# Non-Concave Utility Maximization without the Concavification Principle

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# Motivation (1)

- The classical expected utility maximization model (e.g. CRRA/CARA utility) is a concave optimization problem
- In contrast, many investment objectives are related to non-concave utility, e.g.
  - The goal-reaching problem of Browne (1999, Advances in Applied Probability): fund managers maximize the probability of beating some benchmark
  - The S-shaped utility: Kahneman and Tversky (1979, Econometrica), Berkelaar, Kouwenberg and Post (2004, The Review of Economics and Statistics)
  - Convex compensation schemes: Carpenter (2000, JF), Basak, Pavlova and Shapiro (2007, RFS), He and Kou (2018, MF)
  - Aspiration utility: Lee, Zapatero and Giga (2018)

# Motivation (2)

- Previous literature: ignore portfolio constraints (such as no-short-sale, no borrowing, etc)
- Shortage: Unrealistic high leverage

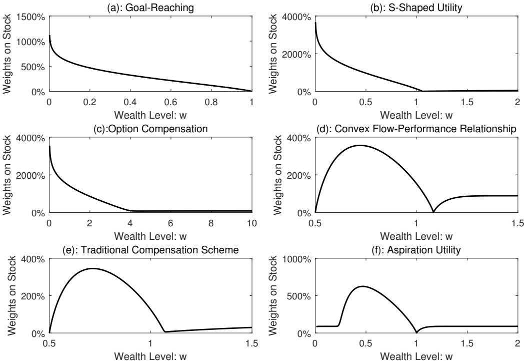


Figure: Time to Maturity $\frac{1}{12}$

# Motivation (3)

- Ignore portfolio constraints $\Rightarrow$ concavification principle $\Rightarrow$ concave value function
- With and without portfolio constraints: an example with S-shaped utility

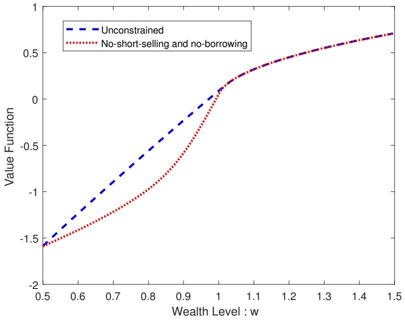


Figure: Time to Maturity $\frac{1}{12}$

# Our focus

- Non-concave portfolio optimization without the concavification principle
  - general leverage constraints (e.g. no short sale, no borrowing)
  - discontinuous utility

- Concavification principle does not apply
  - numerical methods: scheme? convergence?

- Joint impact of portfolio constraints and non-concave utility on optimal policy

# Our findings

- General findings
  - The value function is not necessarily globally concave before maturity
  - Investors are not myopic w.r.t. portfolio constraints in the sense that they may take risker leverage ratios in anticipation of portfolio constraints
  - Investors may short sale stock despite positive risk premium for higher volatility
    The intuition is that in concave utility case, the volatility is a burden which is need to be offset by the higher return.
    However, in convex utility case, the volatility is actually a resource

- Those model-specific findings in literature remain valid to some extent.

# Theoretical contribution

- We prove the comparison principle even in the presence of portfolio constraints and discontinuous utility
  - The value function is the unique discontinuous viscosity solution to the associated HJB equation
  - Convergence of the standard monotone finite difference scheme

# The model

- A riskfree bond $B$ with interest rate $0$ and a risky stock $S$

$$dS_t/S_t = \mu dt + \sigma d\mathcal{B}_t$$

- $W^\pi(t)$: the self-financing wealth process

$$dW^\pi(t) = \mu\pi(t)W^\pi(t)dt + \sigma\pi(t)W^\pi(t)d\mathcal{B}_t,$$

where $\pi(t)$ : the proportion of wealth in stock

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$$V(t,w) = \sup_{d\leq\pi\leq u,\ W^\pi(T)\geq 0} E\left[U(W^\pi(T))|W(t)=w\right],$$

where

- $d \leq \pi \leq u$: portfolio constraints, if $d = 0$, no short-selling; if $u = 1$, no borrowing
- $U(\cdot)$: utility function, not necessarily concave or continuous

# Examples of non-concave utilities: Goal-reaching problem

- A fund manager maximizes the probability of beating some benchmark:

$$U(W_T) = 1_{\{W_T \geq H\}},$$

where $H$ is the target level

# Examples of non-concave utilities: S-shaped utility

- Tversky and Kahneman (1979, Econometrica)'s S-Shaped utility:

$$U(W_T) = \begin{cases} (W_T - W_0)^p & \text{for } W_T > W_0 \\ -\lambda(W_0 - W_T)^p & \text{for } W_T \leq W_0 \end{cases}$$

