# Testing Whether Volatility Can Be Written

# As a Function of the Asset Price

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# 1. Stochastic vs. Local Volatility

- Discrete-time ARCH, then GARCH: time-varying volatility is an important empirical feature in economic and financial data (Engle (1982))

- In continuous-time, models with stochastic volatility became prevalent
  - Require the use of an additional state variable $Y$, often latent, to drive the volatility of the variable of interest, $X$
  - $X$ on its own is typically no longer Markovian
  - Costly implications: higher dimension, plus $Y$ requires filtering or learning

- Local volatility models
  - Make the volatility of $X$ a function of $X$ itself
  - Retains the time variation of the volatility of $X$, and its Markov character, but without the additional stochasticity of $Y$
  - Such models, although more restrictive, remain very popular in financial applications

- This paper
  - Tests using high frequency observations whether the volatility of $X$ can be written as a function of $X$ only
  - Or whether some additional variable $Y$ is needed
  - Can

$$
\begin{aligned}
dX_t &= b_t dt + Y_t dW_t \\
dY_t &= m_t dt + v_t dW'_t
\end{aligned}
$$

be reduced to

$$dX_t = b_t dt + a(X_t) dW_t \text{ ?}$$

- Related question: whether $X$ can be written on its own as a Markov process
  - If $X$ is continuous, then $X$ will be Markovian if its drift and volatility functions are functions of $X$ (and time) only
  - Time-series tests examine the necessary (but not sufficient) condition that the conditional density of the process satisfies the Chapman-Kolmogorov condition (see Aït-Sahalia (1996) and Aït-Sahalia et al. (2010)); whether conditional independence between consecutive durations holds (see de Matos and Fernandes (2007)); and whether the conditional characteristic function satisfies the requirement imposed by the Markov property (see Chen and Hong (2012)).

- This paper tests directly whether the volatility of $X$ is a function of $X$ only
  - At high frequency over a finite time span, the drift is not identified
  - So only properties regarding the volatility of $X$ can be tested anyways
  - If $X$ can jump, additional separate restrictions on the structure of the jump component of $X$ are imposed by the Markov property.

# 2. The Model

$$X_t = X_0 + \int_0^t b_s \, ds + \int_0^t c_s^{1/2} \, dW_s + J_t,$$
$$J_t = \sum_{n \geq 1} \Upsilon_m 1_{\{T_m \leq t\}}.$$

- $W$ is a Brownian motion, $T_m$ jump times, $\Upsilon_m$ jump sizes
- Can be extended to Itô semimartingale with degree of activity strictly less than 1
- The local volatility case ($\mathrm{H}_0$):

  $$c_t = a(X_t)$$

  for some positive $\mathcal{C}^1$ function $a$ on $\mathbb{R}$
- Markov adds: $b_t = \mu(X_t)$ and a special structure on the jumps

- Observations on a finite $[0, T]$, sampling interval $\Delta_n \to 0$ as $n \to \infty$

- With $a(x) > 0$ is assumed, and when the jump process $J_t$ vanishes, the law of $X$ under ($\mathsf{H}_0$) in restriction to any finite time interval $[0, t]$ is equivalent to the law of the Markov process

$$\bar{X}_t = X_0 + \int_0^t a(\bar{X}_s)^{1/2} \, dW_s.$$

  Then, since $X$ is observed on a finite $[0, T]$ only, it is impossible to discriminate between $X$ and $\bar{X}$, hence ($\mathsf{H}_0$) is equivalent for the econometrician to assuming the Markov property, plus the smoothness of $a$

- Microstructure noise: white noise plus rounding

# 3. Testing the Null Hypothesis

## 3.1. Without Noise

- We wish to determine whether the spot variance $c_t$ can be written in the form $c_t = a(X_t)$ for some smooth function $a(\cdot)$
- Estimators localized in $x$
  - Choose a nonnegative kernel function $f$ on $\mathbb{R}$ with support in $[-1, 1]$ and with Lebesgue integral equal to 1, bandwidths $h_n$ and thresholds $v_n > 0$ satisfying, for some $\varepsilon > 0$:

$$\frac{h_n^3}{\Delta_n} \to 0, \qquad \frac{\Delta_n}{h_n^2} \to 0, \qquad v_n \to 0, \qquad \frac{\Delta_n^{1/2-\varepsilon}}{v_n} \to 0,$$

