Hermite Expansion for Transition Densities of Irreducible Diffusions with an Application to Option Pricing

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III. Hermite Expansion for Irreducible Diffusions

IV. Explicit Approximations for Option Prices

V. Relations to Existing Density Approximations

VI. Numerical Experiments

I. Background and Motivation

What is an irreducible diffusion?

- Consider an m-dim diffusion process X satisfying

$$dX(s) = \mu^X(s, X(s))ds + \sigma^X(s, X(s))dW(s), \qquad X(t) = x,\ s \geq t$$

where $X(s) \in D_X \subset \mathbb{R}^m$, $\mu^X(s, \xi) \in \mathbb{R}^m$, $\sigma^X(s, \xi) \in \mathbb{R}^{m \times d}$ and $\{W(s); s \geq 0\}$ is a d-dim standard Brownian motion.

- The diffusion X is said to be **reducible** if there exists a one-to-one map $\lambda(s, \xi)$ such that $Y(s) = \lambda(s, X(s))$ satisfying (Aït-Sahalia, 2008):

$$dY(s) = \mu^Y(s, Y(s))ds + dW(s), \qquad Y(t) = y,\ s \geq t;$$

otherwise X is irreducible.

- Univariate diffusions are always reducible because of the existence of the Lamperti transform: BS, OU, CIR, CEV

Background and Related Literature

Backgroud:

- The diffusion processes are widely used in asset pricing, derivatives pricing, term structure modelling, etc.
- The explicit form of transition density allows us to
  - perform MLE of model parameters based on discretely observed data
  - derive option pricing formulas in closed-form
- Most multivariate diffusions do not have explicit transition densities
  - (multi-factor) stochastic volatility models: Heston/GARCH/CEVSV
  - multivariate term structure models

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Related literature:

- Aït-Sahalia (2002) presents the Hermite expansion for reducible models to deriving a series representation of the transition density
- Several methods are proposed to derive small-time expansion for (multivariate) irreducible diffusions. Typical ones are
  - Aït-Sahalia (2008): the Kolmogorov method
  - Li (2013): the pathwise expansion based on Malliavin calculus
  - Yang, Chen and Wan (2019): the Itô-Taylor (delta) expansion

Motivations

(i) Can we naturally extend the Hermite method to irreducible diffusions at least in the sense of small-time expansion?

- Aït-Sahalia (2008) points out: "*The Hermite method requires, however, that the diffusion be reducible*"
- The advantages of Hermite expansion: calculating moments (Lee, Song and Lee, 2014) and option prices (Xiu, 2014), etc.

(ii) What is the relationship among various expansion methods?

- Deep understanding of different methods
- Guidelines for choosing an appropriate method
- Partial answers are provided by Yang, Chen and Wan (2019)
  - For reducible case, they show in Proposition 5.1 that both the Hermite expansion and the expansion of Yang, Chen and Wan (2019) with $\mu_0 = 0$ lead to the same formulas
  - Using symbolic computations, they verify that expansions of Li (2013) and Yang, Chen and Wan (2019) coincides with each other for general one- and two-dimsional models
  - They further conjecture that the expansions of Li (2013) and Yang, Chen and Wan (2019) are the same for multivariate models

Motivations and Contributions

Motivations:

- Can we naturally extend the Hermite method to irreducible diffusions at least in the sense of small-time expansion?
- What is the relationship among various expansion methods?

Contributions: in this work we provide affirmative answers to above questions and contribute to the literature as follows

- developing the Hermite expansion for transition densities of irreducible diffusions which admitting explicit formulas
- deriving explicit approximation formulas for European option prices, which is also an illustration for the advantage of the Hermite expansion
- showing that the derived Hermite expansion unifies the path expansion of Li (2013) and the Itô-Taylor (delta) expansion of Yang, Chen and Wan (2019)

II. Review of Hermite Expansion for Reducible Diffusions

Review of Hermite Expansion for the Reducible Case

Consider the following time-homogenous 1-dim model ($m = d = 1$)

$$dX(s) = \mu^X(X(s))ds + \sigma^X(X(s))dW(s).$$

Step 1: Do the Lamperti transform $X \to Y$
Define a new process

$$Y := \lambda(X) = \int^X \frac{1}{\sigma^X(\xi)} d\xi.$$

Let $\mu^Y(y) = [\frac{\mu^X(x)}{\sigma^X(x)} - \frac{1}{2}\frac{\partial \sigma^X(x)}{\partial x}]_{x=\lambda^{-1}(y)}$. Then,

$$dY(s) = \mu^Y(Y(s))ds + dW(s).$$

Review for the Reducible Case (cont'd)

Step 2: Expand $p_Y(t', y'|t, y)$ using Hermite polynomials $\{H_j(\gamma)\}$ as an orthonormal basis (ONB) (Theorem 1, Aït-Sahalia, 2002)

$$p_Y^{(J)}(t', y'|t, y) := \frac{1}{\sqrt{\Delta}}\phi(\gamma)\sum_{j=0}^{J}\eta^{(j)}(\Delta|t, y)H_j(\gamma) \xrightarrow{J\to\infty} p_Y(t', y'|t, y),$$

where $\Delta = t' - t$, $\gamma = \frac{y'-y}{\sqrt{\Delta}}$, $\phi(\gamma) = \frac{1}{\sqrt{2\pi}}e^{-\gamma^2/2}$

Review for the Reducible Case (cont'd)

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where $\Delta = t' - t$, $\gamma = \frac{y'-y}{\sqrt{\Delta}}$, $\phi(\gamma) = \frac{1}{\sqrt{2\pi}}e^{-\gamma^2/2}$

Step 3: Calculate $\eta^{(j)}(\Delta|t, y)$ via the Itô-Taylor expansion

$$\begin{aligned}
\eta^{(j)}(\Delta|t, y) &= \frac{1}{j!}\mathbb{E}\Big[H_j\Big(\frac{Y(t+\Delta)-y}{\sqrt{\Delta}}\Big)\Big|Y(t) = y\Big] \\
&= \underbrace{\frac{1}{j!}\sum_{k=0}^{K} \frac{\Delta^k}{k!}\Big(\Big(\mathcal{L}_\zeta\Big)^k \cdot H_j\Big(\frac{\zeta - y}{\sqrt{\Delta}}\Big)\Big)\Big|_{\zeta=y}}_{\eta^{(j,K)}(\Delta|t,y)} + \mathcal{O}(\Delta^{K+1-\frac{j}{2}}) \\
&= \frac{1}{j!}\sum_{k=0}^{\infty} \frac{\Delta^k}{k!}\Big(\Big(\mathcal{L}_\zeta\Big)^k \cdot H_j\Big(\frac{\zeta - y}{\sqrt{\Delta}}\Big)\Big)\Big|_{\zeta=y}
\end{aligned}$$

where $\mathcal{L}_\zeta^Y := \mu^Y(\zeta)\partial_\zeta + \partial_\zeta^2/2$. The last equality holds if Y is stationary and L_ζ^Y has purely discrete spectrum.

