



Quantized Vortex Stability and Dynamics in Superfluidity and Superconductivity



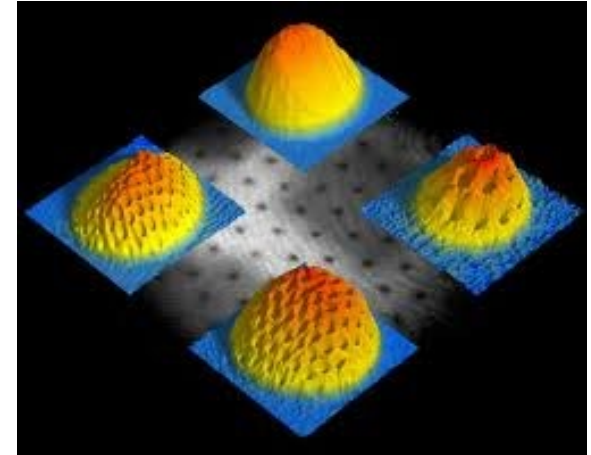
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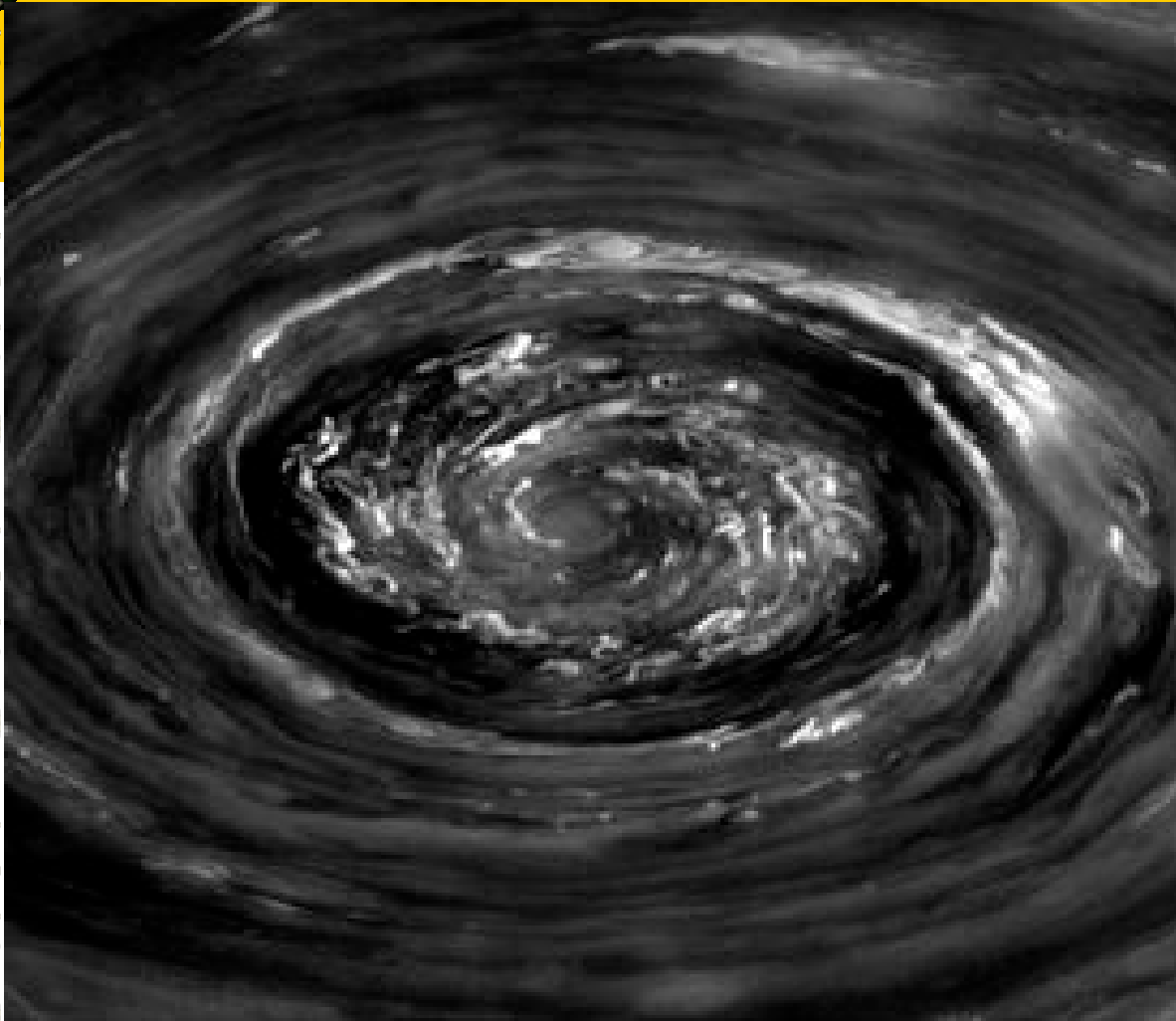
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- Qinglin Tang (Sichuan Univ); Zhihuo Xu & Shaoyun Shi (Jida); Teng Zhang (NUS)