– Approximation of the Dirac mass at 0

$$f_n(x) = \frac{1}{h_n} f\left(\frac{x}{h_n}\right).$$

- Discrete increments (e.g., log-returns)

$$\Delta_i^n X = X_{i\Delta_n} - X_{(i-1)\Delta_n}$$

- Define the statistics for integers $p$, typically in the set $\mathcal{P} = \{0, 2, 4\}$

$$U(x,p)_t^n = \Delta_n^{1-p/2} \frac{1}{m_p} \sum_{i=1}^{[t/\Delta_n]} f_n(X_{(i-1)\Delta_n} - x)\, |\Delta_i^n X|^p \, 1_{\{|\Delta_i^n X| \leq v_n\}}.$$

where $m_p =$ the $p$th absolute moment of $\mathcal{N}(0,1)$

- Truncation at level $v_n$ to get rid of the jumps, if any
  - If we know that $X$ is continuous, omit the indicator function above
  - The truncation can be kept even if we know that $X$ is continuous, in which case it is asymptotically irrelevant

- Local time

  - The limiting behavior of $U$ involves the local times $L^x$ of $X$ at $x$

$$L_t^x = |X_t - x| - |X_0 - x| - \int_0^t \mathrm{sign}(X_s - x)\, dX'_s \\ - \sum_{n \geq 1} \Big( |X_{T_n} - x| - |X_{T_n -} - x| \Big)\, 1_{\{T_n \leq t\}}.$$

  - $X' = X - J$ is the continuous part of $X$
  - $L_t^x > 0$ a.s. if the process $X$ has visited the point $x$ within $[0, T]$
  - This is the semimartingale version of the local time, which differs from the Markov local time by the factor $a(x)$
  - Has the advantage of being defined even when $(\mathrm{H}_0)$ fails

- Theorem:

$$U(x,p)_t^n \overset{\text{u.c.p.}}{\Longrightarrow} \int_0^t c_s^{p/2-1}\, dL_s^x.$$

- With $p = 2$ : $U(x,2)_t^n \overset{\text{u.c.p.}}{\Longrightarrow} L_t^x$

- Generic limitations
  - As in any other high frequency testing problem:
    - A single path of $X$ over $[0, T]$ is observed
    - Nothing about what happens after $t = T$ can be tested
  - Furthermore:
    - Nothing can be said about what happens to the volatility when the process is outside the (random) set of points visited by $X$ on $[0, T]$

## 3.2. The Idea Behind the Test

- Testing first at a single visited point $\mathcal{X} = \{x\}$
- Under the null, we have $c_s = c_{s'}$ for all $s, s' \in [0, t]$ with $X_s = X_{s'} = x$ and therefore

$$\int_0^t \int_0^t (c_s - c_{s'})^2 \, (c_s)^{-1} \, (c_{s'})^{-1} \, dL_s^x \, dL_{s'}^x = 0$$

- The alternative hypothesis ($\mathrm{H}_1$)
  - There is at least one pair $s, s' \in [0, T]$ such that $X_s = X_{s'}$ and $c_{s'} \neq c_s$.
  - This set is random, as often in high frequency
- The same quantity is strictly positive under the alternative
- This does not lend itself directly to a test since we have no good way of estimating $\int_0^t \int_0^t (c_s - c_{s'})^2 \, (c_s)^{-1} \, (c_{s'})^{-1} \, dL_s^x \, dL_{s'}^x$

- However, under the null we also have

$$\left(\int_0^t c_s \, dL_s^x\right) \left(\int_0^t c_s^{-1} \, dL_s^x\right) - (L_t^x)^2 = 0$$