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where $\mathcal{L}_\zeta^Y := \mu^Y(\zeta)\partial_\zeta + \partial_\zeta^2/2$. The last equality holds if Y is stationary and L_ζ^Y has purely discrete spectrum.
A key observation is that for $K \geq j-1$, we have the following characterization of $\eta^{(j)}$

$$\eta^{(j)}(\Delta|t, y) \sim \eta^{(j,K)}(\Delta|t, y) \sim \mathcal{O}(\Delta^{\frac{j}{2}})$$

Examples of $\eta^{(j,K)}$: OU Model

$dY(s) = a(b - Y(s))ds + dW(s)$ for $s \geq t$ and $Y(t) = y$

$$\eta^{(1)} \sim \eta^{(1,1)} = \Delta^{\frac{1}{2}}\Big(a(b-y)\Big)$$

$$\eta^{(2)} \sim \eta^{(2,2)} = \Delta^{\frac{2}{2}}\Big(a(a(b-y)^2 - 1)\Big)$$

$$\eta^{(3)} \sim \eta^{(3,3)} = \Delta^{\frac{3}{2}}\Big(\frac{1}{2}a^2(b-y)\big(a(2b^2 - 4by - \Delta + 2y^2) - 6\big)\Big)$$

$$\eta^{(4)} \sim \eta^{(4,4)} = \Delta^{\frac{4}{2}}\Big(\frac{1}{2}a^2\big(a^2(b-y)^2(2b^2 - 4by - 7\Delta + 2y^2) - 4a(3b^2 - 6by - \Delta + 3y^2) + 6\big)\Big)$$

$$\eta^{(5)} \sim \eta^{(5,5)} = \Delta^{\frac{5}{2}}\Big(\frac{1}{8}a^3(b-y)\big(a^2\big(8b^4 - 32b^3y + 4b^2(12y^2 - 25\Delta) - 8b(4y^3 - 25\Delta y) + \Delta^2 + 8y^4 - 100\Delta y^2\big) - 5a\left(16b^2 - 32by - 41\Delta + 16y^2\right) + 120\big)\Big)$$

$$\eta^{(6)} \sim \eta^{(6,6)} = \Delta^{\frac{6}{2}}\Big(\frac{1}{8}a^3\big(a^3(b-y)^2\big(8b^4 - 32b^3y + 4b^2(12y^2 - 65\Delta) - 8b(4y^3 - 65\Delta y) + 31\Delta^2 + 8y^4 - 260\Delta y^2\big) - 2a^2\big(60b^4 - 240b^3y + 15b^2(24y^2 - 41\Delta) - 30b(8y^3 - 41\Delta y) + 8\Delta^2 + 60y^4 - 615\Delta y^2\big) + 8a(45b^2 - 90by - 56\Delta + 45y^2) - 120\big)\Big)$$

III. Hermite Expansion for Irreducible Diffusions

Quasi-Lamperti Transform for the Irreducible Diffusion

Consider the multivariate time-inhomogeneous diffusion

$$dX(s) = \mu^X(s, X(s))ds + \sigma^X(s, X(s))dW(s)$$

Step 1: Introduce a novel quasi-Lamperti transform $X \to Y$
Given t and $X(t) = x$, define a new process Y as follows

$$Y(s) := \nu_0^{-1/2} X(s), \qquad s \geq t,$$

where $\nu_0 := \nu^X(t, x)$ and $\nu^X(s, \xi) := \sigma^X(s, \xi) \left(\sigma^X(s, \xi)\right)^\top$. Then,

$$dY(s) = \mu^Y(s, Y(s))ds + \sigma^Y(s, Y(s))dW(s), \quad Y(t) = y,$$

Specifically, $\nu^Y(t, y) = Id_m$ where $\nu^Y(s, \zeta) := \sigma^Y(s, \zeta)(\sigma^Y(s, \zeta))^\top$

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For the irreducible case, **the quasi-Lamperti transform lies at the heart of the whole analysis**, which allows us to

- show the small-time convergence of the Hermite expansion
- derive explicit approximations for option prices
- compare various expansion methods analytically

Hermite Expansion for the Transformed Diffusion Y

Step 2: Formally expand $p_Y(t', y'|t, y)$ using multivariate Hermite polynomials $\{H_h(\gamma)\}$ as an ONB

$$p_Y^{(J)}(t', y'|t, y) := \Delta^{-\frac{m}{2}}\phi(\gamma)\sum_{j=0}^{J}\sum_{|h|=j}\eta^{(h)}(\Delta|t, y)\cdot H_h(\gamma),$$

where $\Delta = t' - t$, $\gamma = \frac{y'-y}{\sqrt{\Delta}}$, $h = (h_1, h_2, \cdots, h_m) \in \mathbb{Z}_+^m$ with $|h| := h_1 + h_2 + \cdots + h_m$, and $H_h(\gamma) = \prod_{i=1}^{m} H_{h_i}(\gamma_i)$. The coefficient $\eta^{(h)}(\Delta|t, y)$ is given by the conditional expectation as follow:

$$\eta^{(h)}(\Delta|t, y) = \frac{1}{h!}\mathbb{E}\left[H_h\left(\frac{Y(t+\Delta) - y}{\sqrt{\Delta}}\right)\middle| Y(t) = y\right]$$

Questions?

- How about the expansion error?
- How to calculate the explicit expansion coefficients?

Convergence of the Hermite Expansion

- Recall the key observation for reducible diffusions: $\eta^{(j)} = \mathcal{O}(\Delta^{\frac{j}{2}})$
- For irreducible diffusions, is the expansion coefficients $\eta^{(h)}$ a high order term of Δ?

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  Yes! Actually $\eta^{(h)} \approx \mathcal{O}(\Delta^{\frac{|h|}{6}})$

Step 3: Calculate $\eta^{(h)}(\Delta|t,y)$ via the Itô-Taylor expansion and analyze its order

Still use the Itô-Taylor expansion to calculation the expansion coefficients:

$$\eta^{(h)}(\Delta|t,y) = \underbrace{\frac{1}{h!}\sum_{k=0}^{K}\frac{\Delta^k}{k!}\left(\left(\mathcal{L}^Y_{s,\zeta}\right)^k \cdot H_h\left(\frac{\zeta - y}{\sqrt{\Delta}}\right)\right)\Big|_{s=t,\zeta=y}}_{\eta^{(h,K)}(\Delta|t,y)} + \mathcal{O}(\Delta^{K+1-\frac{|h|}{2}}),$$

Carefully analysing $\eta^{(h,K)}$ above, we have its explicit expression, and we can show that for $K \geq 2|h|/3$,

$$\eta^{(h,K)} = \mathcal{O}\left(\Delta^{\frac{1}{2}\lfloor\frac{j}{3}\rfloor}\right) \approx \mathcal{O}(\Delta^{\frac{|h|}{6}}),$$

with $j = |h|$ and $|h| := h_1 + h_2 + \cdots + h_m$.

