Open-Loop and Closed-Loop Solvabilities for Stochastic LQ Optimal Control Problems of Markov Regime-Switching System

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Outline

- Literature Review
- Formulation
- Open-loop Solvability
- Closed-loop Solvability
- A Weak Closed-loop
- Future Works

Literature Review

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- X. Li and X.Y. Zhou (2002): Indefinite stochastic LQ controls with Markovian jumps in a finite time horizon, *Communications in Information and Systems*, 2, 265–282.
- X. Li, X.Y. Zhou and M. Ait-Rami (2003): Indefinite stochastic linear quadratic control with Markovian jumps in infinite time horizon *Journal of Global Optimization*, 27, 149–175.
- J. Sun, X. Li and J. Yong (2016): Open-loop and closed-loop solvabilities for stochastic linear quadratic optimal control problems *SIAM Journal on Control and Optimization*, 54, 2274–2308.
- J. Sun, H. Wang and J. Yong (2019): Weak closed-loop solvability of stochastic linear-quadratic optimal control problems, *Discrete and Continuous Dynamical Systems*, 39, 2785–2805.

Formulation

Formulation

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a complete filtered probability space on which a standard one-dimensional Brownian motion $W = \{W(t); 0 \leqslant t < \infty\}$ and a continuous time, finite-state, Markov chain $\alpha = \{\alpha(t); 0 \leqslant t < \infty\}$ are defined, where $\mathbb{F} = \{\mathcal{F}_t\}_{t \geqslant 0}$ is the natural filtration of W and α augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$. We identify the state space of the chain α with a finite set $\mathcal{S} \triangleq \{1, 2 \ldots, D\}$, where $D \in \mathbb{N}$ and suppose that the chain is homogeneous and irreducible. To specify the statistical or probabilistic properties of the chain α, we define the generator $\lambda(t) := [\lambda_{ij}(t)]_{i,j=1,2,\ldots,D}$ of the chain under $\mathbb{P}$. Here, for each $i, j = 1, 2, \ldots, D$, $\lambda_{ij}(t)$ is the constant transition intensity of the chain from state i to state j at time t. Note that $\lambda_{ij}(t) \geq 0$, for $i \neq j$ and $\sum_{j=1}^{D} \lambda_{ij}(t) = 0$, so $\lambda_{ii}(t) \leq 0$. In what follows for each $i, j = 1, 2, \ldots, D$ with $i \neq j$, we suppose that $\lambda_{ij}(t) > 0$, so $\lambda_{ii}(t) < 0$. For each fixed $j = 1, 2, \cdots, D$, let $N_j(t)$ be the number of jumps into state j up to time t and set $\widetilde{N}_j(t) := N_j(t) - \lambda_j(t)$ with

$$\lambda_j(t) := \int_0^t \lambda_{\alpha(s-)\, j} I_{\{\alpha(s-) \neq j\}} ds = \sum_{i=1, i \neq j}^{D} \int_0^t \lambda_{ij}(s) I_{\{\alpha(s-)=i\}} ds.$$

Formulation

Let $0 \leqslant t < T$ and consider the following controlled Markovian regime switching linear stochastic differential equation (SDE, for short) over a finite time horizon $[t, T]$:

$$\begin{cases} dX(s) = \Big[A(s,\alpha(s))X(s) + B(s,\alpha(s))u(s) + b(s,\alpha(s))\Big]ds \\ \qquad\qquad + \Big[C(s,\alpha(s))X(s) + D(s,\alpha(s))u(s) + \sigma(s,\alpha(s))\Big]dW(s), \qquad s \in [t,T], \\ X(t) = x, \qquad \alpha(t) = i, \end{cases} \tag{1}$$

where $A, C : [0,T] \times \mathcal{S} \to \mathbb{R}^{n\times n}$ and $B, D : [0,T] \times \mathcal{S} \to \mathbb{R}^{n\times m}$ are given deterministic functions, called the coefficients of the *state equation* (1); $b, \sigma : [0,T] \times \mathcal{S} \times \Omega \to \mathbb{R}^n$ are $\mathbb{F}$-progressively measurable processes, called the *nonhomogeneous terms*; and $(t, x, i) \in [0,T) \times \mathbb{R}^n \times \mathcal{S}$ is called the *initial pair*.

In the above, the process $u(\cdot)$, which belongs to the following space:

$$\mathcal{U}[t,T] \triangleq \left\{ u : [t,T] \times \Omega \to \mathbb{R}^m \;\middle|\; \begin{array}{l} u(\cdot) \text{ is } \mathbb{F}\text{-progressively measurable} \\ \text{and } \mathbb{E}\displaystyle\int_t^T |u(s)|^2 ds < \infty \end{array} \right\}.$$

is called the *control process*, and the solution $X(\cdot)$ of (1) is called the *state process* corresponding to (t, x, i) and $u(\cdot)$.

To measure the performance of the control $u(\cdot)$, we introduce the following quadratic *cost functional*:

$$J(t,x,i;u(\cdot)) \triangleq \mathbb{E}\Bigg\{\Big\langle G(\alpha(T))X(T),X(T)\Big\rangle + 2\Big\langle g(\alpha(T)),X(T)\Big\rangle + \int_t^T \Bigg[\Bigg\langle \begin{pmatrix} Q(s,\alpha(s)) & S(s,\alpha(s))^\top \\ S(s,\alpha(s)) & R(s,\alpha(s)) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \Bigg\rangle + 2\Bigg\langle \begin{pmatrix} q(s,\alpha(s)) \\ \rho(s,\alpha(s)) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \Bigg\rangle \Bigg] ds \Bigg\}, \tag{2}$$

where $G(i) \in \mathbb{R}^{n\times n}$ is a symmetric constant matrix, and $g(i)$ is an $\mathcal{F}_T$-measurable random variable taking values in $\mathbb{R}^n$, with $i \in \mathcal{S}$; $Q : [0,T] \times \mathcal{S} \to \mathbb{R}^{n\times n}$, $S : [0,T] \times \mathcal{S} \to \mathbb{R}^{m\times n}$ and $R : [0,T] \times \mathcal{S} \to \mathbb{R}^{m\times m}$ are deterministic functions with both Q and R being symmetric; $q : [0,T] \times \mathcal{S} \to \mathbb{R}^n$ and $\rho : [0,T] \times \mathcal{S} \to \mathbb{R}^m$ are $\mathbb{F}$-progressively measurable processes.