- Using the theorem

$$\begin{aligned} & \bar{U}(x)_t^n := U(x, 4)_t^n \, U(x, 0)_t^n - (U(x, 2)_t^n)^2 \\ \xrightarrow{\mathbb{P}} \; & \bar{U}(x)_t := \left(\int_0^t c_s \, dL_s^x\right) \left(\int_0^t c_s^{-1} \, dL_s^x\right) - (L_t^x)^2 \end{aligned}$$

- So

$$\bar{U}(x)_t^n \xrightarrow{\mathbb{P}} \begin{cases} \bar{U}(x)_t = 0 & \text{under } (\mathsf{H}_0) \\ \bar{U}(x)_t > 0 & \text{under } (\mathsf{H}_1) \end{cases}$$

- Testing at an arbitrary finite set of visited points $\mathcal{X}$
  - Sum of $\bar{U}(x)_t^n$ over $x \in \mathcal{X}$, possibly weighted
  - The weight assigned to any given $x$ should reflect the time spent by $X$ at or around $x$, which is roughly proportional to the value of $L_t^x$, which itself is estimated by $U(x, 2)_t^n$
  - So we will consider a statistic of the form

$$\mathcal{U}(\mathcal{X}, r)_t^n = \sum_{x \in \mathcal{X}} \bar{U}(x)_t^n \left(U(x, 2)_t^n\right)^r$$

    for some $r$
  - We have

$$\mathcal{U}(\mathcal{X}, r)_t^n \xrightarrow{\mathbb{P}} \begin{cases} \mathcal{U} = 0 & \text{under } (\mathrm{H}_0) \\ \mathcal{U} > 0 & \text{under } (\mathrm{H}_1) \end{cases}$$

# 3.3. With Noise

- Plug-in $\tilde{X}_{i\Delta_n}$ instead of $X_{i\Delta_n}$ into the statistic, to form $U^{\text{noisy}}(x,p)_t^n$
  - However, even in the simplest white noise case we have

$$\frac{\Delta_n^{p/2}}{v_n^{p+1}} U^{\text{noisy}}(x,p)_t^n \stackrel{\text{u.c.p.}}{\Longrightarrow} \begin{cases} \frac{2}{p m_p} \int_0^t \phi(X_s - x)\, ds & \text{if } p > 0 \\ 2 \int_0^t \phi(X_s - x)\, ds & \text{if } p = 0, \end{cases}$$

  - The limit here has no connection with the volatility of $X$
  - Hence these statistics cannot be used for testing ($\text{H}_0$)

- Instead, we need to resort to a preliminary de-noising procedure
  - We use pre-averaging and plug-in the pre-averaged values of $\tilde{X}$
  - Then the same convergence holds

# 3.4. The Central Limit Theorem

- Theorem

$$\Big(\sqrt{h_n/k_n\Delta_n}\,\bar{U}(x)_t^n\Big)_{x\in\mathcal{X}} \stackrel{\mathcal{L}\text{-s}}{\to} (\bar{Z}(x)_t)_{x\in\mathcal{X}},$$

  where $\bar{Z}$ is $\mathcal{F}$–conditionally centered Gaussian with convariance

$$\tilde{\mathbb{E}}(\bar{Z}(x)_t\,\bar{Z}(y)_t \mid \mathcal{F}) = \frac{8\,\beta}{3}\,1_{\{x=y\}}a(x)\,(L_t^x)^3.$$

  and $\beta = \int_{-1}^{+1} f(x)^2 dx$.

- The variables $\bar{Z}(x)_t$ for distinct values of $x$ are $\mathcal{F}$–conditionally independent.