Examples of $\eta^{(h,K)}$: CIR after Quasi-Lamperti Transform

$\underline{dY(s) = a(b - Y(s))dt + \sqrt{Y(s)/y}\,dW(s)}$ for $s \geq t$ and $Y(t) = y$

$$\begin{cases} \eta^{(0,0)} = \Delta^{\frac{0}{2}} = 1 \\ \eta^{(1,1)} = \Delta^{\frac{1}{2}}\Big(a(b-y)\Big) \\ \eta^{(2,2)} = \Delta^{\frac{2}{2}}\left(\frac{a\left(2ab^2y - 4aby^2 + 2ay^3 + b - 3y\right)}{2y}\right) \end{cases}$$

$$\begin{cases} \eta^{(3,2)} = \Delta^{\frac{1}{2}}\left(\frac{3\left(a^2\Delta y(b-y)+1\right)}{2y}\right) \\ \eta^{(4,3)} = \Delta^{\frac{2}{2}}\left(\frac{\left(6a^3\Delta y^2(b-y)^2 + a^2\Delta y(3b-7y) + 6ay(b-y) + 3\right)}{y^2}\right) \\ \eta^{(5,4)} = \Delta^{\frac{3}{2}}\left(-\frac{5\left(a^4\Delta y^3(-(b-y))\left(24b^2 - 48by - \Delta + 24y^2\right) - 12a^3\Delta y^2\left(3b^2 - 11by + 8y^2\right) - 2a \cdots\right.}{8y^3} \cdots\right. \end{cases}$$

$$\begin{cases} \eta^{(6,4)} = \Delta^{\frac{2}{2}}\left(\frac{15\left(32a^4b\Delta y^6 - 8a^4\Delta y^7 + 2a^3\Delta y^5\left(-24ab^2 + 7a\Delta + 48\right) + 4a^3b\Delta y^4\left(8ab^2 - 7a\Delta - \cdots\right.\right.}{\cdots}\right. \\ \eta^{(7,5)} = \Delta^{\frac{3}{2}}\left(\frac{7\left(-120a^5b\Delta y^8 + 24a^5\Delta y^9 - 30a^4\Delta y^7\left(-8ab^2 + 5a\Delta + 18\right) - 30a^4b\Delta y^6\left(8ab^2 - 15\cdots\right.\right.}{\cdots}\right. \\ \eta^{(8,6)} = \Delta^{\frac{4}{2}}\left(-\frac{7\left(-288a^6b\Delta y^{10} + 48a^6\Delta y^{11} - 60a^5\Delta y^9\left(-12ab^2 + 13a\Delta + 30\right) - 120a^5b\Delta y^8\left(\cdots\right.\right.}{\cdots}\right. \end{cases}$$

Examples of $\eta^{(h,K)}$: CIR after Quasi-Lamperti Transform (cont'd)

$\underline{dY(s) = a(b - Y(s))dt + \sqrt{Y(s)/y}dW(s)}$ for $s \geq t$ and $Y(t) = y$

$$\left.\begin{aligned} \eta^{(0,0)} &= \Delta^{\frac{0}{2}} \\ \eta^{(1,1)} &= \Delta^{\frac{1}{2}} \cdot (\cdots) \\ \eta^{(2,2)} &= \Delta^{\frac{2}{2}} \cdot (\cdots) \end{aligned}\right\} \longrightarrow \quad \frac{0}{2} = \frac{1}{2}\lfloor \frac{|h|}{3} \rfloor, \quad h = 0, 1, 2;$$

$$\left.\begin{aligned} \eta^{(3,2)} &= \Delta^{\frac{1}{2}} \cdot (\cdots) \\ \eta^{(4,3)} &= \Delta^{\frac{2}{2}} \cdot (\cdots) \\ \eta^{(5,4)} &= \Delta^{\frac{3}{2}} \cdot (\cdots) \end{aligned}\right\} \longrightarrow \quad \frac{1}{2} = \frac{1}{2}\lfloor \frac{|h|}{3} \rfloor, \quad h = 3, 4, 5;$$

$$\left.\begin{aligned} \eta^{(6,4)} &= \Delta^{\frac{2}{2}} \cdot (\cdots) \\ \eta^{(7,5)} &= \Delta^{\frac{3}{2}} \cdot (\cdots) \\ \eta^{(8,6)} &= \Delta^{\frac{4}{2}} \cdot (\cdots) \end{aligned}\right\} \longrightarrow \quad \frac{2}{2} = \frac{1}{2}\lfloor \frac{|h|}{3} \rfloor, \quad h = 6, 7, 8;$$

Select $\frac{1}{2}\lfloor \frac{|h|}{3} \rfloor$ because it is nondecreasing in $|h|$

Explicit Expansion Coefficients for Y

Thus, we have the following key lemma for the explicit formulas of the expansion coefficients which ensuring the convergence

Lemma

For each integer $j \geq 1$, $|h| = j$ and $K \geq 2j/3$, then

$$\eta^{(h,K)}(\Delta|t,y) = \sum_{n=\lceil 2j/3 \rceil}^{K} \frac{\Delta^{n-\frac{j}{2}}}{n!} w_{n,h}^{Y}(t,y) + \mathcal{O}(\Delta^{K+1-\frac{j}{2}}),$$

where $w_{n,h}^{Y}(t,y)$ is defined below. Moreover, we have

$$\eta^{(h)}(\Delta|t,y) = \mathcal{O}\left(\Delta^{\frac{1}{2}\lfloor\frac{j}{3}\rfloor}\right) \approx \mathcal{O}(\Delta^{\frac{|h|}{6}})$$

The introduction of the quasi-Lamperti transform is the key to prove this lemma.

The Weights Function $w^Y_{n,h}(s,\zeta)$

For a non-negative integer n and an m-dimensional integer valued vector $h=(h_1,\dots,h_m)$, the weights function $w^Y_{n,h}(s,\zeta)$, defined for each $(s,\zeta)\in[0,\infty)\times D_Y$, satisfies: for $n=0$, $w^Y_{0,0}(s,\zeta)=1$ and $w^Y_{0,h}(s,\zeta)=0$ for $h\neq 0$; for $n\geq 1$, $w^Y_{n,h}(s,\zeta)\equiv 0$ if either $\min\{h_1,\dots,h_m\}<0$, either $h=0$ or $|h|>2n$; for $n\geq 1$ and $h\in\mathbb{Z}^m_+$,

$$\begin{aligned} w^Y_{n,h}(s,\zeta) =& \mathcal{L}^Y_{s,\zeta} w^Y_{n-1,h}(s,\zeta) + \sum_{i=1}^m \mathcal{B}^{Y,i}_{s,\zeta} w^Y_{n-1,h-e_i}(s,\zeta) \\ & + \frac{1}{2}\sum_{i,l=1}^m (\nu^Y_{il}(s,\zeta)-\nu^Y_{il}(t,x)) w^Y_{n-1,h-e_i-e_l}(s,\zeta), \end{aligned}$$

where $\nu^Y_{il}(t,x)) = \mathbf{1}_{\{i=l\}}$ by definition and

$$\mathcal{L}^Y_{s,\zeta} = \partial_s + \sum_{i=1}^m \mu^Y_i(s,\zeta)\partial^{e_i}_\zeta + \frac{1}{2}\sum_{i,l=1}^m \nu^Y_{il}(s,\zeta)\partial^{e_i+e_l}_\zeta$$

$$\mathcal{B}^{Y,i}_{s,\zeta} := \mu^Y_i(s,\zeta) + \sum_{l=1}^m \nu^Y_{il}(s,\zeta)\cdot\partial^{e_l}_\zeta, \quad i=1,\dots,m.$$