Formulation

Problem (M-SLQ). For any given initial pair $(t, x, i) \in [0, T) \times \mathbb{R}^n \times \mathcal{S}$, find a control $u^*(\cdot) \in \mathcal{U}[t, T]$, such that

$$J(t, x, i; u^*(\cdot)) = \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x, i; u(\cdot)), \qquad \forall u(\cdot) \in \mathcal{U}[t, T]. \tag{3}$$

The above is called a *stochastic linear quadratic optimal control problem* of the Markovian regime switching system. Any $u^*(\cdot) \in \mathcal{U}[t, T]$ satisfying (3) is called an *open-loop optimal control* of Problem (M-SLQ) for the initial pair (t, x, i); the corresponding state process $X(\cdot) = X(\cdot\,; t, x, i, u^*(\cdot))$ is called an *optimal state process*; and the function $V(\cdot, \cdot, \cdot)$ defined by

$$V(t, x, i) \triangleq \inf_{u(\cdot) \in \mathcal{U}[t,T]} J(t, x, i; u(\cdot)), \qquad (t, x, i) \in [0, T] \times \mathbb{R}^n \times \mathcal{S}, \tag{4}$$

is called the *value function* of Problem (M-SLQ).

Formulation

Note that in the special case when $b(\cdot,\cdot), \sigma(\cdot,\cdot), g(\cdot), q(\cdot,\cdot), \rho(\cdot,\cdot) = 0$, the state equation (1) and the cost functional (2), respectively, become

$$\begin{cases} dX(s) = \Big[A(s,\alpha(s))X(s) + B(s,\alpha(s))u(s)\Big]ds \\ \qquad\qquad + \Big[C(s,\alpha(s))X(s) + D(s,\alpha(s))u(s)\Big]dW(s), \qquad s \in [t,T], \\ X(t) = x, \qquad \alpha(t) = i, \end{cases} \tag{5}$$

and

$$J^0(t,x,i;u(\cdot)) = \mathbb{E}\Big\{\Big\langle G(\alpha(T))X(T), X(T)\Big\rangle + \int_t^T \Big\langle \begin{pmatrix} Q(s,\alpha(s)) & S(s,\alpha(s))^\top \\ S(s,\alpha(s)) & R(s,\alpha(s)) \end{pmatrix} \begin{pmatrix} X(s) \\ u(s) \end{pmatrix}, \begin{pmatrix} X(s) \\ u(s) \end{pmatrix} \Big\rangle ds \Big\}. \tag{6}$$

We refer to the problem of minimizing (6) subject to (5) as the homogeneous LQ problem associated with Problem (M-SLQ), denoted by **Problem (M-SLQ)**^{0}. The corresponding value function is denoted by $V^0(t,x,i)$. Moreover, all the coefficients of (1) and (2) are independent of the regime switching term $\alpha(\cdot)$.

The First Example

Consider the following one-dimensional state equation

$$\begin{cases} dX(s) = \big[- \alpha(s)X(s) + u(s) \big] ds + \sqrt{2\alpha(s)} X(s) dW(s), \qquad s \in [t, 1], \\ X(t) = x, \quad \alpha(t) = i, \end{cases}$$

and the nonnegative cost functional

$$J(t, x, i; u(\cdot)) = \mathbb{E}\big[X(1)^2 \big].$$

The First Example

We construct the control $\bar{u}(\cdot)$ as

$$\bar{u}(s) \equiv \frac{x}{t-1} \cdot \exp\left\{-2\int_t^s \alpha(r)dr + \int_t^s \sqrt{2\alpha(r)}dW(r)\right\}, \qquad s \in [t,1].$$

By the variation of constants formula, the state process $\bar{X}(\cdot)$, corresponding to (t,x,i), can be presented by

$$\bar{X}(s) = \exp\left\{-2\int_t^s \alpha(r)dr + \int_t^s \sqrt{2\alpha(r)}dW(r)\right\} \cdot \left[x + \frac{s-t}{t-1}x\right], \qquad s \in [t,1],$$

which satisfies $\bar{X}(1) = 0$. Hence,

$$J(t,x,i;\bar{u}(\cdot)) = \mathbb{E}\big[\bar{X}(1)^2\big] = 0.$$

Since the cost functional is nonnegative, the control $\bar{u}(\cdot)$ is optimal for the initial pair (t,x,i).

Question

Is the optimal control $\bar{u}(\cdot)$ open-loop or closed-loop?

Assumptions

The following standard assumptions will be in force throughout this paper.

