Convex Duality in Portfolio Theory

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Introduction

Efficient trade-off between utility and risk happens on a convex curve in the two dimensional space of utility and risk. This is a general pattern that unifies a number of widely used portfolio theory:

- Markowitz portfolio theory (Markowitz 1952),
- Capital market asset pricing model (Sharpe 1964),
- Growth optimal portfolio theory (Kelly 1956, Lintena 1966),
- Leverage space portfolios (Vince 1990).

We illustrate with a one period financial market model that convex duality plays a crucial role in such portfolio problems.

Economy

- State of an economy is represented as a probability space (Ω, P) with a finite sample space $\Omega = \{\omega_1, \ldots, \omega_N\}$.
- We use $RV(\Omega, P)$ to denote the Hilbert space of all random variables endowed with the inner product

$$\langle x, y \rangle = \mathbb{E}(xy) = \sum_{\omega \in \Omega} x(\omega) y(\omega) P(\omega)$$

- Elements in $RV(\Omega, P)$ represent the payoffs of assets.
- We consider the one period model in which transaction can only take place at either the beginning $(t = 0)$ or the end of the period $(t = 1)$.

Financial Markets

- A financial market is modeled by random vectors

$$S_t = (S_t^0, S_t^1, \dots, S_t^M), t = 0, 1 \ on \ \Omega.$$

- S_t^0 represent the price of a risk free bond so that $R = S_1^0/S_0^0$, is a constant.
- and $\hat{S}_t = (S_t^1, \dots, S_t^M)$ represent the prices of risky assets at time t.
- We assume that S_0^i is a constant (current price is known) and S_1^i is a random variable (ending price depends on economic status).

Portfolio

Portfolio

A portfolio is a vector $x = [x_0, x_1, \ldots, x_M]^\top \in \mathbb{R}^{M+1}$ where x_i signifies the number of units of the ith asset. The value of a portfolio x at time t is $S_t \cdot x$, where notation "$\cdot$" signifies the usual dot product in $\mathbb{R}^{M+1}$. The risky part is $\hat{x} = [x_1, \ldots, x_M]^\top$

Utility

Utility functions $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ are usually assumed to satisfy some of the following properties.

- (u1) (Profit Seeking) The utility function u is an increasing function.
- (u2) (Diminishing Marginal Utility) The utility function u is concave.
- (u2s) (Strict Diminishing Marginal Utility) The utility function u is strictly concave.
- (u3) (Bankrupcy Forbidden) For $t < 0$, $u(t) = -\infty$.
- (u4) (Unlimited Growth) For $t \to +\infty$, we have $u(t) \to +\infty$.

Risk

A continuous risk function $\mathfrak{r} : A \to [0, +\infty)$ where A is a convex set of admissible portfolios satisfies: assumptions.

(r1) (Riskless Asset Contributes No risk) The risk measure $\mathfrak{r}(x) = \widehat{\mathfrak{r}}(\widehat{x})$ is a function of only the risky part of the portfolio, where $x = (x_0, \widehat{x})^\top$.

(r1n) (Normalization) There is at least one portfolio of purely bonds in A. Furthermore, $\mathfrak{r}(x) = 0$ if and only if x contains only riskless bonds, i.e. $x = (x_0, \widehat{0})^\top$ for some $x_0 \in \mathbb{R}$.

(r2) (Diversification Reduces Risk) The risk function $\mathfrak{r}$ is convex.

(r2s) (Diversification Strictly Reduces Risk) The risk function $\widehat{\mathfrak{r}}$ is strictly convex.

(r3) (Positive homogeneous) For $t > 0$, $\widehat{\mathfrak{r}}(t\widehat{x}) = t\widehat{\mathfrak{r}}(\widehat{x})$.

Trade-off risk and reward

- Markowitz poineer the idea of trade-off between risk and rewards for portfolio analysis.
- For Markowitz, risk is variance (or equivalently standard deviation) and reward is expected return. The math problem is:

$$\min\{\sigma(S_1 \cdot x) : \mathbb{E}[S_1 \cdot x] = \mu, S_0 \cdot x = 1\}.$$

We consider more general trade-off between a convex risk measure $\mathfrak{r}$ and the expected utility corresponding to a given (concave) utility function u.

