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Modeling Large Societies with Uncertainty

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Following the suggestion of Von-Neumann-Morgenstern, and Kuhn-Tucker,

Milnor-Shapley (1961) and Aumann (1964) used an atomless measure space (in particular, the Lebesgue unit interval) to model the space of many "small" players interacting with each other, where each individual has negligible influence.

When a large society with uncertainty is modeled, one needs to work with processes which are functions from the **joint agent-sample space** to some target space (a continuum of random variables).

Even the simplest model with only iid individual risks has serious problems.

The exact law of large numbers

had been used in a large literature without a proper foundation.

Let f be a function on $I \times \Omega$ (called a *process*). For a fixed ω, the function $f(\cdot, \omega)$ on I is denoted by f_ω, called a *sample function*. For each $i \in I$, $f_i(\cdot) = f(i, \cdot)$ is a random variable with distribution μ.

If the f_i's are independent of each other, the distributions of the sample functions f_ω should be μ for $\omega \in \Omega$ with probability one.

Existence Problem for a continuum of iid random variables

solved by *Kolmogorov's existence theorem*

Let $I = [0,1]$ with Lebesgue measure λ, $\Omega = \mathbb{R}^{[0,1]}$ and P be a product measure on Ω constructed on probability distributions on $\mathbb{R}$.

For $i \in I$, let X_i be the i-th coordinate function, i.e., $X_i(\omega) = \omega(i)$, where $\omega \in \Omega$ is a function on I.

Measurability Problem

It was already observed in *Doob (Trans. AMS, 1937, Theorem 2.2 on p. 113)* that for any real-valued function h on $[0, 1]$,

$M_h = \{\omega : X_\omega(i) = \omega(i) = h(i)$
except for countably many $i \in I\}$

has P-outer measure 1, which will have probability one under some extension $\bar{P}$ of P.

• If h is non-Lebesgue measurable, then so is any $g \in M_h$, and thus
$\bar{P}(\omega : X_\omega$ is not Lebesgue measurable$) = 1$
and $\bar{P}(\omega : X_\omega$ is Lebesgue measurable$) = 0$.

Further Interpretations

– taking $h \equiv c$, constant,
$\bar{P}(\omega : X_\omega \equiv c) = \bar{P}(\omega : \mathbb{E}X_\omega = c) = 1$,

– The exact LLN as stated may be based on an absurd claim! (coin-tossing, variability of sample realizations?)

• This may appear to be **a weak straw to clutch** (Judd, JET, 1985, p.24)!

One can also construct examples of a continuum of iid random variables with common mean m to claim that almost no sample mean is m.

Essential Difficulty in the Continuum Approach

• Independence and joint measurability are never compatible with each other except for the trivial case.

Proposition 1. (Doob, S.) Let X be a process on any product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. Assume that λ is atomless. Assume that X is essentially pairwise independent in the sense that for λ-almost all $s \in I$, the random variables X_s and X_i are independent for λ-almost all $i \in I$. If X is jointly measurable with respect to the usual produce σ-algebra $\mathcal{I} \otimes \mathcal{F}$, then, for almost all i, X_i is a constant random variable.

In the discrete setting, the general condition for proving the law of large numbers is the countable additivity of probability spaces.

♣ **Question:** what will be an analogous general condition in the continuum setting?

• Since independence and joint measurability are never compatible, one has to go beyond the usual measure-theoretic framework to study independence in the continuum setting!

What one needs is a rich product space
(1) extending the usual product,
(2) retaining the Fubini property.

Definition of Fubini Extension

Let $(I \times \Omega, \mathcal{W}, Q)$ be a probability space extending the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. The extension is said to be a **Fubini extension** if it retains the Fubini property, i.e., for any real-valued $\mathcal{W}$-integrable function h on $I \times \Omega$,

$$\int_{I\times\Omega} h dQ = \int_I \left(\int_\Omega h_i dP \right) d\lambda$$
$$= \int_\Omega \left(\int_I h_\omega d\lambda \right) dP.$$

Such an extension is denoted by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Exact Law of Large Numbers for Random Variables

Let X be a measurable process in a Fubini extension.

Proposition 2. (S., 2006) (1) If the random variables X_i are essentially pairwise independent (not required to have identical distributions), then

$$P\left(\omega \in \Omega : \lambda X_\omega^{-1} = (\lambda \boxtimes P)X^{-1}\right) = 1.$$

(2) If the random variables X_i are real-valued and essentially uncorrelated, then

$$P\left(\omega \in \Omega : \int_I X_\omega d\lambda = \int_{I\times\Omega} X d(\lambda \boxtimes P)\right) = 1.$$

Exact Law of Large Numbers for Stochastic Processes

Let F be a real-valued process on $(I \times \Omega \times T)$ that is $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{T}$-measurable. The space T represents discrete or continuous time.

Proposition 3. (S., 2006) Assume that for λ-almost all $i \in I$, the stochastic processes F_i on $\Omega \times T$ have the same finite dimensional distributions with a fixed stochastic process Φ.

If the stochastic processes F_i are essentially pairwise independent, then for P-almost all $\omega \in \Omega$, the empirical process F_ω on $I \times T$ has the same finite dimensional distributions with Φ.

Coalitional Aggregate Certainty

The empirical process F_ω essentially has the same finite dimensional distributions as a given stochastic process Φ at the coalitional level.

Fubini Extension:

the Only Right Framework for Idiosyncratic Shocks

Theorem 1. Let F be a process on $I \times \Omega \times T$. Among the following three conditions on F, any two conditions imply the third one.

(1) There is a Fubini extension in which F is measurable.

(2) The stochastic processes F_i are essentially pairwise independent.