(H1) For every $i \in \mathcal{S}$, the coefficients of the state equation satisfy the following

$$\begin{cases} A(\cdot,i) \in L^1(0,T;\mathbb{R}^{n\times n}), & B(\cdot,i) \in L^2(0,T;\mathbb{R}^{n\times m}), \\ C(\cdot,i) \in L^2(0,T;\mathbb{R}^{n\times n}), & D(\cdot,i) \in L^\infty(0,T;\mathbb{R}^{n\times m}), \\ b(\cdot,i) \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)), & \sigma(\cdot,i) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n). \end{cases}$$

(H2) For every $i \in \mathcal{S}$, the weighting coefficients in the cost functional satisfy the following

$$\begin{cases} Q(\cdot,i) \in L^1(0,T;\mathbb{S}^n), & S(\cdot,i) \in L^2(0,T;\mathbb{R}^{m\times n}), & R(\cdot,i) \in L^\infty(0,T;\mathbb{S}^m), \\ G(i) \in \mathbb{S}^n, & g(i) \in L^2_{\mathcal{F}_T}(\Omega;\mathbb{R}^n), \\ \rho(\cdot,i) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), & q(\cdot,i) \in L^2_{\mathbb{F}}(\Omega;L^1(0,T;\mathbb{R}^n)). \end{cases}$$

Definition 1

(i) An element $u^*(\cdot) \in \mathcal{U}[t,T]$ is called an *open-loop optimal control* of Problem (M-SLQ) for the initial pair $(t,x,i) \in [0,T] \times \mathbb{R}^n \times \mathcal{S}$ if

$$J(t,x,i;u^*(\cdot)) \leqslant J(t,x,i;u(\cdot)), \qquad \forall u(\cdot) \in \mathcal{U}[t,T]. \tag{7}$$

(ii) A pair $(\Theta^*(\cdot), v^*(\cdot)) \in L^2(t,T;\mathbb{R}^{m\times n}) \times \mathcal{U}[t,T]$ is called a *closed-loop optimal strategy* of Problem (M-SLQ) on $[t,T]$ if

$$J(t,x,i;\Theta^*(\cdot)X^*(\cdot)+v^*(\cdot)) \leqslant J(t,x,i;u(\cdot)), \;\; \forall (x,i) \in \mathbb{R}^n\times\mathcal{S}, \;\; u(\cdot) \in \mathcal{U}[t,T], \tag{8}$$

where $X^*(\cdot)$ is the strong solution to the following closed-loop system:

$$\begin{cases} dX^*(s) = \Big\{\big[A(s,\alpha(s)) + B(s,\alpha(s))\Theta^*(s)\big]X^*(s) \\ \qquad\qquad +B(s,\alpha(s))v^*(s) + b(s,\alpha(s))\Big\}ds \\ \qquad\quad +\Big\{\big[C(s,\alpha(s)) + D(s,\alpha(s))\Theta^*(s)\big]X^*(s) \\ \qquad\qquad +D(s,\alpha(s))v^*(s) + \sigma(s,\alpha(s))\Big\}dW(s), \\ X^*(t) = x. \end{cases} \tag{9}$$

FBSDE

To simply notation of our further analysis, we introduce the following forward-backward stochastic differential equation (FBSDE for short) on a finite horizon $[t, T]$:

$$\begin{cases} dX^u(s) = \big[A(s,\alpha(s))X^u(s) + B(s,\alpha(s))u(s) + b(s,\alpha(s))\big]ds \\ \qquad\qquad + \big[C(s,\alpha(s))X^u(s) + D(s,\alpha(s))u(s) + \sigma(s,\alpha(s))\big]dW(s), \\ dY^u(s) = -\big[A(s,\alpha(s))^\top Y^u(s) + C(s,\alpha(s))^\top Z^u(s) \\ \qquad\qquad + Q(s,\alpha(s))X^u(s) + S(s,\alpha(s))^\top u(s) + q(s,\alpha(s))\big]ds \\ \qquad\qquad + Z^u(s)dW(s) + \sum_{k=1}^{D} \Gamma_k^u(s)d\widetilde{N}_k(s), \qquad s \in [t, T], \\ X^u(t) = x, \quad \alpha(t) = i, \quad Y^u(T) = G(\alpha(T))X^u(T) + g(\alpha(T)). \end{cases} \tag{10}$$

FBSDE

The solution of the above FBSDE system is denoted by $(X^u(\cdot), Y^u(\cdot), Z^u(\cdot), \Gamma^u(\cdot))$, where $\Gamma^u(\cdot) := (\Gamma_1^u(\cdot), \cdots, \Gamma_D^u(\cdot))$.

If the control $u(\cdot)$ is chose as $\Theta(\cdot)X(\cdot) + v(\cdot)$, we will use the notation

$$(X^{\Theta,v}(\cdot), Y^{\Theta,v}(\cdot), Z^{\Theta,v}(\cdot), \Gamma^{\Theta,v}(\cdot))$$

denoting by the solution of the above FBSDE.

If $b(\cdot,\cdot) = \sigma(\cdot,\cdot) = q(\cdot,\cdot) = g(\cdot) = 0$, the solution of the above FBSDE is denoted by

$$(X_0^u(\cdot), Y_0^u(\cdot), Z_0^u(\cdot), \Gamma_0^u(\cdot)).$$

Open-loop Solvability

Open-loop Solvability

We first present the equivalence between the open-loop solvability and the corresponding forward-backward differential equation system.

Theorem 1

Let (H1)–(H2) *hold and* $(t, x, i) \in [t, T] \times \mathbb{R}^n \times \mathcal{S}$ *be given. An element* $u(\cdot) \in \mathcal{U}[t, T]$ *is an open-loop optimal control of Problem* (M-SLQ) *if and only if* $J^0(t, 0, i; v(\cdot)) \geq 0, \forall v(\cdot) \in \mathcal{U}[t, T]$ *and the following stationary condition hold:*

$$
\begin{aligned}
&B(s, \alpha(s))^\top Y^u(s; t, x, i) + D(s, \alpha(s))^\top Z^u(s; t, x, i) \\
&+ S(s, \alpha(s)) X^u(s; t, x, i) + R(s, \alpha(s)) u(s) + \rho(s, \alpha(s)) = 0, \quad s \in [t, T],
\end{aligned} \tag{11}
$$

where $(X^u(\cdot\,; t, x, i), Y^u(\cdot\,; t, x, i), Z^u(\cdot\,; t, x, i))$ *is the adapted solution to the FBSDE* (10).