The Explicit Expansion Formulas for Y

- Recall the Hermite expansion $p_Y^{(J)}$ and the expansion coefficients

$$p_Y^{(J)}(t', y'|t, y) = \Delta^{-\frac{m}{2}} \phi(\gamma) \sum_{j=0}^{J} \sum_{|h|=j} \eta^{(h)}(\Delta|t, y) \cdot H_h(\gamma),$$

$$\eta^{(h)}(\Delta|t, y) = \eta^{(h,K)}(\Delta|t, y) + \mathcal{O}(\Delta^{K+1-\frac{|h|}{2}})$$

- Then we have the explicit expansion formulas

$$p_Y^{(J)}(t', y'|t, y) = \underbrace{\frac{\phi(\gamma)}{\Delta^{\frac{m}{2}}} \sum_{j=0}^{J} \sum_{|h|=j} \eta^{(h,K)}(\Delta|t, y) H_h(\gamma)}_{p_Y^{(J,K)}(t', y'|t, y)} + \mathcal{O}(\Delta^{K+1-\frac{J}{2}-\frac{m}{2}})$$

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- Taking $J = 3L$ and $K \geq 2L \implies K+1-\frac{J}{2} \geq 2L+1-\frac{3L}{2} = \frac{L}{2}+1$
- Define $p_Y^{(L,\Delta)}$ by throwing away terms whose order higher than $\Delta^{L/2-m/2}$ in $p_Y^{(3L,K)}$

$$p_Y^{(L,\Delta)}(t',y'|t,y) := \frac{\phi(\gamma)}{\Delta^{\frac{m}{2}}}\sum_{l=0}^{L}\Delta^{\frac{l}{2}}\sum_{n=\lceil\frac{l}{2}\rceil}^{2l}\sum_{|h|=2n-l}\frac{1}{n!}w_{n,h}^{Y}(t,y)\cdot H_h(\gamma)$$

The Explicit Expansion Formulas for Y: An Alternative

Theorem

The rearranged Hermite expansion $p_Y^{(L,\Delta)}$ *is given by*

$$p_Y^{(L,\Delta)}(t',y'|t,y) = \frac{\phi(\gamma)}{\Delta^{\frac{m}{2}}} \sum_{l=0}^{L} \Delta^{\frac{l}{2}} \sum_{n=\lceil l/2 \rceil}^{2l} \frac{1}{n!} \sum_{|h|=2n-l} w_{n,h}^{Y}(t,y) \cdot H_h(\gamma),$$

where $\Delta = t'-t$, $\gamma = \frac{y'-y}{\sqrt{\Delta}}$, $w_{n,h}^{Y}(t,y)$ *are known explicitly.*
Moreover, under mild conditions, for $L > m$, *as* $\Delta \to 0$, *we have*

$$\sup_{(t,y,y')\in[0,T]\times D_Y^c \times D_Y} \left| p_Y(t',y'|t,y) - p_Y^{(L,\Delta)}(t',y'|t,y) \right| = \mathcal{O}\left(\Delta^{\frac{L+1}{2}-\frac{m}{2}}\right);$$

for $J \geq 3m$, *as* $\Delta \to 0$, *the Hermite expansion* $p_Y^{(J)}$ *converges as follows:*

$$\sup_{(t,y,y')\in[0,T]\times D_Y^c \times D_Y} \left| p_Y(t',y'|t,y) - p_Y^{(J)}(t',y'|t,y) \right| = \mathcal{O}\left(\Delta^{\frac{1}{2}(\lfloor \frac{J}{3} \rfloor+1)-\frac{m}{2}}\right).$$

We can express the approximation for the original process X by defining a sequence of approximations to p_X as

$$p_X^{(L,\Delta)}(t',x'|t,x) := \det(\nu_0)^{-1/2} p_Y^{(L,\Delta)}(t',\nu_0^{-1/2}x'|t,\nu_0^{-1/2}x).$$

The Explicit Expansion Formulas for X

Theorem

The density expansion for X has the following representation:

$$p_X^{(L,\Delta)}(t',x'|t,x) = \frac{\phi(z;\nu_0)}{\Delta^{\frac{m}{2}}} \sum_{l=0}^{L} \Delta^{\frac{l}{2}} \sum_{n=\lceil \frac{l}{2} \rceil}^{2l} \frac{1}{n!} \sum_{|h|=2n-l} w_{n,h}^X(t,x) H_h(z;\nu_0),$$

where $w_{n,h}^X$ are defined similarly to $w_{n,h}^Y$ (explicitly known), $z = \frac{x'-x}{\sqrt{\Delta}}$,

$$\phi(z;\nu_0) = \frac{\exp\left(-\frac{1}{2} z^\top \nu_0^{-1} z\right)}{\sqrt{(2\pi)^m \det(\nu_0)}}, \quad H_h(z;\nu_0) = (-1)^{|h|} \phi^{-1}(z;\nu_0) \partial_z^h \phi(z;\nu_0).$$

Moreover, under mild conditions, for $L > m$, as $\Delta \to 0$, we have

$$\sup_{(t,x,x')\in[0,T]\times D_X^c \times D_X} \left| p_X(t',x'|t,x) - p_X^{(L,\Delta)}(t',x'|t,x) \right| = \mathcal{O}\left(\Delta^{\frac{L+1}{2} - \frac{m}{2}}\right).$$

$p_X^{(L,\Delta)}(t',x'|t,x)$ is the same as the Itô-Taylor (delta) expansion (22) in Yang, Chen and Wan (2019) under the choice of $\mu_0 = 0$.

IV. Explicit Approximations for Option Prices

European Option Pricing via the Density Expansion for X

Assume X is defined under the risk-neutral measure $\mathbb{Q}$. At time t with $X(t) = x$, the price of European option with payoff $f(\cdot)$ and maturity t' is given below:

$$C(t,x) = e^{-r\Delta} \int_{\mathbb{R}^m} p_X(t', x'|t, x) f(x') dx'.$$

Using the expansion $p_X^{(L,\Delta)}$, we have an approximation as

$$C^{(L)}(t,x) := e^{-r\Delta} \int_{\mathbb{R}^m} p_X^{(L,\Delta)}(t', x'|t, x) f(x') dx'.$$

The structure of the expansion $p_X^{(L,\Delta)}$ simplify the above integral into a linear combination of the following integrals:

$$I_h(f) := \int_{\mathbb{R}^m} \frac{1}{\Delta^{m/2}} \phi\left(\frac{x'-x}{\sqrt{\Delta}}; \nu_0\right) H_h\left(\frac{x'-x}{\sqrt{\Delta}}; \nu_0\right) f(x') dx', \quad h \in \mathbb{Z}_+^m.$$