Markowitz portfolio: graphic illustration

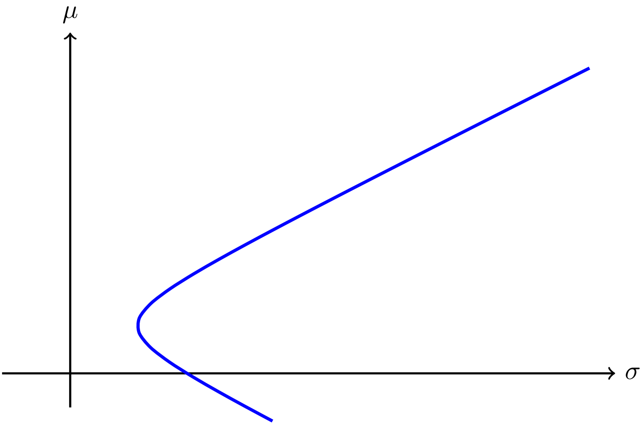


Figure: Markowitz Bullet

Convex duality

Our more general framework leads to a convex programming problem

$$\min\{\mathfrak{r}(x) : \mathbb{E}[u(S_1 \cdot x)] \geq \mu, x \in A\}.$$

Duality plays an important role in linking the risk-reward space and the portfolio space. Applications of convex duality are not limited to portfolio theory:

Peter Carr and Qiji J. Zhu

Convex Duality and Financial Mathematics

Springer 2018.

Efficient frontier

Define

$$\mathcal{G}(\mathfrak{r}, u; A) := \{(r, \mu) : \exists x \in A \ s.t. \ r \geq \mathfrak{r}(x), \mu \leq \mathbb{E}[u(S_1 \cdot x)]\} \subset \mathbb{R}^2.$$

Then the efficient frontier is the Pareto efficient points of $\mathcal{G}(\mathfrak{r}, u; A)$ and is denoted $\mathcal{G}_{eff}(\mathfrak{r}, u; A)$.

Efficient frontier in pictures

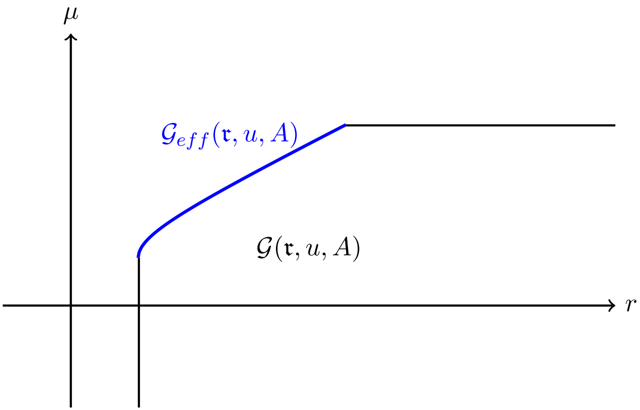


Figure: Finite efficient frontier.

Representing Efficient frontier

Define

$$\gamma(\mu) := \inf\{\mathfrak{r}(x) : \mathbb{E}[u(S_1 \cdot x)] \geq \mu, x \in A\}.$$

Then $\gamma(\mu)$ is convex and

$$\text{graph } \gamma(\mu)$$

contains the efficient frontier.

Efficient frontier in pictures

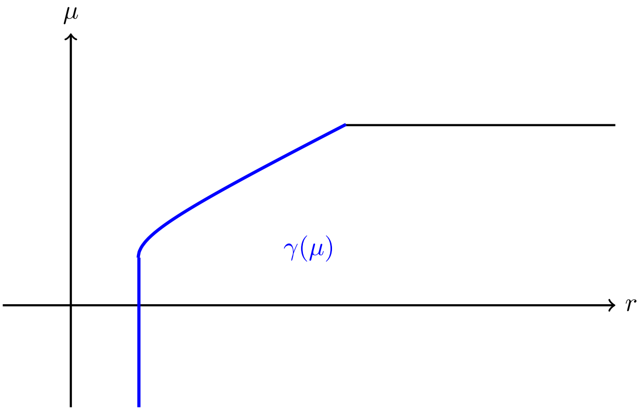


Figure: Efficient frontier in graph γ.