Open-loop Solvability

The standard conditions:

$$G(i) \geqslant 0, \quad R(s,i) \geqslant \delta I, \quad Q(s,i) - S(s,i)^\top R(s,i)^{-1} S(s,i) \geqslant 0, \qquad i \in \mathcal{S}, \quad \text{a.e. } s \in [0,T]. \tag{12}$$

Proposition 1

Let (H1)–(H2) and (12) hold. Then for any $(t,i) \in [0,T) \times \mathcal{S}$, the map $u(\cdot) \mapsto J^0(t,0,i;u(\cdot))$ is uniformly convex.

Theorem 2

Let (H1)–(H2) hold. Suppose the map $u(\cdot) \mapsto J^0(t,0,i;u(\cdot))$ is uniformly convex. Then Problem (M-SLQ) is uniquely open-loop solvable, and there exists a constant $\gamma \in \mathbb{R}$ such that

$$V^0(t,x,i) \geqslant \gamma |x|^2, \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{13}$$

Note that in the above, the constant γ does not have to be nonnegative.

Closed-loop Solvability

In the following, we first introduce some notation and the Riccati equation.

Let

$$\begin{aligned}\hat{S}(s,i) &:= B(s,i)^\top P(s,i) + D(s,i)^\top P(s,i)C(s,i) + S(s,i),\\ \hat{R}(s,i) &:= R(s,i) + D(s,i)^\top P(s,i)D(s,i).\end{aligned} \tag{14}$$

The Riccati equation associated with Problem (M-SLQ) is

$$\begin{cases}\dot{P}(s,i) + P(s,i)A(s,i) + A(s,i)^\top P(s,i) + C(s,i)^\top P(s,i)C(s,i)\\ -\hat{S}(s,i)^\top \hat{R}(s,i)^\dagger \hat{S}(s,i) + Q(s,i) + \sum_{k=1}^{D} \lambda_{ik}(s)P(s,k) = 0, \quad \text{a.e. } s \in [0,T],\\ P(T,i) = G(i).\end{cases} \tag{15}$$

Closed-loop Solvability

Definition 2

A solution $P(\cdot,\cdot) \in C([0,T] \times \mathcal{S}; \mathbb{S}^n)$ of (15) is said to be *regular* if

$$\begin{cases} \mathcal{R}\big(\hat{S}(s,i)\big) \subseteq \mathcal{R}\big(\hat{R}(s,i)\big), & \text{a.e. } s \in [0,T], \\ \hat{R}(\cdot,\cdot)^{\dagger}\hat{S}(\cdot,\cdot) \in L^2(0,T;\mathbb{R}^{m\times n}), \\ \hat{R}(s,i) \geqslant 0, & \text{a.e. } s \in [0,T]. \end{cases} \tag{16}$$

A solution $P(\cdot,\cdot)$ of (15) is said to be *strongly regular* if

$$\hat{R}(s,i) \geqslant \lambda I, \qquad \text{a.e. } s \in [0,T], \tag{17}$$

for some $\lambda > 0$. The Riccati equation (15) is said to be (*strongly*) *regularly solvable*, if it admits a (strongly) regular solution.

Theorem 3

Let (H1)–(H2) *hold. Problem* (M-SLQ) *is closed-loop solvable on* $[0, T]$ *if and only if the Riccati equation* (15) *admits a regular solution* $P(\cdot,\cdot) \in C([0,T] \times \mathcal{S}; \mathbb{S}^n)$ *and the solution* $(\eta(\cdot), \zeta(\cdot), \xi_1(\cdot), \cdots, \xi_D(\cdot))$ *of the following BSDE:*

$$\begin{cases} d\eta(s) = -\Big\{\big[A(s,\alpha(s))^\top - \hat{S}(s,\alpha(s))^\top \hat{R}(s,\alpha(s))^\dagger B(s,\alpha(s))^\top\big]\eta(s) \\ \qquad + \big[C(s,\alpha(s))^\top - \hat{S}(s,\alpha(s))^\top \hat{R}(s,\alpha(s))^\dagger D(s,\alpha(s))^\top\big]\zeta(s) \\ \qquad + \big[C(s,\alpha(s))^\top - \hat{S}(s,\alpha(s))^\top \hat{R}(s,\alpha(s))^\dagger D(s,\alpha(s))^\top\big] P(s,\alpha(s))\sigma(s,\alpha(s)) \\ \qquad - \hat{S}(s,\alpha(s))^\top \hat{R}(s,\alpha(s))^\dagger \rho(s,\alpha(s)) + P(s,\alpha(s))b(s,\alpha(s)) + q(s,\alpha(s))\Big\} ds \\ \qquad + \zeta(s)dW(s) + \displaystyle\sum_{k=1}^{D} \xi_k(s) d\widetilde{N}_k(s), \quad s \in [0,T], \\ \eta(T) = g(i), \end{cases} \tag{18}$$

Closed-loop Solvability

Theorem 3 (Continuous)

satisfies

$$\begin{cases} \hat{\rho}(s,i) \in \mathcal{R}(\hat{R}(s,i)), & \text{a.e. a.s.} \\ \hat{R}(s,i)^{\dagger}\hat{\rho}(s,i) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m), \end{cases} \tag{19}$$

with

$$\hat{\rho}(s,i) = B(s,i)^{\top}\eta(s) + D(s,i)^{\top}\zeta(s) + D(s,i)^{\top}P(s,i)\sigma(s,i) + \rho(s,i). \tag{20}$$