Explicit Approximation Formulas for European Options

The price of the European call option with the payoff function $f(x_1') = (e^{x_1'} - A)^+$ has the following approximation:

$$C^{(L)}(t,x) = e^{-r\Delta} I_0 + e^{-r\Delta} \sum_{l=1}^{L} \Delta^{\frac{l}{2}} \sum_{n=\lceil (l+1)/2 \rceil}^{2l} \frac{1}{n!} w^X_{n,2n-l}(t,x) I_{2n-l},$$

where $w^X_{n,l}(t,x) \equiv w^X_{n,(l,0,\cdots,0)}(t,x)$, $I_0 = e^{x_1 + \frac{1}{2}\bar{\sigma}^2\Delta} \cdot \Phi(d_2) - A \cdot \Phi(d_1)$ and for $l \geq 1$

$$I_l = \sqrt{\Delta}^l e^{x_1 + \frac{1}{2}\bar{\sigma}^2\Delta} \cdot \Phi(d_2) + A \sum_{1 \leq i \leq l-1} \sqrt{\Delta}^i \left(\bar{\sigma}\right)^{-(l-i)} H_{l-1-i}(-d_1)\phi(d_1).$$

Here $d_1 = \frac{x_1 - \ln A}{\bar{\sigma}\sqrt{\Delta}}$, $d_2 = d_1 + \bar{\sigma}\sqrt{\Delta}$, and $\bar{\sigma} := \sqrt{\nu^X_{11}(t,x)}$
Moreover, let D^c_X be a compact subsect of D_X. Under mild conditions, for $L > m$, as $\Delta \to 0$, we have

$$\sup_{(t,x) \in [0,T] \times D^c_X} \left| C(t,x) - C^{(L)}(t,x) \right| = \mathcal{O}\left(\Delta^{\frac{L+1}{2}} \right).$$

V. Relations to Existing Density Approximations

Relations to Existing Density Approximations

- We prove that the Hemite expansion derived in this paper unifies the expansions of Li (2013) and Yang, Chen and Wan (2019), that is,

Theorem (Equivalence)

The following three expansion formulas are the same:

(i) *the Hermite expansion* $p_X^{(L,\Delta)}$ *in this paper;*
(ii) *the pathwise expansion (3.21) in Li (2013);*
(iii) *the Itô-Taylor (delta) expansion (22) in Yang, Chen and Wan (2019) under the choice of* $\mu_0 = 0$.

- The equivalence between the Hermite expansion and the Itô-Taylor expansion, i.e., "(i)⇔(iii)"
  - For the reducible (univariate) case, it is proved in Proposition 5.1, Yang, Chen and Wan (2019)
  - For irreducible case, it is proved in the previous theorem(s), i.e., the explicit expansion formulas for X and/or Y

- Different from the Kolmogorov method of Aït-Sahalia (2008)

Main Ideas: the Hermite Expansion ⇔ the Expansion of Li (2013)

- Li (2013) develops an expansion for transition density of a time-homogenous diffusion
  - **providing explicit algorithm to compute high order terms**
  - the transition density can be as the conditional expectation of the Dirac delta function below:
    $$p_Y(t', x'|t, x) = \mathbb{E}[\delta(X(t') - x')|X(t) = x]$$
  - expanding the above conditional expectation via the pathwise expansion, i.e., Watanabe (1987)'s theory in Malliavin calculus
- The following steps are used to prove the equivalence.
  (a) We further derive **explicit formulas** of Li's expansion (relying on the quasi-Lamperti transform) and express it in terms of the Hermite polynomials
  (b) Derive the explicit formulas for $\eta^{(h)}$ (the coefficient of the Hermite expansion) via the pathwise expansion
  (c) Using (a) and (b), we prove that the Hermite expansion calculated via the pathwise expansion is the same as that of Li (2013)
  (d) The Hermite expansion derived via the Itô-Taylor and pathwise expansions are the same (Because $\eta^{(h)}$ derived in two methods are both $\sqrt{\Delta}$-expansion of $\eta^{(h)}$)

  (c) + (d)⇒ two expansions are the same

VI. Numerical Experiments

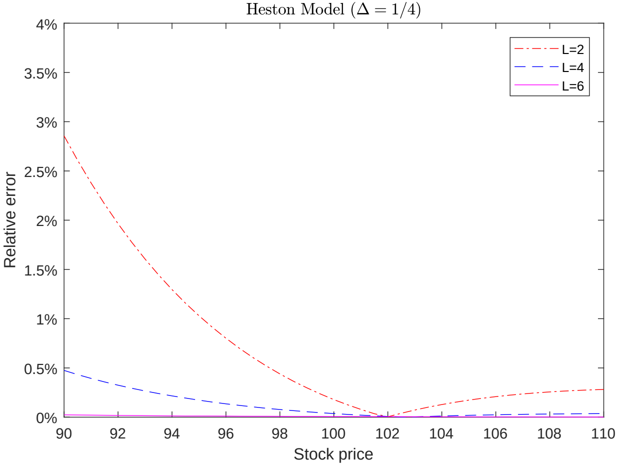
The Stochastic Volatility Models

- Consider the following general stochastic volatility model:

$$d\ln S(t) = \big((r-\delta) - \frac{1}{2}V(t)\big)dt + \sqrt{V(t)}dW_1(t),$$
$$dV(t) = \kappa(\alpha - V(t))dt + \sigma V^{\beta}(t)\big(\rho dW_1(t) + \sqrt{(1-\rho^2)}dW_2(t)\big),$$

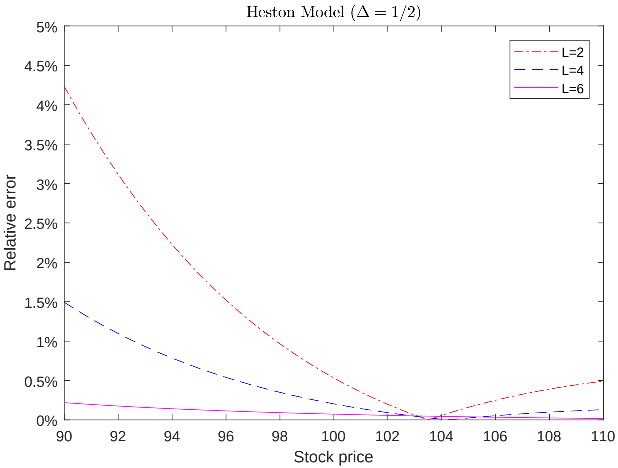
- The above process nests three kinds of models:
  - $\beta = 1/2$: the Heston stochastic volatility model
  - $\beta = 1$: the GARCH stochastic volatility model
  - $\beta \in (1/2, 1)$: the stochastic CEV model (SVCEV)

Relative Errors for the Heston Model with $\Delta = 1/4$



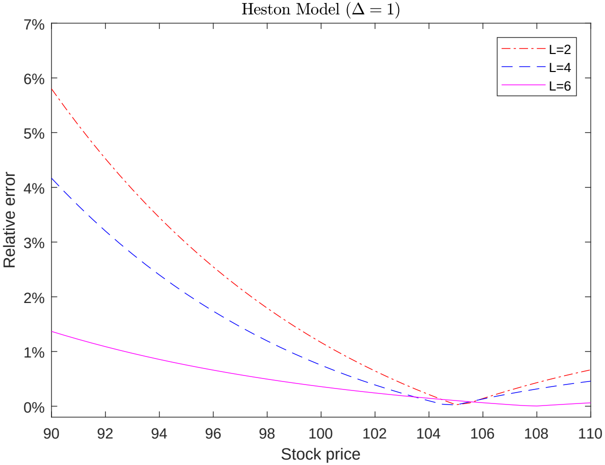
The relative errors for European call is calculated via $\mathrm{RE} = \frac{|\mathrm{FDM}-\mathrm{Appr.}|}{\mathrm{FDM}}$, where "FDM" denotes the benchmark is computed via the finite different method and "Appr" calculated via the L-th order of the approximation. Values of model parameters: $(r, \delta, \kappa, \alpha, \sigma, \rho, \beta) = (0.04, 0.015, 3, 0.1, 0.25, -0.8, 0.5)$, the initial volatility $V(0) = 0.1$ and the strike is 100.