Vortex: From macroscale to microscale



Cloud vortex — Saturn hexagon at the north pole of the planet Saturn

http://en.wikipedia.org/wiki/Saturn%27s_hexagon

Vortex: From macroscale to microscale



Vortex galaxy – vortex in cosmos

Vortex: From macroscale to microscale



Tornado – Vortex in air

Vortex: From macroscale to microscale



Vortex in water – generated by a boat

Vortex: From macroscale to microscale



Vortex in water – generated by an airplane

Vortex: From macroscale to microscale



BLUE ENERGY TORNADO BY JURI HAHHALEV WWW.CREStock.COM

Magnetic vortex –vortex in plasma

Vortex: From macroscale to microscale



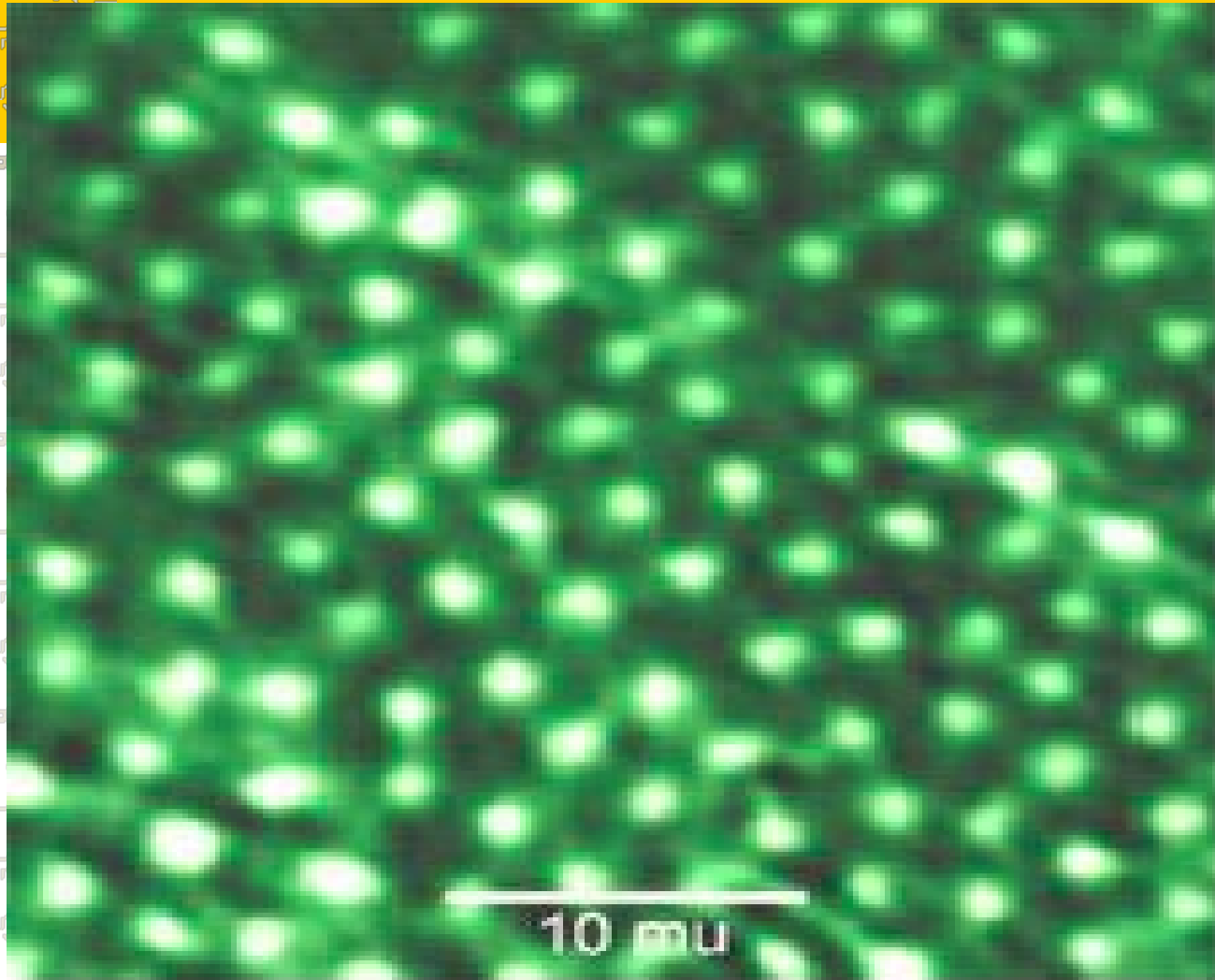
Vortex in plant

Vortex: From macroscale to microscale



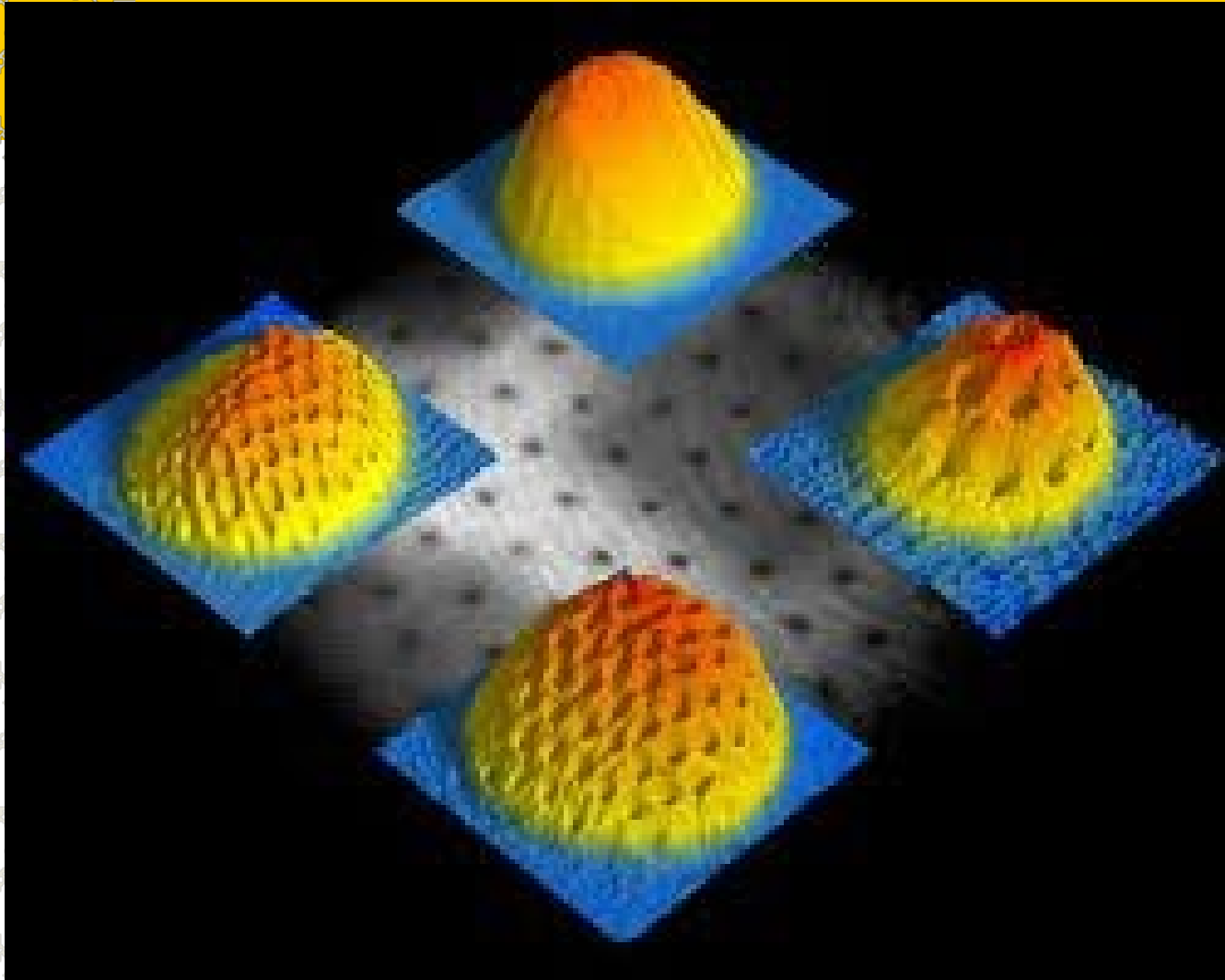
Quantized Vortex in liquid Helium 3

Vortex: From macroscale to microscale

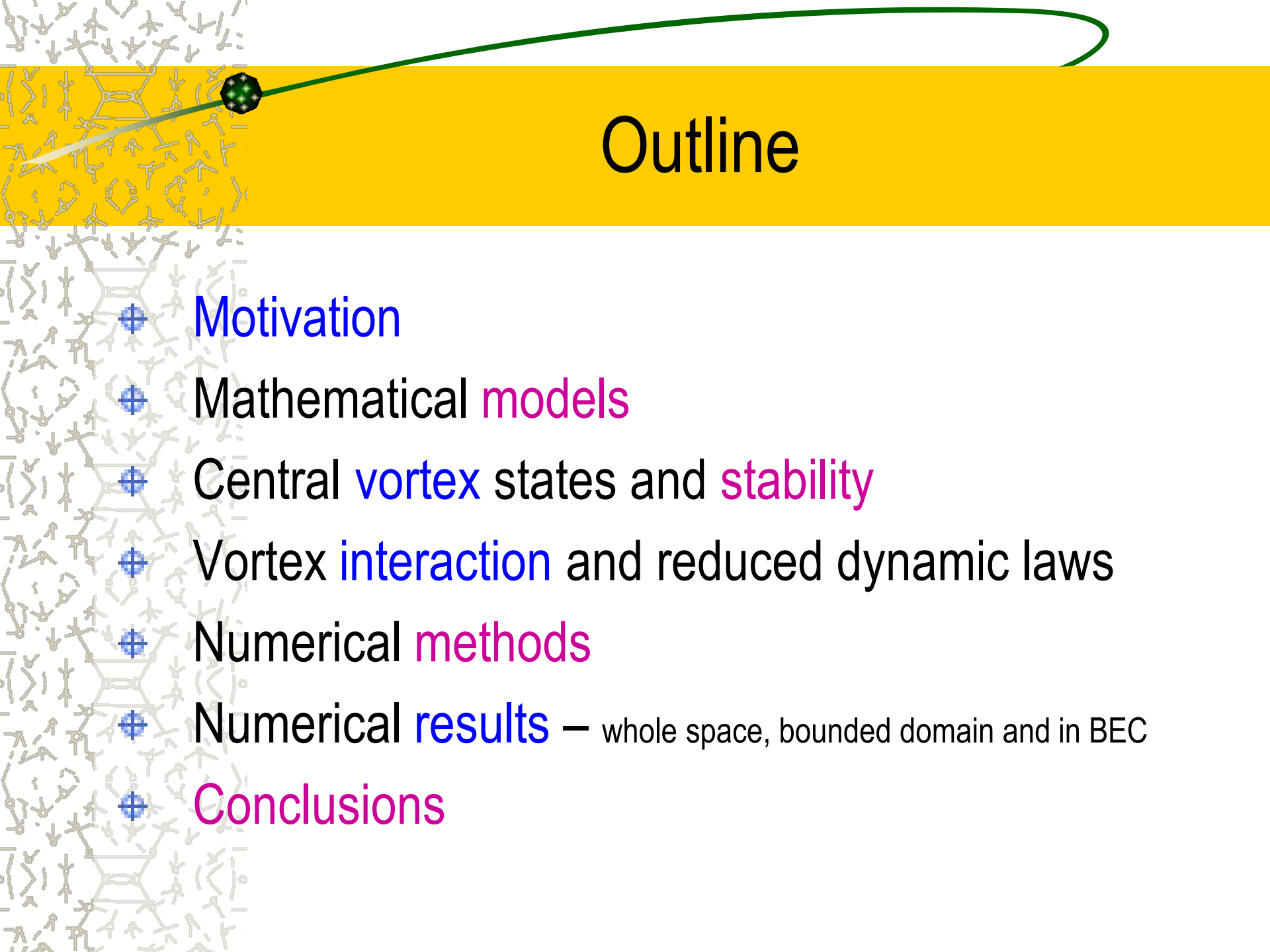