In this case, Problem (M-SLQ) *is closed-loop solvable on any* $[t,T]$, *and the closed-loop optimal strategy* $(\Theta^*(\cdot), v^*(\cdot))$ *admits the following representation:*

$$\begin{cases} \Theta^*(s) = -\hat{R}(s,\alpha(s))^{\dagger}\hat{S}(s,\alpha(s)) + \big[I - \hat{R}(s,\alpha(s))^{\dagger}\hat{R}(s,\alpha(s))\big]\Pi(s), \\ v^*(s) = -\hat{R}(s,\alpha(s))^{\dagger}\hat{\rho}(s,\alpha(s)) + \big[I - \hat{R}(s,\alpha(s))^{\dagger}\hat{R}(s,\alpha(s))\big]\nu(s), \end{cases} \tag{21}$$

for some $\Pi(\cdot) \in L^2(t,T;\mathbb{R}^{m\times n})$ *and* $\nu(\cdot) \in L^2_{\mathbb{F}}(t,T;\mathbb{R}^m)$, *and the value function is given by*

Closed-loop Solvability

Theorem 3 (Continuous)

The value function is given by

$$\begin{aligned} V(t,x,i) =& \mathbb{E}\Big\{ \langle P(t,i)x,x\rangle + 2\langle \eta(t),x\rangle \\ &+ \int_t^T \Big[\hat{P}(s,\alpha(s)) - \langle \hat{R}(s,\alpha(s))^{\dagger}\hat{\rho}(s,\alpha(s)),\hat{\rho}(s,\alpha(s))\rangle\Big] ds \Big\}, \end{aligned} \tag{22}$$

where

$$\hat{P}(s,i) := \langle P(s,i)\sigma(s,i) + 2\zeta(s),\sigma(s,i)\rangle + 2\langle \eta(s),b(s,i)\rangle.$$

Closed-loop Solvability

Now we present the equivalence between the uniform convexity of the cost functional and the strongly regular solution of the Riccati equation.

Theorem 4

Let (H1)–(H2) *hold. Then the following statements are equivalent:*

(i) *The map* $u(\cdot) \mapsto J^0(t, 0; u(\cdot))$ *is uniformly convex, i.e., there exists a* $\lambda > 0$ *such that*

$$J^0(t, 0, i; u(\cdot)) \geqslant \lambda \mathbb{E} \int_t^T |u(s)|^2 ds, \qquad \forall u(\cdot) \in \mathcal{U}[t, T].$$

(ii) *The Riccati equation* (15) *admits a strongly regular solution* $P(\cdot, \cdot) \in C([0, T] \times \mathcal{S}; \mathbb{S}^n)$.

The First Example

Since $Q(\cdot,i)=0, R(\cdot,i)=0, D(\cdot,i)=0$ for every $i\in\mathcal{S}$ and $0^\dagger=0$, we have the following Riccati equations

$$\begin{cases} \dot{P}(s,i)+\sum_{k=1}^{D}\lambda_{ik}(s)P(s,k)=0, \quad \text{a.e. } s\in[t,T], \\ P(T,i)=1. \end{cases} \tag{23}$$

Clearly, the unique solution of the above ordinary differential equation system is

$$P(s,i)\equiv 1, \text{ for } (s,i)\in[t,T]\times\mathcal{S}.$$

Thus, for any $(s,i)\in[t,T]\times\mathcal{S}$, we have

$$\begin{aligned} \mathcal{R}\big(\hat{S}(s,i)\big)&=\mathcal{R}(e^i)=\mathcal{R}(1)=\mathbb{R}, \\ \mathcal{R}\big(\hat{R}(s,i)\big)&=\mathcal{R}(0)=\{0\}, \end{aligned} \quad \Longrightarrow \quad \mathcal{R}\big(\hat{S}(s,i)\big)\nsubseteq\mathcal{R}\big(\hat{R}(s,i)\big).$$

Now from Theorem 3, we can deduce that this problem is not closed-loop solvable.

Weak Closed-loop Solvability

Weak Closed-loop Solvability

Definition 3 (Weak closed-loop)

Let $\Theta : [t, T) \to \mathbb{R}^{m\times n}$ be a locally square-integrable deterministic function and $v : [t, T) \times \Omega \to \mathbb{R}^m$ be a locally square-integrable $\mathbb{F}$-progressively measurable process, i.e., $\Theta(\cdot)$ and $v(\cdot)$ are such that for any $T' \in [t, T)$,

$$\int_t^{T'} |\Theta(s)|^2 ds < \infty, \qquad \mathbb{E}\int_t^{T'} |v(s)|^2 ds < \infty.$$

We call $(\Theta(\cdot), v(\cdot))$ a *weak closed-loop strategy* on $[t, T)$ if for any initial state $(x, i) \in \mathbb{R}^n \times \mathcal{S}$, the outcome $u(\cdot) \equiv \Theta(\cdot)X(\cdot) + v(\cdot)$ belongs to $\mathcal{U}[t, T] \equiv L^2_{\mathbb{F}}(t, T; \mathbb{R}^m)$, where $X(\cdot)$ is the solution to the *weak closed-loop system*:

$$\begin{cases} dX(s) = \Big\{\big[A(s,\alpha(s))+B(s,\alpha(s))\Theta(s)\big]X(s)+B(s,\alpha(s))v(s)+b(s,\alpha(s))\Big\}ds \\ \qquad\qquad + \Big\{\big[C(s,\alpha(s)) + D(s,\alpha(s))\Theta(s)\big]X(s) \\ \qquad\qquad\qquad +D(s,\alpha(s))v(s) + \sigma(s,\alpha(s))\Big\}dW(s), \qquad s \in [t, T], \\ X(t) = x. \end{cases} \tag{24}$$

The set of all weak closed-loop strategies is denoted by $\mathscr{C}_w[t, T]$.