Representing Efficient frontier

Symmetrically, define

$$\nu(r) := \sup\{\mathbb{E}[u(S_1 \cdot x)] : \mathfrak{r}(x) \leq r, x \in A\}.$$

Then $\nu(r)$ is concave and

$$\text{graph } \nu(r)$$

contains the efficient frontier.

Efficient frontier in pictures

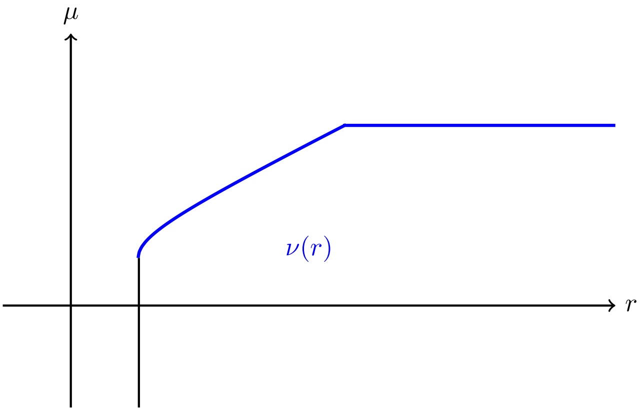


Figure: Efficient frontier in graph ν.

Representing Efficient frontier

Introducing the exchange operator $\hat{P} : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\hat{P}(x, y) = (y, x)$ we have the following representation

Representation of Efficient Frontier

$$\mathcal{G}_{eff}(\mathfrak{r}, u; A) = \hat{P}[\text{graph } \gamma] \cap \text{graph } \nu.$$

Efficient frontier in pictures

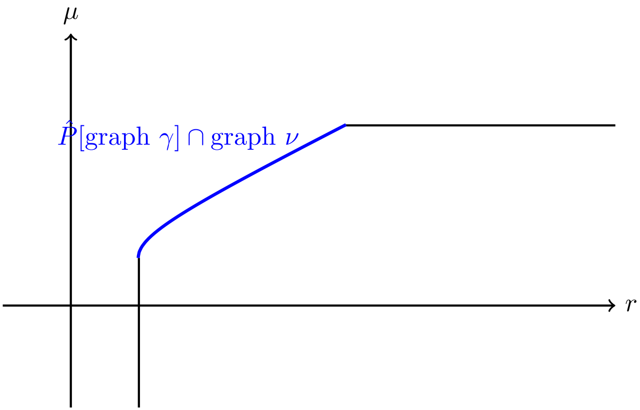


Figure: Finite efficient frontier.

Efficient portfolios

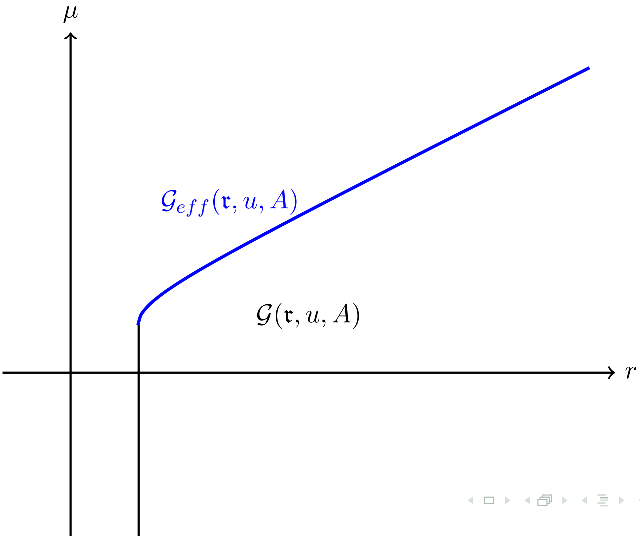
Uniqueness of efficient portfolio

Under reasonable conditions for the risk measure and utility function.