  - $W_0$: the initial wealth, distinguishing the gain and loss
  - $0 < p < 1$: the degree of risk aversion, e.g. $p = 0.88$
  - $\lambda > 1$: pain from loss > pleasure from gain, e.g. $\lambda = 2.25$
  - The utility is convex for loss $W_T < W_0$ and concave for gain $W_T > W_0$

# Examples of non-concave utilities: Option compensation

- Carpenter (2000, JF): A risk averse manager compensated with a call option over the fund he controls

$$U(W_T) = (m \max\{W_T - K, 0\} + C)^p$$

  - $0 < p < 1$: the risk aversion degree
  - $K > 0$: the strike price of the option
  - $m$: the number of options
  - $C > 0$: the constant compensation

# Theoretic analysis of constrained non-concave problem

Consider the HJB equation:

$$\frac{\partial V}{\partial t} + \sup_{d \le \pi_t \le u} \left\{ \frac{1}{2} \pi_t^2 w^2 \sigma^2 \frac{\partial^2 V}{\partial w^2} + \pi_t w \mu \frac{\partial V}{\partial w} \right\} = 0, \qquad (1)$$

with the boundary condition

$$V(t, 0) = U(0) \qquad (2)$$

and an asymptotic condition at maturity:
(i) if $[d, u]$ is unbounded, it degenerates to the standard case

$$\lim_{(t,\zeta) \to (T-, w)} V(t, \zeta) = \hat{U}(w), \qquad (3)$$

where $\hat{U}$ is the concave envelope of $U$
(ii) if $[d, u]$ is bounded, (discontinuity!)

$$\lim_{(t,\zeta) \to (T-, w)} V(t, \zeta) - U(w-) - 2\Phi(y) \left( U(w+) - U(w-) \right) = 0 \qquad (4)$$

where $y = \frac{0 \wedge (\ln \zeta - \ln w)}{\max\{-d, u\} \sigma \sqrt{T-t}}$ and $\Phi$ is the CDF of a standard normal random variable

# Theoretic analysis of constrained non-concave problem

## Theorem (Comparison Principle)

*(i) Assume* $[d, u]$ *is bounded (unbounded). Let* $v^*$ *and* $v_*$ *be separately viscosity* subsolution *and* supersolution *to* (1) *with boundary conditions* (2) *and* (4) *(with* (2) *and* (3)*)*

*Suppose* $|v^*|$, $|v_*| \leq C_1 w^p + C_2$, *for some* $0 < p < 1$, $C_1, C_2 > 0$

*Then* $v^* \leq v_*$ *for all* $w \geq B$ *and* $0 < t < T$

- This theorem covers both the continuous and the discontinuous case
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$$\text{Comparison principle} \Rightarrow \begin{cases} \text{uniqueness of viscosity solution} \\ \text{convergence of numerical schemes} \end{cases}$$

# Theoretic analysis of constrained non-concave problem

**Theorem**

*$V(t,w)$ is the unique viscosity solution of the HJB equation to (1) with boundary conditions (2) and (4) (with (2) and (3)) which satisfies*

$$|v| \leq C_1 w^p + C_2, \text{ for some } 0 < p < 1, \; C_1, C_2 > 0 \tag{5}$$

**Theorem (Numerical Scheme Convergence)**

*The numerical solution of a fully implicit finite difference scheme with upwind treatment for the HJB equation converges to the value function as the discretization size tends to zero*

# General findings (1)

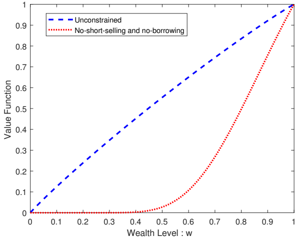



Figure: Goal reaching

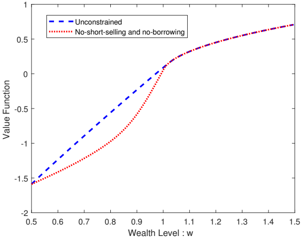
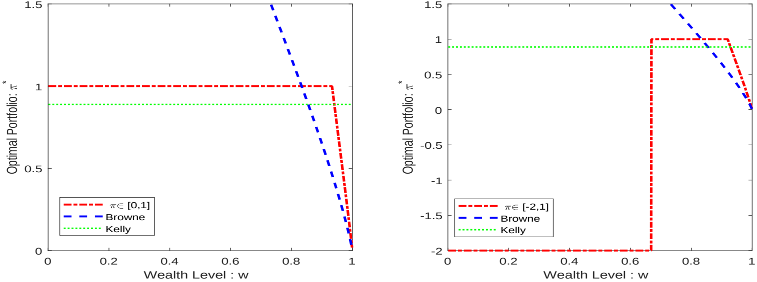