Weak Closed-loop Solvability

Let $\Theta_\varepsilon : [0,T] \to \mathbb{R}^{m\times n}$ and $v_\varepsilon : [0,T]\times\Omega \to \mathbb{R}^m$ be

$$\Theta_\varepsilon(s) = -[\hat{R}_\varepsilon(s,\alpha(s)) + \varepsilon I_m]^{-1}\hat{S}_\varepsilon(s,\alpha(s)), \tag{25}$$

$$v_\varepsilon(s) = -[\hat{R}_\varepsilon(s,\alpha(s)) + \varepsilon I_m]^{-1}\hat{\rho}_\varepsilon(s,\alpha(s)), \tag{26}$$

with

$$\hat{\rho}_\varepsilon(s,i) = B(s,i)^\top \eta_\varepsilon(s) + D(s,i)^\top \zeta_\varepsilon(s) + D(s,i)^\top P_\varepsilon(s,i)\sigma(s,i) + \rho(s,i).$$

We prove that the family $\{\Theta_\varepsilon(\cdot)\}_{\varepsilon>0}$ and $\{v_\varepsilon(\cdot)\}_{\varepsilon>0}$ defined by (25) (26) are locally convergent in $[0,T)$.

Weak Closed-loop Solvability

Proposition 2

Let (H1) and (H2) hold. Suppose that Problem (M-SLQ)^0 is open-loop solvable. Then the family $\{\Theta_\varepsilon(\cdot)\}_{\varepsilon>0}$ defined by (25) converges in $L^2(0,T';\mathbb{R}^{m\times n})$ for any $0<T'<T$; that is, there exists a locally square-integrable deterministic function $\Theta^*(\cdot):[0,T)\to\mathbb{R}^{m\times n}$ such that

$$\lim_{\varepsilon\to 0}\int_0^{T'}|\Theta_\varepsilon(s)-\Theta^*(s)|^2ds=0,\qquad \forall\ 0<T'<T.$$

Proposition 3

Let (H1) and (H2) hold. Suppose that Problem (M-SLQ) is open-loop solvable. Then the family $\{v_\varepsilon(\cdot)\}_{\varepsilon>0}$ defined by (26) converges in $L^2(0,T';\mathbb{R}^{m})$ for any $0<T'<T$; that is, there exists a locally square-integrable deterministic function $v^*(\cdot):[0,T)\to\mathbb{R}^{m}$ such that

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_0^{T'}|v_\varepsilon(s)-v^*(s)|^2ds=0,\qquad \forall\ 0<T'<T.$$

Weak Closed-loop Solvability

Theorem 5

Let (H1) and (H2) hold. If Problem (M-SLQ) is open-loop solvable, then the limit pair $(\Theta^*(\cdot), v^*(\cdot))$ *obtained in Propositions 2 and 3 is a weak closed-loop optimal strategy of Problem (M-SLQ) on any* $[t, T)$*. Consequently, the open-loop and weak closed-loop solvability of Problem (M-SLQ) are equivalent.*

The Second Example

Let $T = 1$ and $D = 2$, that is, the state space of $\alpha(\cdot)$ is $\mathcal{S} = \{1, 2\}$. For the generator $\lambda(s) \triangleq [\lambda_{ij}(s)]_{i,j=1,2}$, note that $\sum_{j=1}^{2} \lambda_{ij}(s) = 0$ for $i \in \mathcal{S}$, then

$$\lambda(s) = \begin{pmatrix} \lambda_{11}(s) & \lambda_{12}(s) \\ \lambda_{21}(s) & \lambda_{22}(s) \end{pmatrix} = \begin{pmatrix} \lambda_{11}(s) & -\lambda_{11}(s) \\ -\lambda_{22}(s) & \lambda_{22}(s) \end{pmatrix}, \quad s \in [0, 1].$$

Consider the following Problem (M-SLQ) with one-dimensional state equation

$$\begin{cases} dX(s) = \Big[-\alpha(s)X(s) + u(s) + b(s, \alpha(s)) \Big] ds + \sqrt{2\alpha(s)} X(s) dW(s), & s \in [t, 1], \\ X(t) = x, \quad \alpha(t) = i, \end{cases} \tag{27}$$

and the cost functional

$$J(t, x, i; u(\cdot)) = \mathbb{E}\big[X(1)^2\big],$$

where the nonhomogeneous term $b(\cdot, \cdot)$ is given by

$$b(s, \alpha(s)) = \begin{cases} \dfrac{1}{\sqrt{1-s}} \cdot \exp\left\{ \displaystyle\int_0^s \sqrt{2\alpha(r)} dW(r) - 2 \int_0^s \alpha(r) dr \right\}, & s \in [0, 1); \\ 0, \quad s = 1. \end{cases}$$

The Second Example

It is easy to see that $b(\cdot, i) \in L^2_{\mathbb{F}}(\Omega; L^1(0,1;\mathbb{R}))$ for each $i \in \mathcal{S}$. In fact,

$$\mathbb{E}\left(\int_0^1 |b(s,\alpha(s))|ds\right)^2 \leq 4\mathbb{E}\left(\sup_{0\leqslant s\leqslant 1} \exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - \int_0^s \alpha(r)dr\right\}\right)^2.$$