- Optimal Rate
  - The (not fully achievable, but almost) optimal rates are $1/\Delta_n^{1/3}$ without noise and $1/\Delta_n^{1/6}$ with noise, respectively.
  - In the no-noise case, this is the typical non-parametric rate
  - And, as in other situations, the presence of noise result in replacing the rate without noise by its square-root

# 3.5. Construction of the Test

- From the theorem, under (H$_0$):

$$\left(\frac{h_n}{k_n \Delta_n}\right)^{1/2} \mathcal{U}(\mathcal{X}, r)^n \overset{\mathcal{L}\text{-s}}{\Longrightarrow} \mathcal{Z}(\mathcal{X}, r) := (L^x)^r \sum_{x \in \mathcal{X}} \bar{Z}(x)$$

- $\mathcal{Z}(\mathcal{X}, r)_t$ is $\mathcal{F}$–conditionally centered Gaussian with conditional variance:

$$\Sigma(\mathcal{X}, r)_t = \frac{8\,\beta}{3} \sum_{x \in \mathcal{X}} a(x)(L^x_t)^{3+2r}$$

- Under (H$_0$) an estimator of $\Sigma(\mathcal{X}, r)_t$ is

$$\Sigma(\mathcal{X}, r)^n_t = \frac{8\,\beta}{3} \sum_{x \in \mathcal{X}} U(x, 0)^n_t \; (U(x, 4)^n_t)^2 \; (U(x, 2)^n_t)^{2r}$$

- Standardized test statistic

$$\mathcal{T}_t^n = \left(\frac{h_n}{k_n \Delta_n}\right)^{1/2} \frac{\mathcal{U}(\mathcal{X}, r)_t^n}{\left(\Sigma(\mathcal{X}, r)_t^n\right)^{1/2}}$$

- Which is $\mathcal{N}(0,1)$ under the null
- We use the set $\mathcal{P} = \{0, 2, 4\}$ of values of $p$, other choices are possible.

## 3.6. Tuning parameters

- The localization kernel $f$
  - Rather immaterial in practice, we suggest

$$f(x) = \frac{15}{16}\left(x^2 - 1\right)^2$$

  - In which case $\beta = 5/7$.

- The bandwidth $h_n$, plus the pre-averaging window $k_n$ in the noisy case
  - The rate of convergence for $\mathcal{U}(\mathcal{X}, r)_t^n$ gets faster when we increase $h_n$ and/or when we decrease $k_n$
  - A good choice: $h_n \asymp \Delta_n^{1/3} / \log(1/\Delta_n)$ in the no-noise case, leading to the rate $(\log(1/\Delta_n))^{1/2} / \Delta_n^{1/3}$
  - In the noisy case we cannot use $k_n \asymp \Delta_n^{1/2}$, which is the optimal choice in pre-averaging for estimating the volatility for example
  - A good choice: take $k_n \asymp \Delta_n^{1/2} \log(1/\Delta_n)$ and $h_n \asymp \Delta_n^{1/6}$, which leads to the rate $(\log(1/\Delta_n))^{1/2} / \Delta_n^{1/6}$

- The jump truncation threshold $v_n$
  - Choose $v_n \approx \gamma\Delta_n^{1/2}$ if there is no noise
  - And $v_n \approx \gamma\,(k_n\Delta_n)^{1/2}$ in the presence of noise, where $\gamma$ is 3 to 5 times a rough average of the volatility.

- The weighting power $r$
  - The choice of $r$ is asymptotically irrelevant
  - Increasing $r$ puts more weight on values visited often
  - Small sample discrepancies affect sites not visited often, so to avoid this problem $r$ should be taken relatively large
  - Increasing $r$ typically decreases the power of the test
  - Under the alternative, $\bar{U}(x)_t^n$ roughly varies as $(L_t^x)^2$, so "equal weights" means taking $r = -2$, which is good for the power of the test, but bad for the null
  - A good compromise is perhaps simply to take $r = 0$: then $\mathcal{U}_t^n$ would be a proxy for

$$\frac{1}{h_n^2} \int_0^t \int_0^t (c_s - c_{s'})^2 \, 1_{\{|X_s - X_{s'}| \leq h_n\}} \, ds \, ds'$$

    which is exactly the type of quantity which we want to test whether it vanishes or not

- The set where we test the property
  - When the cardinal of $\mathcal{X}$ is small, the procedure tests the property only at a limited number of values of $x$
  - This lowers the power of the test
  - So take $\mathcal{X}$ relatively large, but the minimal distance between two points of $\mathcal{X}$ should be bigger than $2h_n$
  - In practice, use a grid