Figure: S-shaped utility

In general the value function is not globally concave before maturity

# General findings (2)

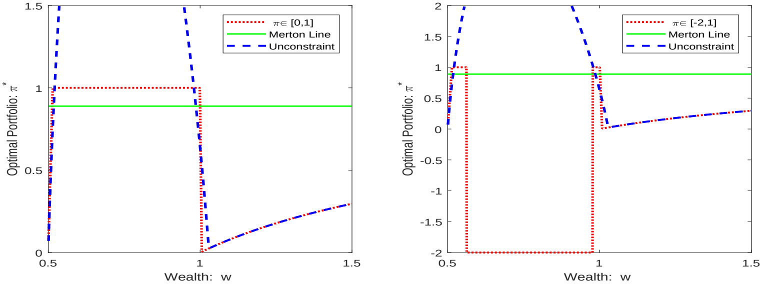



Goal reaching problem $(r = 0.07, \mu = 0.15, \sigma = 0.3, T = 1)$

Investors are not myopic with respect to portfolio constraints

Investors may gamble by short-selling (borrowing) stock even with positive (negative) risk premium

# General findings (3)



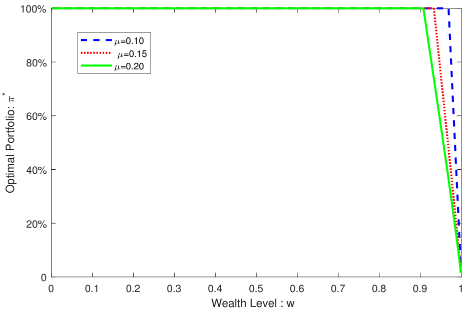

S-shaped utility maximization

$$(r = 0.03, \mu = 0.07, \sigma = 0.3, p = 0.5, \lambda = 2.25, W_0 = 1, \\ T = 1/12, B = 0.5)$$

Investors are not myopic with respect to portfolio constraints

Investors may gamble by short-selling (borrowing) stock even with positive (negative) risk premium

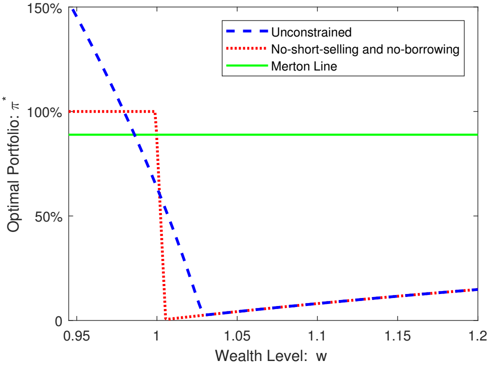
# Goal-reaching problem




Optimal strategy for different $\mu$, while
$r = 0.07, \sigma = 0.3, T = 1, B = 0$

- The optimal goal-reaching strategy is no longer equivalent to the replicating strategy of a digital option rather than the assertion in Browne (1999)

# S-shaped utility($W_0 = 1$)




$$r = 0.03,\ \mu = 0.07,\ \sigma = 0.3,\ p = 0.5,\ \lambda = 2.25,\ W_0 = 1,$$
$$T = 1/12,\ B = 0.5$$

- Reduce more stock near reference point: a more conservative strategy compared to Berkelaar, Kouwenberg and Post (2004)

# Option compensation

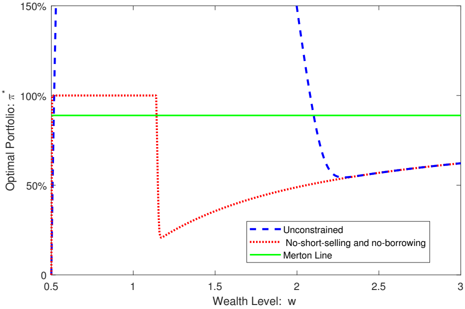



Figure: $r = 0.03$, $\mu = 0.07$, $\sigma = 0.3$, $p = 0.5$, $K = 1$, $\alpha = 0.2$, $C = 0.02$, $W_0 = 1$, $T - t = 1/12$, $B = 0.5$

- Convex incentives may reduce stock investment in more scenarios compared to Carpenter (2000, JF)

# Conclusion(1)

- Non-concave portfolio optimization without the concavification principle: portfolio constraints, discontinuous utility

- We prove comparison principle by introducing an asymptotic condition at maturity. This implies
  - the convergence of the standard monotone finite difference method
  - uniqueness of discontinuous viscosity solutions to the associate HJB equations

# Conclusion (2)

- Three general findings
  - The concavification technique no longer applies, and in general the value function is not globally concave before maturity
  - Investors may take action in anticipation of future portfolio constraints being binding
  - Investors may gamble against market trend in the case of underperformance

- Those model-specific findings hold to some extent