Since the term $\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - \int_0^s \alpha(r)dr\right\}$ is a square-integrable martingale, note that $\alpha(\cdot)$ belongs to $\mathcal{S} = \{1, 2\}$, it follows from Doob's maximal inequality that

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant s\leqslant 1} \exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - \int_0^s \alpha(r)dr\right\}\right)^2 \\
&\leq 4\mathbb{E}\exp\left\{2\int_0^1 \sqrt{2\alpha(r)}dW(r) - 2\int_0^1 \alpha(r)dr\right\} \\
&\leq 4e^4.
\end{aligned}$$

Hence,

$$\mathbb{E}\left(\int_0^1 |b(s,\alpha(s))|ds\right)^2 \leq 16e^4,$$

which implies that $b(\cdot, i) \in L^2_{\mathbb{F}}(\Omega; L^1(0,1;\mathbb{R}))$ for each $i \in \mathcal{S}$.

The Second Example: Not closed-loop solvable

We first claim that this (M-SLQ) problem is not closed-loop solvable on any $[t,1]$. Indeed, the generalized Riccati equation associate with this problem reads

$$\begin{cases} \dot{P}(s,1) + \lambda_{11}(s)P(s,1) - \lambda_{11}(s)P(s,2) = 0, \quad \text{a.e. } s \in [t,1], \\ P(1,1) = 1, \end{cases} \qquad \text{for } i = 1,$$

and

$$\begin{cases} \dot{P}(s,2) - \lambda_{22}(s)P(s,1) + \lambda_{22}(s)P(s,2) = 0, \quad \text{a.e. } s \in [t,1], \\ P(1,2) = 1, \end{cases} \qquad \text{for } i = 2,$$

whose solutions are $P(s,1) = P(s,2) = 1$, or $P(s,i) \equiv 1$, for $(s,i) \in [0,1] \times \mathcal{S}$. Then for any $s \in [t,1]$ and $i \in \mathcal{S}$, we have

$$\begin{aligned} &\mathcal{R}\big(\hat{S}(s,i)\big) = \mathcal{R}(1) = \mathbb{R}, \\ &\mathcal{R}\big(\hat{R}(s,i)\big) = \mathcal{R}(0) = \{0\}, \end{aligned} \qquad \Longrightarrow \qquad \mathcal{R}\big(\hat{S}(s,i)\big) \nsubseteq \mathcal{R}\big(\hat{R}(s,i)\big).$$

where

$$\begin{aligned} &\hat{S}(s,i) \triangleq B(s,i)^\top P(s,i) + D(s,i)^\top P(s,i)C(s,i) + S(s,i), \\ &\hat{R}(s,i) \triangleq R(s,i) + D(s,i)^\top P(s,i)D(s,i). \end{aligned} \qquad (28)$$

Therefore, the range inclusion condition is not satisfied, which deduce that our claim holds.

The Second Example

Without loss of generality, we consider only the open-loop solvability at $t = 0$. To this end, let $\varepsilon > 0$ be arbitrary and consider Riccati equations (15), which, in our example, read:

$$\begin{cases} \dot{P}_\varepsilon(s,1) - \dfrac{1}{\varepsilon} P_\varepsilon(s,1)^2 + \lambda_{11}(s) P_\varepsilon(s,1) - \lambda_{11}(s) P_\varepsilon(s,2) = 0, & \text{a.e. } s \in [t,1], \\ P_\varepsilon(1,1) = 1, \end{cases}$$

and

$$\begin{cases} \dot{P}_\varepsilon(s,2) - \dfrac{1}{\varepsilon} P_\varepsilon(s,2)^2 - \lambda_{22}(s) P_\varepsilon(s,1) + \lambda_{22}(s) P_\varepsilon(s,2) = 0, & \text{a.e. } s \in [t,1], \\ P_\varepsilon(1,2) = 1. \end{cases}$$

Solving the above equations yields

$$P_\varepsilon(s,1) = P_\varepsilon(s,2) = \frac{\varepsilon}{\varepsilon + 1 - s}, \qquad s \in [0,1].$$

Or,

$$P_\varepsilon(s,i) = \frac{\varepsilon}{\varepsilon + 1 - s}, \qquad (s,i) \in [0,1] \times \mathcal{S}.$$

Note that the state space of $\alpha(s)$ is $\mathcal{S} = \{1,2\}$, we let

$$\begin{aligned} \Theta_\varepsilon(s) &\triangleq -[\hat{R}_\varepsilon(s,\alpha(s)) + \varepsilon I_m]^{-1} \hat{S}_\varepsilon(s,\alpha(s)) \\ &= -\frac{P_\varepsilon(s,\alpha(s))}{\varepsilon} = -\frac{1}{\varepsilon + 1 - s}, \qquad s \in [0,1]. \end{aligned} \tag{29}$$

The Second Example

Then the corresponding BSDE (18) reads

$$\begin{cases} d\eta_\varepsilon(s) = -\Big\{\big[\Theta_\varepsilon(s)-\alpha(s)\big]\eta_\varepsilon(s)+\sqrt{2\alpha(s)}\zeta_\varepsilon(s)+P_\varepsilon(s,\alpha(s))b(s)\Big\}ds \\ \qquad\qquad +\zeta_\varepsilon(s)dW(s)+\sum_{k=1}^{2}\xi_k^\varepsilon(s)d\widetilde{N}_k(s), \qquad s\in[0,1], \\ \eta_\varepsilon(1)=0. \end{cases}$$