- Each point $(r, \mu) \in \mathcal{G}_{eff}(\mathfrak{r}, u; A)$ corresponding to a unique efficient portfolis $x(r, \mu)$.
- The mapping $(r, \mu) \to x(r, \mu)$ is continuously.

When $\mathfrak{r}, u$ satisfy a certain additional structure conditions so do the optimal portfolios.

Typical Efficient frontiers



Typical Efficient frontiers

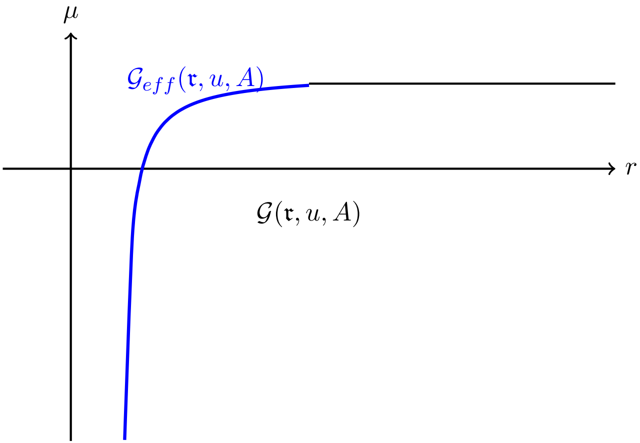


Figure: Efficient frontier with non zero minimum risk.

Typical Efficient frontiers

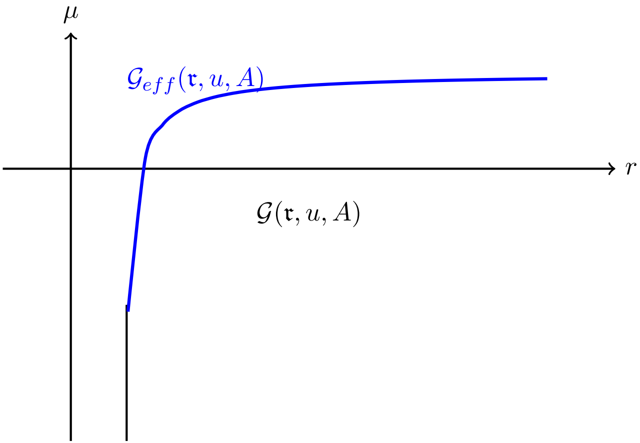
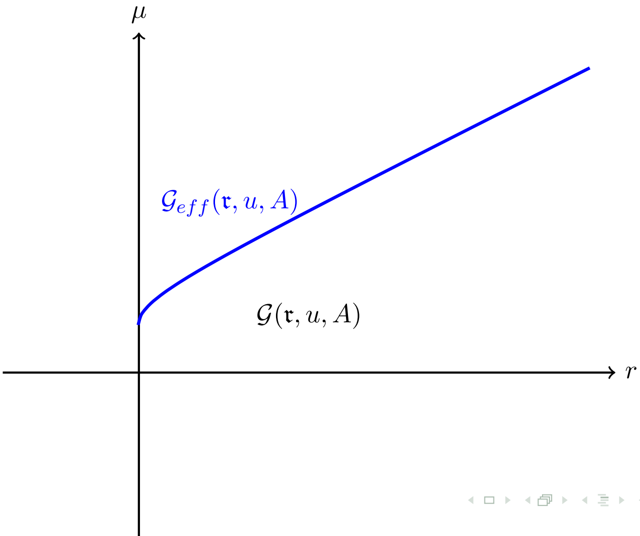


Figure: Efficient frontier with bounded return.