Relative Errors for the Heston Model with $\Delta = 1/2$



The relative errors for European call is calculated via $\mathrm{RE} = \frac{|\mathrm{FDM} - \mathrm{Appr.}|}{\mathrm{FDM}}$, where "FDM" denotes the benchmark is computed via the finite different method and "Appr" calculated via the L-th order of the approximation. Values of model parameters: $(r, \delta, \kappa, \alpha, \sigma, \rho, \beta) = (0.04, 0.015, 3, 0.1, 0.25, -0.8, 0.5)$, the initial volatility $V(0) = 0.1$ and the strike is 100.

Relative Errors for the Heston Model with $\Delta = 1$



The relative errors for European call is calculated via $\mathrm{RE} = \frac{|\mathrm{FDM} - \mathrm{Appr.}|}{\mathrm{FDM}}$, where "FDM" denotes the benchmark is computed via the finite different method and "Appr" calculated via the 6-th order of the approximation. Values of model parameters: $(r, \delta, \kappa, \alpha, \sigma, \rho, \beta) = (0.04, 0.015, 3, 0.1, 0.25, -0.8, 0.5)$, the initial volatility $V(0) = 0.1$ and the strike is 100.

Percentage Relative Errors (%) with Different S_0

	Heston			GARCH			SVCEV		
S_0	$1/52$	$1/12$	$1/4$	$1/52$	$1/12$	$1/4$	$1/52$	$1/12$	$1/4$
97	0.005	0.002	0.009	0.006	0.006	0.137	0.004	0.004	0.076
98	0.002	0.001	0.009	0.001	0.003	0.145	0.001	0.001	0.082
99	0.001	0.000	0.008	0.001	0.001	0.150	0.001	0.000	0.086
100	0.009	0.002	0.006	0.013	0.003	0.151	0.017	0.003	0.086
101	0.001	0.000	0.006	0.001	0.000	0.151	0.001	0.001	0.086
102	0.001	0.000	0.005	0.001	0.000	0.148	0.001	0.001	0.083
103	0.001	0.001	0.005	0.001	0.000	0.143	0.001	0.000	0.077

The relative errors for European call is calculated via $\mathrm{RE} = \frac{|\mathrm{FDM} - \mathrm{Appr.}|}{\mathrm{FDM}}$, where "FDM" denotes the benchmark is computed via the finite different method and "Appr" calculated via the 6-th order of the approximation. The values of parameter vector $(r, \delta, \kappa, \alpha, \sigma, \rho, \beta)$ for three models are Heston: $(0.04, 0.015, 3, 0.1, 0.25, -0.8, 0.5)$; GARCH: $(0.04, 0.015, 1.6, 0.07, 2.2, -0.75, 1)$; SVCEV: $(0.04, 0.015, 4, 0.05, 0.75, -0.75, 0.8)$. The strike price is 100 for all options. For each model, the default initial volatility is $V(0) = \alpha$.

Percentage Relative Errors (%) with Different V_0

	Heston			GARCH			SVCEV		
V_0	$1/52$	$1/12$	$1/4$	$1/52$	$1/12$	$1/4$	$1/52$	$1/12$	$1/4$
0.08	0.011	0.004	0.070	0.011	0.003	0.211	0.011	0.002	0.264
0.1	0.009	0.002	0.006	0.009	0.002	0.386	0.009	0.004	0.352
0.12	0.008	0.001	0.054	0.008	0.000	0.638	0.008	0.005	0.435
0.14	0.006	0.000	0.092	0.007	0.002	0.973	0.007	0.007	0.513
0.16	0.006	0.001	0.126	0.006	0.008	1.394	0.006	0.008	0.587
0.18	0.005	0.001	0.158	0.005	0.020	1.901	0.005	0.010	0.658
0.2	0.005	0.002	0.189	0.005	0.041	2.495	0.005	0.011	0.725

The relative errors for European call is calculated via $\mathrm{RE} = \frac{|\mathrm{FDM}-\mathrm{Appr.}|}{\mathrm{FDM}}$, where "FDM" denotes the benchmark is computed via the finite different method and "Appr" calculated via the 6-th order of the approximation. The values of parameter vector $(r, \delta, \kappa, \alpha, \sigma, \rho, \beta)$ for three models are Heston: $(0.04, 0.015, 3, 0.1, 0.25, -0.8, 0.5)$; GARCH: $(0.04, 0.015, 1.6, 0.07, 2.2, -0.75, 1)$; SVCEV: $(0.04, 0.015, 4, 0.05, 0.75, -0.75, 0.8)$. The strike price is 100 for all options. For each model, the default initial stock price is $S(0) = 100$.

Conclusions

In this work, we contribute to the literature in the following aspects:

- developing the Hermite expansion for transition densities of irreducible diffusions which admitting explicit formulas
- deriving explicit approximation formulas for European option prices, which is also an illustration for the advantage of the Hermite expansion
- showing that the derived Hermite expansion unifies the pathwise expansion of Li (2013) and the Itô-Taylor (delta) expansion of Yang, Chen and Wan (2019)

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Thank you!

Questions and comments are welcome!

The full paper is available at SSRN
https://ssrn.com/abstract=3413376

Backup Slides

The Pathwise Expansion of Li (2013)

- Li (2013) develops the pathwise expansion for the transition density of a time-homogeneous diffusion X satisfying

$$dX(s) = \mu^X(X(s))ds + \sigma^X(X(s))dW(s), \quad X(t) = x \qquad (1)$$

He provides an **explicit algorithm** to compute the high order terms.
- To facilitate the comparison, we also contribute to the pathwise expansion as follows:
  - deriving explicit expansion formulas
  - express the pathwise expansion in terms of the Hermite polynomials
- A sketch of the derivation
  - **The quasi-Lamperti transform**. Letting $Y(s) := \left(\sigma^X(x)\right)^{-1} X(s)$ for $s \geq t$ and $X(t) = x$, we have

$$dY(s) = \mu^Y(Y(s))ds + \sigma^Y(Y(s))dW(s), \ \ Y(t) = y,$$

    where $\sigma^Y(y) = \nu^Y(y) \equiv Id_m$.
  - Define $Y^\epsilon(s) := Y(\epsilon^2 s + t)$ and $t' = \epsilon^2 + t$ (i.e. $\epsilon = \sqrt{\Delta}$)

$$dY^\epsilon(s) = \epsilon^2 \mu^Y(Y^\epsilon(s))ds + \epsilon\sigma^Y(Y^\epsilon(s))dW(s), \ Y^\epsilon(0) = y$$