Let $f(s)=\frac{1}{\sqrt{1-s}}$. Using the variation of constants formula for BSDEs, and noting that $W(\cdot)$ and $\widetilde{N}_k(\cdot)$ are $(\mathbb{F},\mathbb{P})$-martingales, we obtain

$$\eta_\varepsilon(s)=\frac{\varepsilon}{\varepsilon+1-s}\exp\left\{\int_0^s\sqrt{2\alpha(r)}dW(r)-2\int_0^s\alpha(r)dr\right\}\int_s^1 f(r)dr, \quad s\in[0,1].$$

The Second Example

Now let

$$v_\varepsilon(s) \triangleq -[\hat{R}_\varepsilon(s,\alpha(s)) + \varepsilon I_m]^{-1}\hat{\rho}_\varepsilon(s,\alpha(s)) = -\frac{\eta_\varepsilon(s)}{\varepsilon}$$

$$= -\frac{1}{\varepsilon+1-s}\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\}\int_s^1 f(r)dr, \quad s \in [0,1]. \tag{30}$$

Then the corresponding closed-loop system can be written as

$$\begin{cases} dX_\varepsilon(s) = \Big\{\big[\Theta_\varepsilon(s) - \alpha(s)\big]X_\varepsilon(s) + v_\varepsilon(s) + b(s,\alpha(s))\Big\}ds + \sqrt{2\alpha(s)}X_\varepsilon(s)dW(s), \\ X_\varepsilon(0) = x. \end{cases}$$

By the variation of constants formula for SDEs, we get

$$\begin{aligned} X_\varepsilon(s) &= (\varepsilon+1-s)\cdot\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\} \\ &\quad \cdot \int_0^s \left[\frac{1}{\varepsilon+1-r}\exp\left\{-\int_0^r \sqrt{2\alpha(\bar r)}dW(\bar r) + 2\int_0^r \alpha(\bar r)d\bar r\right\}\big(v_\varepsilon(r) + b(r,\alpha(r))\big)\right]dr \\ &\quad + x\cdot\frac{\varepsilon+1-s}{\varepsilon+1}\cdot\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\}, \quad s \in [0,1]. \end{aligned}$$

The Second Example

The family $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ is defined by

$$
\begin{aligned}
u_\varepsilon(s) &\triangleq \Theta_\varepsilon(s)X_\varepsilon(s) + v_\varepsilon(s) \\
&= -\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\} \\
&\quad \cdot \int_0^s \left[\frac{1}{\varepsilon+1-r}\exp\left\{-\int_0^r \sqrt{2\alpha(\bar r)}dW(\bar r) + 2\int_0^r \alpha(\bar r)d\bar r\right\}\big(v_\varepsilon(r) + b(r,\alpha(r))\big)\right]dr \\
&\quad - \frac{x}{\varepsilon+1}\cdot\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\} + v_\varepsilon(s), \qquad s\in[0,1],
\end{aligned} \tag{31}
$$

is bounded in $L^2_{\mathbb{F}}(0,1;\mathbb{R})$.

Simplifying (31) by Fubini's theorem yields

$$
\begin{aligned}
u_\varepsilon(s) &= -\left(\frac{x}{\varepsilon+1} + \frac{1}{\varepsilon+1}\int_0^1 f(r)dr\right)\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\} \\
&= -\frac{x+2}{\varepsilon+1}\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\}.
\end{aligned} \tag{32}
$$

The Second Example: Open-loop solvable

A short calculation gives

$$\mathbb{E}\int_0^1 |u_\varepsilon(s)|^2 ds = \left(\frac{x+2}{\varepsilon+1}\right)^2 \leq (x+2)^2, \qquad \forall \varepsilon > 0.$$

Therefore $\{u_\varepsilon(\cdot)\}_{\varepsilon>0}$ is bounded in $L^2_{\mathbb{F}}(0,1;\mathbb{R})$.

Now, let $\varepsilon \to 0$ in (32), we get an open-loop optimal control:

$$u^*(s) = -(x+2)\exp\left\{\int_0^s \sqrt{2\alpha(r)}dW(r) - 2\int_0^s \alpha(r)dr\right\}, \qquad s \in [0,1].$$

From the above discussion, similar to the state process $X(\cdot)$ of (27), the open-loop optimal control $u^*(\cdot)$ also depends on the regime switching term $\alpha(\cdot)$. That is to say, as the value of the switching $\alpha(\cdot)$ varies, the open-loop optimal control $u^*(\cdot)$ will be changed too.

Finally, we let $\varepsilon \to 0$ in (29) and (30) to get a weak closed-loop optimal strategy $(\Theta^*(\cdot), v^*(\cdot))$:

$$\Theta^*(s) = \lim_{\varepsilon \to 0} \Theta_\varepsilon(s) = -\frac{1}{1-s}, \qquad s \in [0,1),$$
$$v^*(s) = \lim_{\varepsilon \to 0} v_\varepsilon(s) = -\frac{2}{\sqrt{1-s}} \exp\left\{ \int_0^s \sqrt{2\alpha(r)} dW(r) - 2\int_0^s \alpha(r) dr \right\}, \quad s \in [0,1).$$

We put out that neither $\Theta^*(\cdot)$ and $v^*(\cdot)$ is square-integrable on $[0,1)$. Indeed, one has

$$\int_0^1 |\Theta^*(s)|^2 ds = \int_0^1 \frac{1}{(1-s)^2} ds = \infty,$$
$$\mathbb{E}\int_0^1 |v^*(s)|^2 ds = \mathbb{E}\int_0^1 \frac{4}{1-s} \exp\left\{ 2\int_0^s \sqrt{2\alpha(r)} dW(r) - 4\int_0^s \alpha(r) dr \right\} ds$$
$$= \mathbb{E}\int_0^1 \frac{4}{1-s} ds = \infty.$$

Future Works

- Random coefficients
- Infinite time horizon
- Discrete-time linear-quadratic control

Thank You