Typical Efficient frontiers



Typical Efficient frontiers

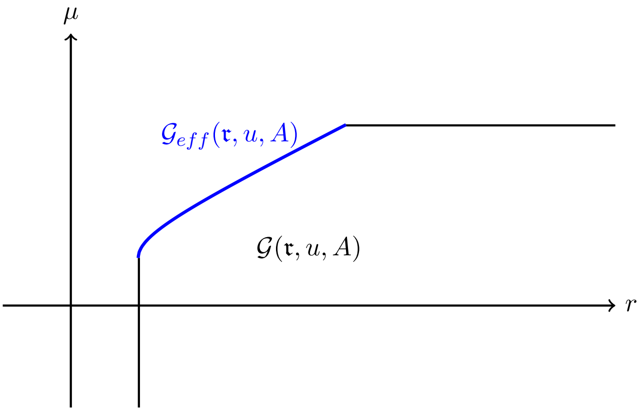


Figure: Finite efficient frontier.

Parameterization of efficient portfolios

The efficient portfolios can be parameterized in r and μ, respectively.

Parameterization of the efficient portfolios

For $r \in \mathrm{dom}\ \nu \cap \mathrm{range}\ \gamma$,

$$r \to x(r, \nu(r))$$

is a continuous parameterization of the efficient portfolios.
Similarly, For $\mu \in \mathrm{dom}\ \gamma \cap \mathrm{range}\ \nu$,

$$\mu \to x(\gamma(\mu), \mu)$$

is a continuous parameterization of the efficient portfolios.

Typical Efficient frontiers

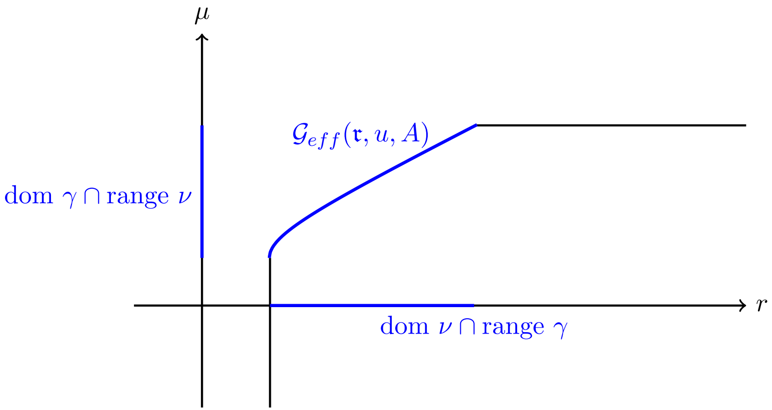


Figure: Parameterization of efficient frontier.

The role of duality

Convex programming and its dual help us to unify different portfolio theories and provide new perspectives.

- Primal space –portfolio (higher dimension).
- Dual space – risk-reward (two dimension).
- Strict convexity – unique efficient portfolio that are continuously depend on risk-reward.
- Linear structure in classical portfolio theory is the result of additional structure assumption on risks and rewards.
- Dual solution – in different settings have different financial explanations. They provide additional insights and perspectives.

Markowitz portfolio problem: In portfolio space

$$\text{minimize } \{\mathrm{Var}(\hat{S}_1 \cdot \hat{x}) : \mathbb{E}[\hat{S}_1 \cdot \hat{x}] \geq \mu, \hat{S}_0 \cdot \hat{x} = 1\},$$

that is $A = \{x : x_0 = 0, S_0 \cdot x = 1\}$.
The efficient portfolio $\hat{x}(\gamma(\mu), \mu)$ is an affine function of μ leads to

Two fund Theorem (Tobin)

Any portfolio on the Markowitz frontier can be represented as a linear combination of two distinct portfolios on this frontier.

Math: two point determines a line.
Implications: For investing two good mutual funds suffice.