The Pathwise Expansion of Li (2013) (cont'd)

- Define a random variable Γ^ϵ below. As $\epsilon \to 0$, we have

$$\Gamma^\epsilon := \frac{Y^\epsilon(1) - y}{\epsilon} = \epsilon \int_0^1 \mu^Y(Y^\epsilon(s))ds + \int_0^1 \sigma^Y(Y^\epsilon(s))dW(s) \to W(1)$$

- Using Itô's lemma iteratively leads to the pathwise expansion of Γ^ϵ

$$\Gamma^\epsilon = \sum_{j=0}^{L} \Big(\sum_{\mathbf{i} \in \mathcal{M}_{j+1}} C_{\mathbf{i}}^Y(y) \cdot \mathbb{I}_{\mathbf{i}}(1) \Big) \cdot \epsilon^i + \mathcal{O}(\epsilon^{L+1})$$

$\mathcal{M}_{j+1}$ is some index set. $C_{\mathbf{i}}^Y(y)$ is defined by the derivatives of $\mu^Y(\cdot)$ and $\sigma^Y(\cdot)$ with $\mathbf{i} = (i_1, i_2, \ldots, i_n) \in \{0, 1, \ldots, m\}^n$. Let $W_0(t) = t$. $\mathbb{I}_{\mathbf{i}}(t)$ is the iterated Itô integral, $\mathbb{I}_{\mathbf{i}}(t) = \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} dW_{i_k}(t_k) \cdots dW_{i_2}(t_2) dW_{i_1}(t_1)$

- The transition density is given by the conditional expectation below

$$\begin{aligned} p_Y(t', y'|t, y) =& \mathbb{E}[\delta(Y^\epsilon(1) - y')|Y^\epsilon(0) = y] \\ =& \Delta^{-\frac{m}{2}} \mathbb{E}[\delta(\Gamma^\epsilon - \gamma)|Y(t) = y], \qquad \gamma := (y' - y)/\sqrt{\Delta} \end{aligned}$$

- The pathwise expansion of p_Y is given below

$$p_Y(t', y'|t, y) = \underbrace{\Delta^{-\frac{m}{2}} \phi(\gamma) + \Delta^{-\frac{m}{2}} \sum_{k=1}^{L} \Delta^{\frac{k}{2}} \Omega_k(\gamma; y)}_{p_Y^{(L,LI)}(t', y'|t, y)} + \mathcal{O}(\Delta^{\frac{L+1-m}{2}})$$

The Explicit Formulas for the Expansion of Li (2013)

Proposition B.2, Yang, Chen and Wan (2019) states that the iterated Itô integral can be expressed as Hermite polynomials, i.e.,

$$\mathbb{E}[\mathbb{I}_{\mathbf{i}}(t)|W(t)=y] = \frac{\sqrt{t}^{\|\mathbf{i}\|}}{n!} H_{\mathbf{n_i}}\left(\frac{y}{\sqrt{t}}\right),$$

where $\mathbf{n_i} = (n_{\mathbf{i}}(1), \cdots, n_{\mathbf{i}}(m))$ with $n_{\mathbf{i}}(\alpha)$ being the number of α (for $\alpha = 0, 1, \cdots, m$) in $\mathbf{i}$, and $\|\mathbf{i}\| = 2n_{\mathbf{i}}(0) + \sum_{\alpha=1}^{m} n_{\mathbf{i}}(\alpha)$.

Proposition

The L-th order density expansion of Li (2013) for the diffusion Y can be represented as follows:

$$p_Y^{(L,LI)}(t', y'|t, y) = \Delta^{-\frac{m}{2}}\phi(\gamma) + \Delta^{-\frac{m}{2}}\phi(\gamma)\sum_{j=1}^{3L}\sum_{|h|=j} H_h(\gamma)\,\eta^{(h,LI)}(\Delta|t, y),$$

where $\Delta = t' - t$, $\gamma = (y' - y)/\sqrt{\Delta}$, and the expansion coefficient $\eta^{(h,LI)}(\Delta, y)$ is given by

$$\eta^{(h,LI)}(\Delta|t, y) = \sum_{k=1}^{L} \Delta^{\frac{k}{2}} \sum_{(j_1, j_2, \ldots, j_l)\in\mathcal{S}_k} \frac{1}{l!} \sum_{\mathbf{r}\in\{1,2,\ldots,m\}^l} \sum_{\substack{\mathbf{i}_\omega \in \mathcal{M}_{j_\omega+1} \\ \omega=1,\cdots,l}} \left(\prod_{\omega=1}^{l} C^Y_{\mathbf{i}_\omega, r_\omega}(y)\right) \cdot \tilde{w}_{\mathbf{a},\ddot{\mathbf{i}}} \cdot \frac{1}{(\ell(\ddot{\mathbf{i}}) - |\mathbf{a}|)!} \cdot \mathbf{1}_{\{\mathbf{a}\in\mathbb{Z}^m\}}.$$

Here, $\mathbf{r} := (r_1, \ldots, r_l)$, $\mathbf{b_r} := \sum_{\omega=1}^{l} e_{r_\omega}$, $\mathbf{a} := (\mathbf{n}(\ddot{\mathbf{i}}) + \mathbf{b_r} - h)/2$; the index sets $\mathcal{S}_k$ and $\mathcal{M}_j$, the coefficients $C^Y_{\mathbf{i}_\omega, r_\omega}(y)$ and $\tilde{w}_{\mathbf{a},\ddot{\mathbf{i}}}$ are recursively defined.

Calculating $\eta^{(h)}(\Delta|t,y)$ via the Pathwise Expansion

- Recall the coefficient $\eta^{(h)}(\Delta|t,y)$ in the Hermite expansion.

$$\eta^{(h)}(\Delta|t,y) = \frac{1}{h!}\mathbb{E}\Big[H_h\Big(\frac{Y(t+\Delta)-y}{\sqrt{\Delta}}\Big)\Big|Y(t)=y\Big]$$

$$= \underbrace{\frac{1}{h!}\sum_{k=0}^{K}\frac{\Delta^k}{k!}\Big(\Big(\mathcal{L}^{Y}_{s,\zeta}\Big)^k H_h\Big(\frac{\zeta-y}{\sqrt{\Delta}}\Big)\Big)\Big|_{s=t,\zeta=y}}_{\eta^{(h,K)}(\Delta|t,y)\text{, the Itô-Taylor expansion}} + \mathcal{O}(\Delta^{K+1-\frac{|h|}{2}}),$$

- Rewrite $\eta^{(h)}(\Delta|t,y)$ using Γ^ϵ.

$$\eta^{(h)}(\Delta|t,y) = \frac{1}{h!}\mathbb{E}\big[H_h\big(\Gamma^\epsilon\big)\big|Y^\epsilon(0)=y\big]\Big|_{\Gamma^\epsilon=\frac{Y^\epsilon(1)-y}{\epsilon}}$$

$$= \underbrace{\frac{1}{h!}\mathbb{E}\left[H_h\left(W(1)\right)\right] + \frac{1}{h!}\sum_{k=1}^{L}\tilde{\Omega}_k(y)\Delta^{\frac{k}{2}}}_{\eta^{(h)}_L(\Delta|t,y)\text{, the pathwise expansion}} + \mathcal{O}(\Delta^{\frac{L+1}{2}})$$

- Both $\eta^{(h,K)}(\Delta|t,y)$ and $\eta^{(h)}_L(\Delta|t,y)$ are coefficients of the Taylor expansion of $\eta^{(h)}$ as a function of $\epsilon \equiv \sqrt{\Delta}$, thus they are the same.