Quantized Vortex in a type-II superconductor

Vortex: From macroscale to microscale



Quantized Vortex in Bose-Einstein condensation (BEC)



Outline

• Motivation

• Mathematical models

• Central vortex states and stability

• Vortex interaction and reduced dynamic laws

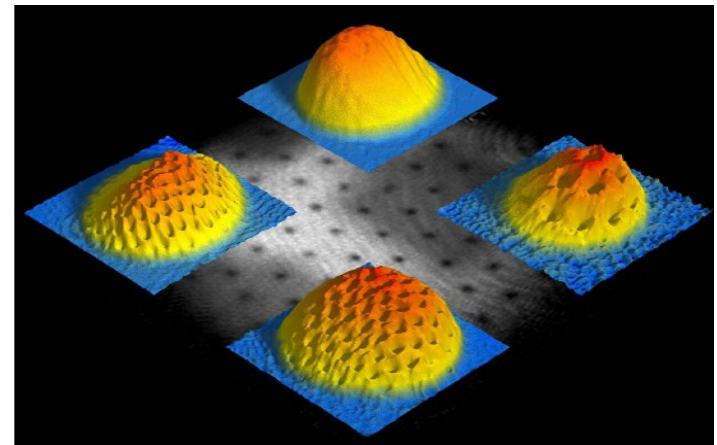
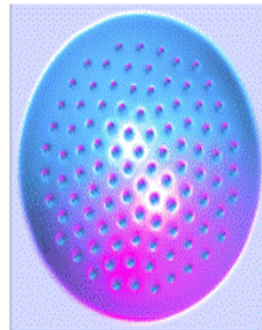
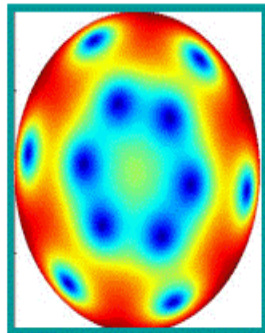
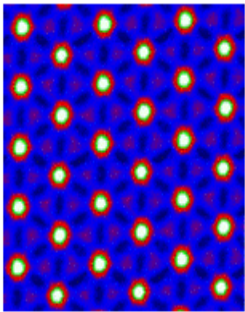
• Numerical methods