(3) The process F satisfies the property of coalitional aggregate certainty.

Large Non-cooperative Games

Definition. Let A be a compact metric space, and $\mathcal{U}_A$ the space of real-valued continuous functions on $(A \times \mathcal{M}(A))$.

A large game $\mathcal{G}$ is a measurable function from an atomless probability space $(I, \mathcal{I}, \lambda)$ to $\mathcal{U}_A$, with agent i having payoff function $u_i(a, \nu)$.

A Nash equilibrium of game $\mathcal{G}$ is a measurable function g from I to A such that for λ-almost all $i \in I$, $u_i(g(i), \lambda g^{-1}) \geq u_i(a, \lambda g^{-1})$ for all $a \in A$.

Mixed-strategy Nash equilibrium in Large Games

$f : I \times \Omega \to A$ is a mixed strategy Nash equilibrium for a large game $\mathcal{G}$ with compact metric action space A if f is an essentially pairwise independent process, and for λ-almost all $i \in I$,

$$\int_\Omega u_i(f_i(\omega), \lambda f_\omega^{-1}) dP \geq \int_\Omega u_i(h(\omega), \lambda f_\omega^{-1}) dP$$

for all $h : \Omega \to A$. Let $\nu^f(\cdot) = \int_I P f_i^{-1}(\cdot) d\lambda(i)$, the average action distribution.

Corollary 1. (Khan-Rath-Sun-Yu, 2015) A mixed-strategy Nash equilibrium is an ex post Nash equilibrium in pure strategy.

Proof: By the exact law of large numbers, for P-almost all $\omega \in \Omega$, $\lambda f_\omega^{-1} = \nu^f$. Thus, for λ-almost all $i \in I$,

$$\int_\Omega u_i(f_i(\omega), \nu^f) dP \geq \int_\Omega u_i(h(\omega), \nu^f) dP$$

for all $h : \Omega \to A$. Hence, for λ-almost all $i \in I$, $f_i(\omega) \in \text{Argmax}_{a \in A} u_i(a, \nu^f)$ holds for P-almost all $\omega \in \Omega$. By the Fubini property, for P-almost all $\omega \in \Omega$, $f_\omega(i) \in \text{Argmax}_{a \in A} u_i(a, \lambda f_\omega^{-1})$ holds for λ-almost all $i \in I$.

Theorem 2. (Hammond and S., 2008) Let g be a process from $I \times \Omega$. The following conditions are equivalent.

1. The process g has a *stochastic macro structure* in the sense that it is essentially pairwise conditionally independent given a countably generated sub-σ-algebra of $\mathcal{F}$.

2. The process g has *event-wise measurable conditional probabilities* in the sense that for each event $A \in \mathcal{F}$ with $P(A) > 0$, the function on I that maps i to the conditional probability $P(g_i^{-1}(B)|A)$ is $\mathcal{I}$-measurable for each $B \in \mathcal{B}$.

A Large Bayesian Game

- $(I, \mathcal{I}, \lambda)$, an atomless probability space of players
- T^0, a complete separable metric space as the common type space for all the players
- A, a compact metric space as the common action space.
- $(T, \mathcal{T}, \rho)$, a probability space modeling all the uncertainty associated with players' types
- f, a type process from $I \times T$ to T^0 such that $f(i, t)$ is agent i's type at the random realization $t \in T$

- $(\Omega, \mathcal{F}, P)$, a probability space modeling all the uncertainty associated with the randomization of strategies

- $u(i, t^0, a, \nu)$, the payoff of agent i at type $t^0 \in T^0$, action $a \in A$, and an empirical type-action distribution

- a mixed-strategy profile: ψ from $I \times \Omega \times T^0$ to A, essentially pairwise independent across players

• Under a type sample realization, $\psi(i, \omega, f(i,t))$, the action to be taken by agent i at her type $f(i,t)$ and strategy sample realization ω, let

$$G(i,t,\omega) = (f(i,t), \psi(i,\omega, f(i,t))),\ G_{(t,\omega)} = G(\cdot, t, \omega).$$

• Let $\lambda(G_{(t,\omega)})^{-1}$ be the empirical type-action distribution under the particular realizations.

• For a mixed strategy ψ^0 of agent i, its expected payoff $U_i(\psi^0)$ is

$$\int_T \int_\Omega u_i\left(f_i(t), \psi^0(\omega, f_i(t)), \lambda(G_{(t,\omega)})^{-1}\right) dP d\rho.$$

• A mixed-strategy profile ψ is a mixed-strategy Bayesian-Nash equilibrium if for agent $i \in I$, $U_i(\psi_i) \geq U_i(\psi^0)$ for any mixed strategy ψ^0.

Ex post Realization of a mixed-strategy Bayesian-Nash equilibrium

Theorem 3. (Strategy-Ex-Post Bayesian-Nash Property) For any mixed-strategy Bayesian-Nash equilibrium ψ, if it is $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{T}^0$-measurable, then for P-almost all $\omega \in \Omega$, ψ_ω is a pure-strategy Bayesian-Nash equilibrium for the same Bayesian game.

There is aggregate uncertainty!

Independent Types

Corollary 2. (Type-Ex-Post Nash Property) Assume that the type process is essentially pairwise independent and measurable in a Fubini extension. Let ψ be any mixed-strategy Bayesian-Nash equilibrium. Then, for ρ-almost $t \in T$, $\psi_i(\omega, f_t(i))$ is a mixed-strategy Nash equilibrium for the ex post complete-information large game.

Corollary 3. Any mixed-strategy Bayesian-Nash equilibrium for the large Bayesian game $\mathcal{G}$ with essentially pairwise independent types has the type-strategy ex post Nash property.

Some References with the above Results

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Thanks!