Markowitz portfolio problem: In risk-award space

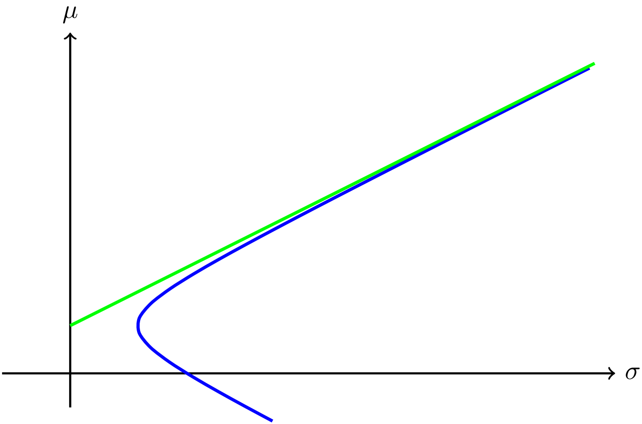


Figure: Markowitz Bullet

Capital asset pricing model (Sharpe): in portfolio space

$$\text{minimize } \{\mathrm{Var}(S_1 \cdot x) : \mathbb{E}[S_1 \cdot x] \geq \mu, S_0 \cdot x = 1\},$$

that is $A = \{x : S_0 \cdot x = 1\}$.
In portfolio space $x(\mu)$ is an affine function of μ leads to

One fund Theorem

Any portfolio on the Efficient frontier of CAPM can be represented as a linear combination of the bond and a pure risky market portfolio that is also on the Markowitz bullet.

Implications: Mix bond and mutual funds in your investment portfolio.

Capital asset pricing model: in risk-reward space

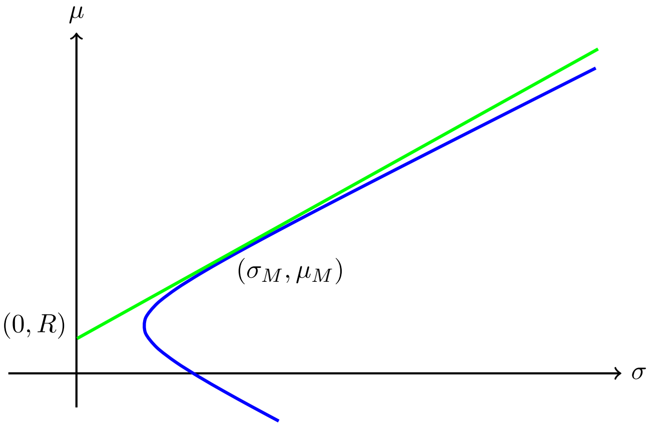


Figure: Capital Market Line and Markowitz Bullet

Generalizations

- The one fund theorem holds for any positive homogeneous risk measures.
- This goes beyond the variance type risk measure that used originally by Markowitz and in CAPM model.
- In particular, it applies to approximation of drawdown type risk measures that are widely used by traders.
- We constructed counter-examples showing that the two fund theorem fail in more general settings.
- Meaning of the dual solution: Sharpe ratio.

Growth optimal portfolio (Lintner)

The portfolio κ corresponding to

$$\max\{\mathbb{E}[\ln(S_1 \cdot x)] : \mathfrak{r}(x) \leq +\infty, x \in A\},$$

is call the growth optimal portfolio. Theoretically it gives the fastest compounded growth of the wealth. In practice, it is known to be too risky that have to be scale back.

The maximum can be used to gauge the effectiveness of an investment strategy– in terms of how much useful information it contains.

Leverage portfolio space (Vince)

Leverage portfolios space theory suggests to scale back from κ in portfolio space along curves connecting κ and pure bond but didn't specify how. It turns out solution $y(r)$ corresponding to

$$\max\{\mathbb{E}[\ln(S_1 \cdot x)] : \mathfrak{r}(x) \leq r, x \in A\},$$

is the reasonable way of implementing. Different choice of $\mathfrak{r}$ will lead to different paths in the leverage portfolio space.