• Numerical results — whole space, bounded domain and in BEC

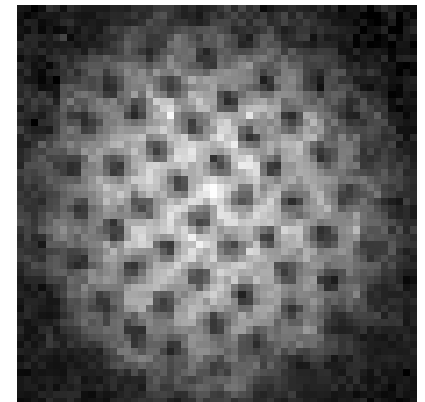
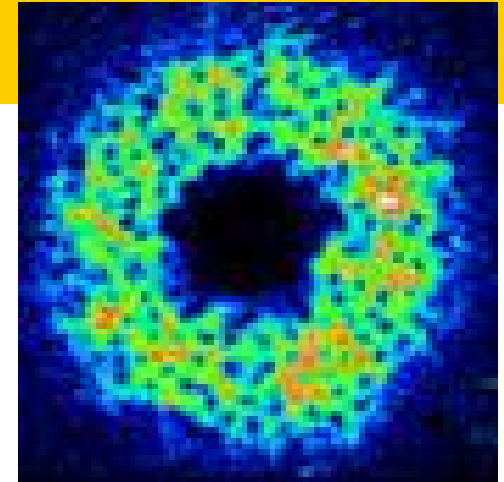
• Conclusions

Motivation

- **Quantized Vortex:** Particle-like (topological) defect
 - Zero of the complex scalar field
 - localized phase singularities with integer topological charge:
$$\psi(\vec{x}_0) = 0, \quad \psi(\vec{x}) = \sqrt{\rho} e^{i\phi}, \quad \int_{\Gamma} d(\arg \psi) = \int_{\Gamma} d\phi = 2\pi n \neq 0$$
 - Key of superfluidity: ability to support dissipationless flow



Motivation



- Existing in

- Superconductors:

- Ginzburg-Landau equations (GLE)

- Liquid helium:

- Two-fluid model
 - Gross-Pitaevskii equation (GPE)

- Bose-Einstein condensation (BEC):

- Nonlinear Schroedinger equation (NLSE) or GPE

- Nonlinear optics & propagation of laser beams

- Nonlinear Schroedinger equation (NLSE)
 - Nonlinear wave equation (NLWE)

Mathematical models

- Ginzburg-Landau equation (GLE): Superconductivity, nonlinear heat flow, etc.

$$\psi_t = \Delta \psi + \frac{1}{\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right) \psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

- Gross-Pitaevskii equation (GPE): nonlinear optics, BEC, superfluidity

$$-i \psi_t = \Delta \psi + \frac{1}{\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right) \psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

- Nonlinear wave equation (NLWE): wave motion; cosmology

$$\psi_{tt} = \Delta \psi + \frac{1}{\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right) \psi, \quad \vec{x} \in \mathbb{R}^2, \quad t > 0,$$

– Here

ψ : complex-valued wave function or order parameter,

$\varepsilon > 0$: dimensionless constant,

$V(\vec{x})$: real-valued external potential

Mathematical models

- 'Free Energy' or Lyapunov functional:

$$E(\psi) = \int_{\mathbb{R}^2} \left[|\nabla \psi|^2 + \frac{1}{2\varepsilon^2} \left(V(\vec{x}) - |\psi|^2 \right)^2 \right] d\vec{x},$$

- Dispersive system (GPE): $-i \frac{\partial \psi}{\partial t} = -\frac{\delta E(\psi)}{\delta \psi^*}$

- Energy conservation: $E(\psi) \equiv E(\psi_0), \quad t \geq 0$

- Density conservation: $\int_{\mathbb{R}^2} (|\psi(\vec{x}, t)|^2 - |\psi_0(\vec{x})|^2) d\vec{x} \equiv 0, \quad t \geq 0$

- Admits particle like solutions: solitons, kinks & vortices

- Dissipative system (GLE): $\frac{\partial \psi}{\partial t} = -\frac{\delta E(\psi)}{\delta \psi^*}$

- Energy diminishing, etc.