The Hermite Expansion ⇔ the Expansion of Li (2013)

Proposition

For $h \in \mathbb{Z}_+^m$ *and* $h \neq 0$*, the coefficients of the pathwise expansion* $\eta^{(h,LI)}(\Delta|t,y)$ *and the Hermite expansion* $\eta_L^{(h)}(\Delta|t,y)$ *satisfies*

$$\eta_L^{(h)}(\Delta|t,y) = \eta^{(h,LI)}(\Delta|t,y)$$

The Itô-Taylor (Delta) Expansion of Yang, Chen and Wan (2019)

- Recall the multivariate time-inhomogeneous diffusion
$$dX(s) = \mu^X(s, X(s))dt + \sigma^X(s, X(s))dW(s)$$
- Select a smooth sequence to approximate the Dirac delta function. Fix μ_0 and ν_0. Define
$$q(t', x'; s, y) = \frac{\exp\left(-\frac{(x'-y-\mu_0(t'-s))^\top \nu_0{}^{-1}(x'-y-\mu_0(t'-s))}{2(t'-s)}\right)}{(2\pi(t'-s))^{m/2}\det(\nu_0)^{1/2}}$$
- Formally, p_X can be expressed as follows:
$$p_X(t', x'|t, x) = \mathbb{E}[\delta(X(t') - x')|X(t) = x] = \lim_{s\uparrow t'} \mathbb{E}^{t,x}[q(t', x'; s, X(s))]$$
$$\approx \lim_{s\uparrow t'} \sum_{N=0}^{J} \frac{(s-t)^N}{N!}\left[(\partial_s + \mathcal{L}^X_{s,\xi})^N q(t', x'; s, \xi)\big|_{s=t,\xi=x}\right]$$
$$= \sum_{N=0}^{J} \frac{\Delta^N}{N!}(\partial_s + \mathcal{L}^X_{s,\xi})^N q(t', x'; s, \xi)\big|_{s=t,\xi=x} + \mathcal{R}_J$$
- **Choose** $\nu_0 = \nu^X(t, x)$ **and keep** μ_0 **free**. Then, the first term on RHS is the Itô-Taylor expansion with error as follows:
$$p_X(t', x'|t, x) = \underbrace{p_X^{(Ito,J)}(t', x'|t, x)}_{\text{Itô-Taylor expansion}} + \mathcal{O}\left(\Delta^{\frac{\lceil (J+1)/2\rceil - m}{2}}\right)$$

The Itô-Taylor (Delta) Expansion of Yang, Chen and Wan (2019) (cont'd)

- Let $z = (x' - x - \mu_0\Delta)/\sqrt{\Delta}$. The general term is

$$(\partial_s + \mathcal{L}_{s,\xi})^N q(t', x'|s, \xi) = \sum_{|h|=1}^{2N} w_{N,h}(s,\xi)\partial_y^h q(t', x'|s,\xi)$$
$$= \sum_{|h|=1}^{2N} \frac{1}{(t'-s)^{|h|/2}} w_{N,h}(s,\xi) \times H_h(z;\nu_0) \times q(t', x'|s,\xi)$$

- Given $\nu_0 = \nu(t, x; \theta)$, we can show that $w_{N,h}(t,x) = w_{N,h}(s,y)|_{s=t,y=x,\nu_0=\nu(t,x;\theta)} = 0$ for $|h| > 3N/2$!
- Then we have **the Itô-Taylor expansion**

$$p_X^{(Ito,J)}(t', x'|t, x) = q(t', x'|t, x)\left(1 + \sum_{N=1}^{J} \sum_{|h|=1}^{\lfloor 3N/2 \rfloor} \frac{w_{N,h}(t,x)H_h(z;\nu_0)}{N!} \Delta^{N-\frac{|h|}{2}}\right)$$

- Collecting terms in $p_X^{(Ito,J)}$ in an ascending order of $\sqrt{\Delta}$ up to the order of $\Delta^{L/2}$, we can arrive at **the delta expansion, i.e.,** $p_X^{(Ito,L,\Delta)}$
- Taking $\mu_0 = 0$, we can show $w_{N,h}(s,\xi)|_{\mu_0=0} = w_{N,h}^X(t,\xi)$, the latter is defined in the Hermite expansion $p_X^{(L,\Delta)}$. Consequently, we have

$$p_X^{(Ito,L,\Delta)}(t', x'|t, x)|_{\mu_0=0} = p_X^{(L,\Delta)}(t', x'|t, x)$$

Yang, Chen and Wan (2019): the Coefficients $w_{N,h}(s,\xi)|_{\nu_0=\nu(t,x;\theta)}$

(i). For any $N \geq 1$, define $w_{N,h}(s,\xi) \equiv 0$ if either $\min\{h_1, \cdots, h_m\} < 0$, $h = 0$, or $|h| > 2N$.
(ii). When $N = 1$,

$$\begin{cases} w_{1,e_i}(s,\xi) = \mu_i(s,\xi) - \mu_{0i}, \quad i = 1, \cdots, m; \\ w_{1,2e_i}(s,\xi) = \frac{1}{2}(\nu_{ii}(s,\xi;\theta) - \nu_{ii}(t,x)), \quad i = 1, \cdots, m; \\ w_{1,e_i+e_j}(s,\xi) = \nu_{ij}(s,\xi) - \nu_{ij}(t,x), \quad i \neq j, \ i,j = 1, \cdots, m, \end{cases}$$

where $\nu_{ij}(\cdot,\cdot;\theta)$ is the (i,j)-element of the diffusion matrix ν.
(iii).When $N > 1$ and all the components in h are nonnegative, $0 < |h| \leq 2N$, define recursively

$$\begin{aligned} w_{N,h}(s,\xi) =& (\partial_s + \mathcal{L}^X_{s,\xi}) w_{N-1,h}(s,\xi) + \sum_{i=1}^{m} \mathcal{A}_i w_{N-1,h-e_i}(s,\xi) \\ &+ \frac{1}{2} \sum_{i,j=1}^{m} (\nu_{ij}(s,\xi;\theta) - \nu_{ij}(t,x;\theta)) w_{N-1,h-e_i-e_j}(s,\xi), \end{aligned}$$

where

$$(\mathcal{A}_i) f(s,\xi) = (\mu_i(s,\xi;\theta) - \mu_{0i}) f(s,\xi) + \sum_{j=1}^{m} \nu_{ij}(s,\xi;\theta) \partial_{e_j} f(s,\xi).$$