(E(K, W) + D(K, W) - W)}{dK} \\ &= q_a(w) - (e'(w) + d'(w) - 1)w \leq q_a. \end{aligned}$$

## More Discussions: Tobin's Average Q and Marginal Q (Continued)

- ▶ If  $w \rightarrow \bar{w}$ , then  $e' \rightarrow 1$  and  $d' \rightarrow 0$ . We then have

$$\frac{e(w)}{e'(w)} + d(w) - w \approx e(w) + d(w) - w = q_a \approx q_m.$$

When the firm is rich in cash reserve, the average q/marginal q can serve as a very good proxy that determines the optimal investment.

- ▶ However, as the firm becomes financially constrained with  $w \ll \bar{w}$ ,  $e'(w) > 1$ . Thus,

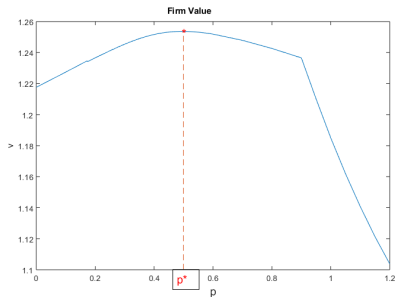
$$\frac{e(w)}{e'(w)} + d(w) - w < e(w) + d(w) - w = q_a.$$

Neither average q nor marginal q remains a valid indicator of the firm's investment for a financially constrained firm. The liquidity concern weighs down the investment.

# Numerical Results: Optimal Capital Structure

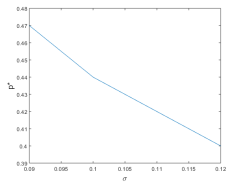
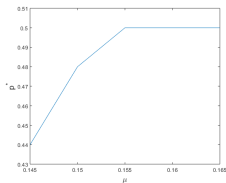
- Assume that given the initial cash level  $w$ , the firm chooses  $p$  to maximize the firm value at  $t = 0$

$$p^* = \arg \max_p [e(w) + d(w)] \quad (6)$$



# Numerical Results: Optimal Capital Structure (Continued)

- Optimal capital structure for different  $\mu$  and  $\sigma$





# Conclusions

- ▶ We propose a unified framework to investigate firms' choices of investment, financing, liquidity management, and default policies in the presence of financial constraints.
- ▶ It shows the central role of liquidity management in corporate decision making.
- ▶ Several directions of the future work:
  - ▶ Debt issuance costs
  - ▶ Jump risk
  - ▶ Implications of the regulatory requirements such as capital and liquidity adequacy
    - ▶ Debt overhang and funding value adjustment (Andersen, Duffie, and Song (2019, JF))