Two kinds of vortices

• **'Bright-tail' vortex:** GLE, GPE & NLWE ('order parameter')

$$V(\vec{x}) \rightarrow 1, \quad \text{when } |\vec{x}| \rightarrow \infty, \Rightarrow |\psi(\vec{x}, t)| \rightarrow 1, \quad |\vec{x}| \rightarrow \infty, \quad (\text{e.g. } \psi \rightarrow e^{im\theta}),$$

– Time independent case (*Neu*, 90'): $\varepsilon = 1$, $V(\vec{x}) \equiv 1$

– Vortex solutions: $\psi(\vec{x}) = \phi_n(\vec{x}) = f_n(r) e^{in\theta}$

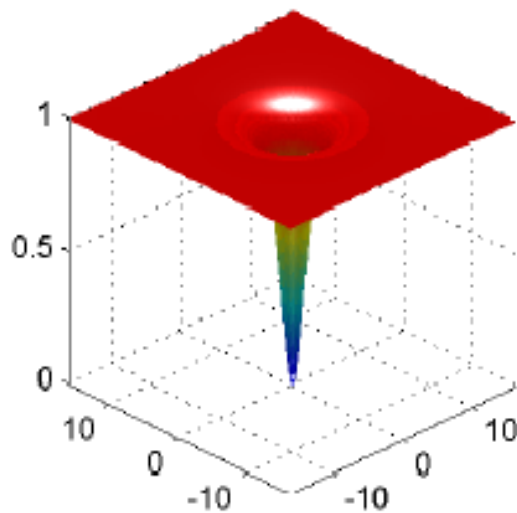
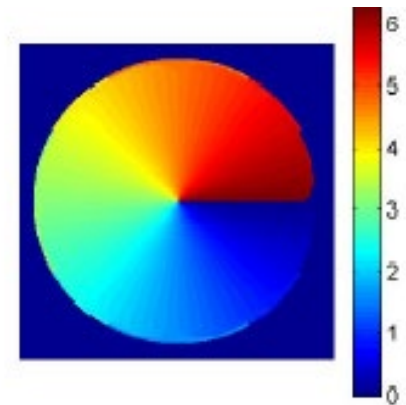
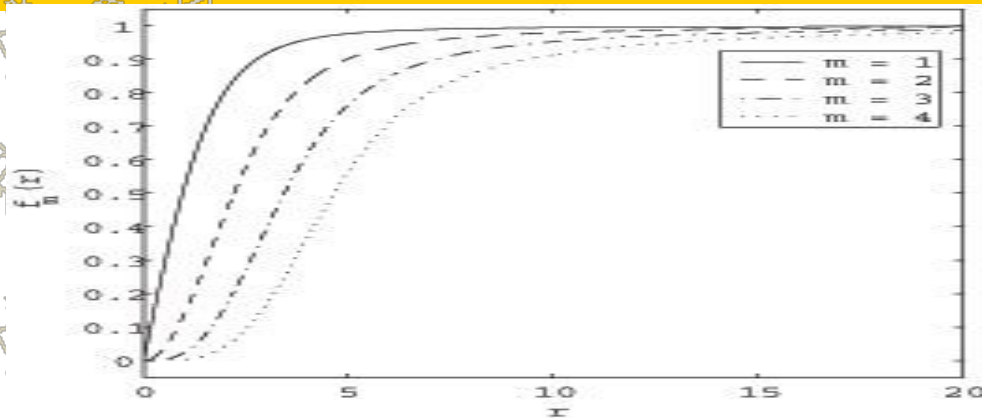
$$f_n''(r) + \frac{1}{r} f_n'(r) - \frac{n^2}{r^2} f_n(r) + (1 - f_n^2(r)) f_n(r) = 0, \quad 0 < r < \infty,$$

$$f_n(0) = 0, \quad f_n(\infty) = 1.$$

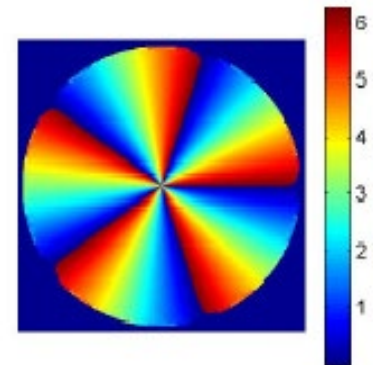
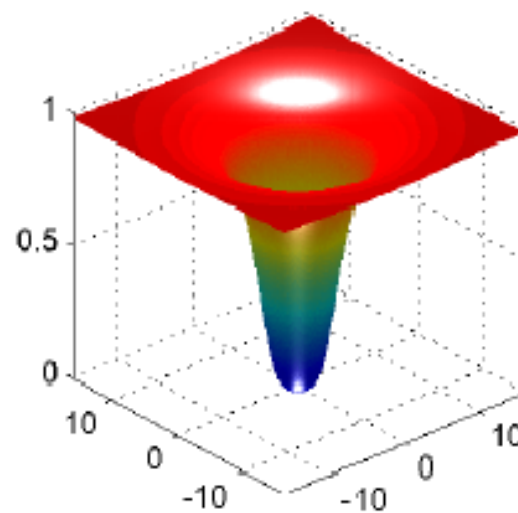
– Asymptotic results

$$f_n(r) \approx \begin{cases} ar^{|n|} + O(r^{|n|+2}), & r \rightarrow 0, \\ 1 - n^2 / 2r^2 + O(1/r^4), & r \rightarrow \infty. \end{cases}$$