Leverage portfolio space in picture

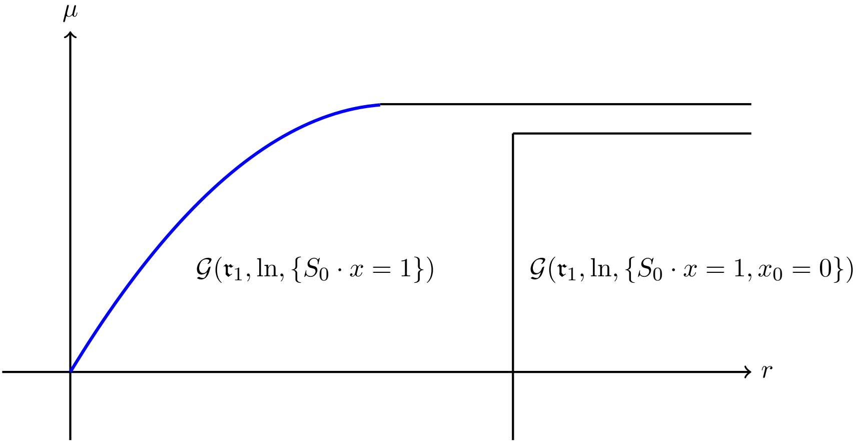


Figure: Separated efficient frontiers

Leverage portfolio space in picture

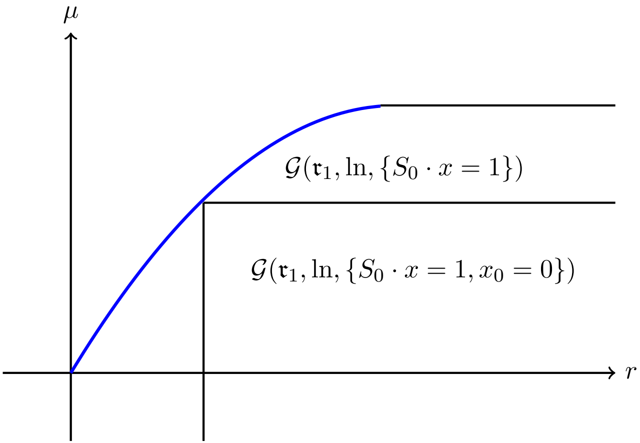


Figure: Touching efficient frontiers

Leverage portfolio space in picture

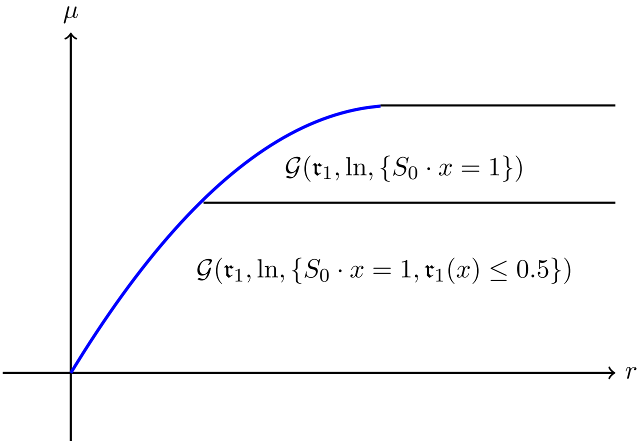


Figure: Shared efficient frontiers

Leverage level for the banking sector

- leverage level in some banks close to the level of GOP which cause desaster.
- Vince - Zhu scale back that leverage level based on explicit consideration of risk and a finite investment horizon.
- resulting leverage level is safer and more realistic.
- sensitivity analysis provide guide to adjust to the interest risk.
- the marginal rate of change of the return with regards to the interest risk is determined by the dual solution.

Multiple risk measures

- In actual bank balance sheet problem we oftern need to discuss two types of risks: interest risk and credit risk.
- Simple cases can be modeled by linear programming.
- Due to the two different kinds of risk the dual space becomes 3 dimension instead of 2.
- The center issue is sensitivity with respect to the two different risk measures.
- They are determined by the solution of the dual problem which also provides pricing for insurance to the corresponding risks.

Concluding remarks

- Portfolio problems can be modeled by convex programming.
- Dual solution always provides additional perspective to the portfolio problem and are worthy explore.

Concluding remarks

Convex duality also plays important roles in many other financial problems beyond portfolio theory, e.g.

- represent and estimate of coherent risk measures.
- fundamental theorems of asset pricing.
- determining super-sub hedging bounds in incomplete markets.
- delta hedging.
- conic finance: model for market with friction.

Peter Carr and Qiji J. Zhu

Convex Duality and Financial Mathematics
Springer 2018.

Thank you

Thank You!