# 4. Monte Carlo Simulations

- Null model

$$dX_t = \kappa(\alpha - X_t)\,dt + \sigma X_t^{1/2} dW_t$$

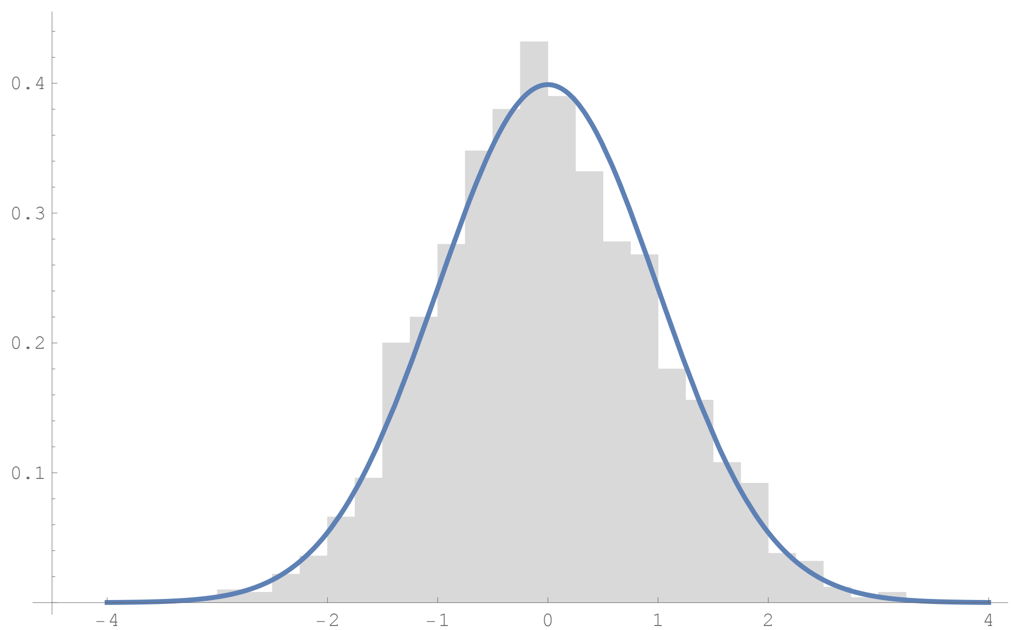
- Alternative

$$\begin{aligned} dX_t &= \kappa(\alpha - X_t)\,dt + \sigma(X_t + Y_t)^{1/2}\,dW_t \\ dY_t &= \beta(\omega - Y_t)\,dt + \eta Y_t^{1/2} dW_t' \\ \mathbb{E}\left[dW_t dW_t'\right] &= \rho dt \end{aligned}$$

- Paths simulated at 5 sec increments, data subsampled 1 to 5 mn, $T = 1$ year

- $\alpha = 10$, parameters give a stationary distribution with 99% of the mass on $[0, 30]$

- $X =$ uniform grid on $[0, 30]$ in increments of 1



# 5. Empirical Results: Interest Rates vs. Stock Prices

- Data
  - ES (S&P500 eMini Futures) 1998-2017 1mn frequency
  - ED (Eurodollar CME Futures) 1982-2017 5mn frequency, converted to interest rate equivalent
- Results
  - Clear rejection of the null for stocks
  - Not so for interest rates
  - Variation over time

# 6. Conclusions

- Testing whether

$$dX_t = b_t dt + Y_t dW_t$$
$$dY_t = m_t dt + v_t dW_t'$$

can be reduced to

$$dX_t = b_t dt + a(X_t) dW_t$$

- Based on a simple standardized statistic, involving sums of powers of increments with truncation (for jumps) and localization (at a given $x$), summed over $x$

- Asymptotically $\mathcal{N}(0, 1)$ under the null

- Interesting contrast of empirical results: stochastic volatility necessary to model stock prices, much less so for interest rates

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