'Bright-tail' vortex



a)



b)

Two kinds of vortices

• 'Dark-tail' vortex: BEC (wave function)

$$i \psi_t = -\Delta \psi + \frac{|\vec{x}|^2}{2} \psi + \beta |\psi|^2 \psi, \quad \vec{x} \in R^2, \quad t > 0, \quad \|\psi\|^2 = \int |\psi|^2 d\vec{x} = 1$$

– Center vortex states: $\psi(\vec{x}, t) = e^{-i\mu_n t} \phi_n(\vec{x}) = e^{-i\mu_n t} f_n(r) e^{in\theta}$

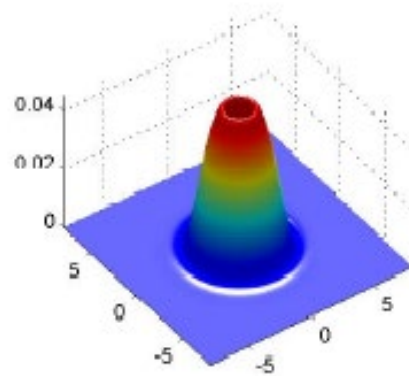
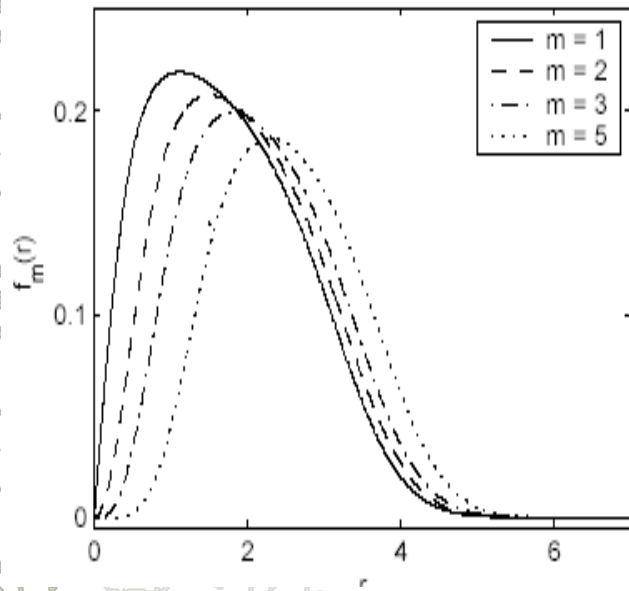
$$\mu_n f_n(r) = -\frac{d}{dr} \left(r \frac{df_n(r)}{dr} \right) + \left(\frac{n^2}{r^2} + \frac{r^2}{2} \right) f_n(r) + \beta f_n^3(r), \quad 0 < r < \infty,$$

$$f_n(0) = 0, \quad f_n(\infty) = 0, \quad 2\pi \int_0^\infty f_n^2(r) r dr = 1$$

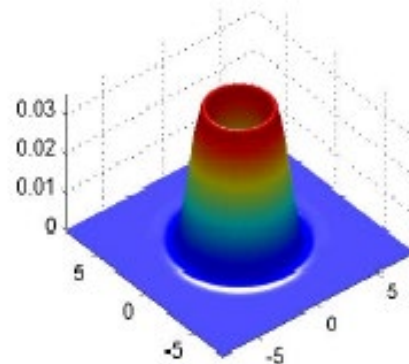
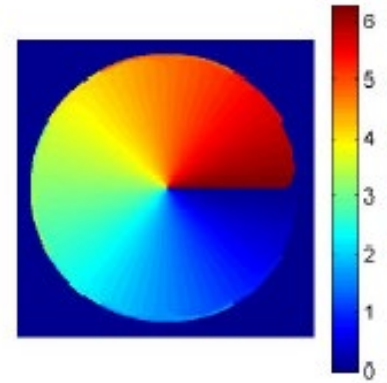
– Asymptotic results

$$f_n(r) \approx \begin{cases} a r^{|n|} + O(r^{|n|+2}), & r \rightarrow 0, \\ b r^{|n|} e^{-r^2}, & r \rightarrow \infty. \end{cases}$$

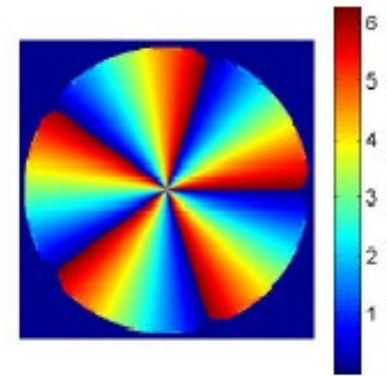
'Dark-tail' vortex



a)



b)



Stability of vortices

- Data chosen

$$\varepsilon = 1, \quad V(\vec{x}, t) \equiv 1 \quad (+W(\vec{x}, t)), \quad \psi_0(\vec{x}) = \phi_n^h(\vec{x}) \quad (+ \text{noise})$$

- Results (Bao, Du & Zhang, Eur. J. Appl. Math., 07')

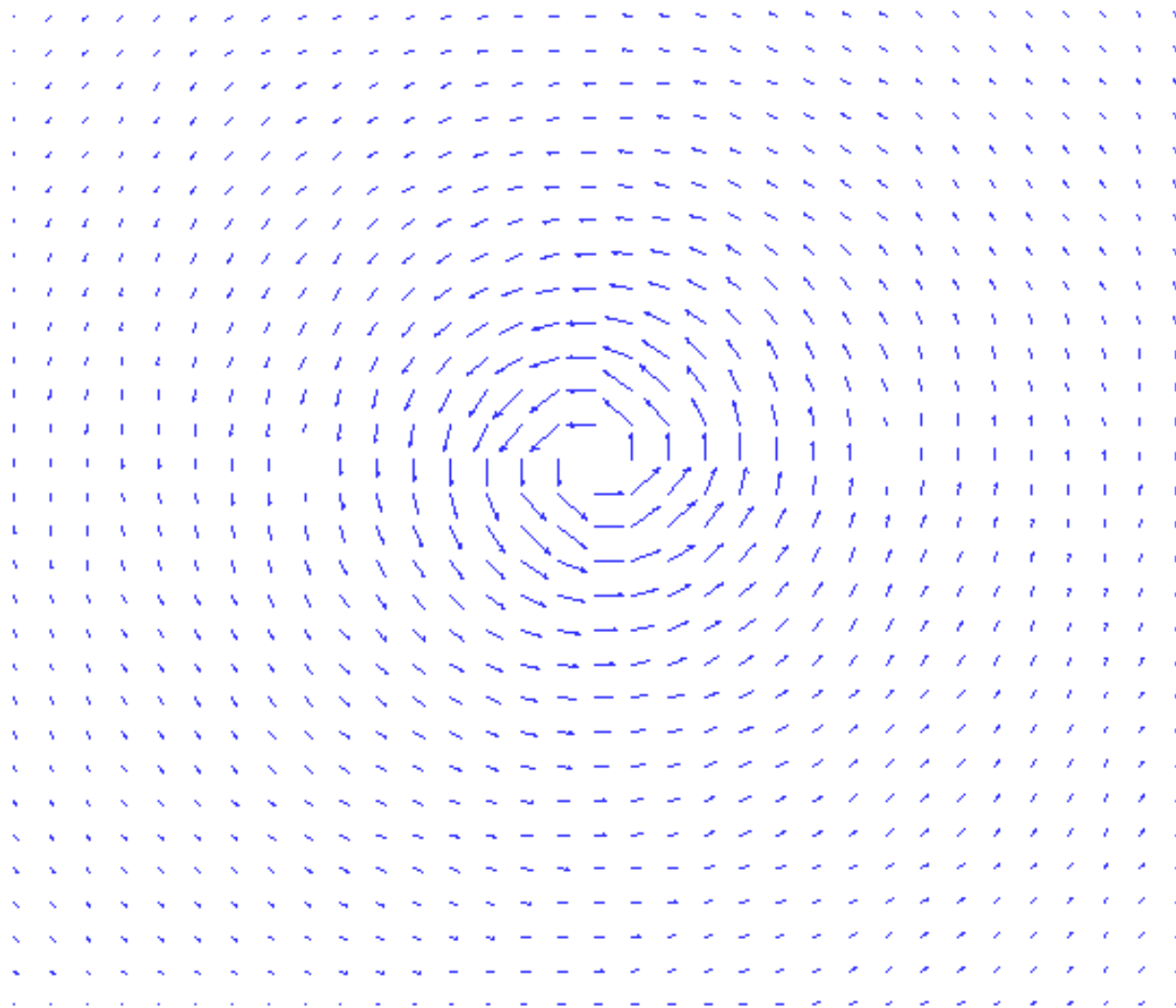
- For GLE or NLWE with initial data perturbed

- n=1: velocity density
- N=3: velocity density

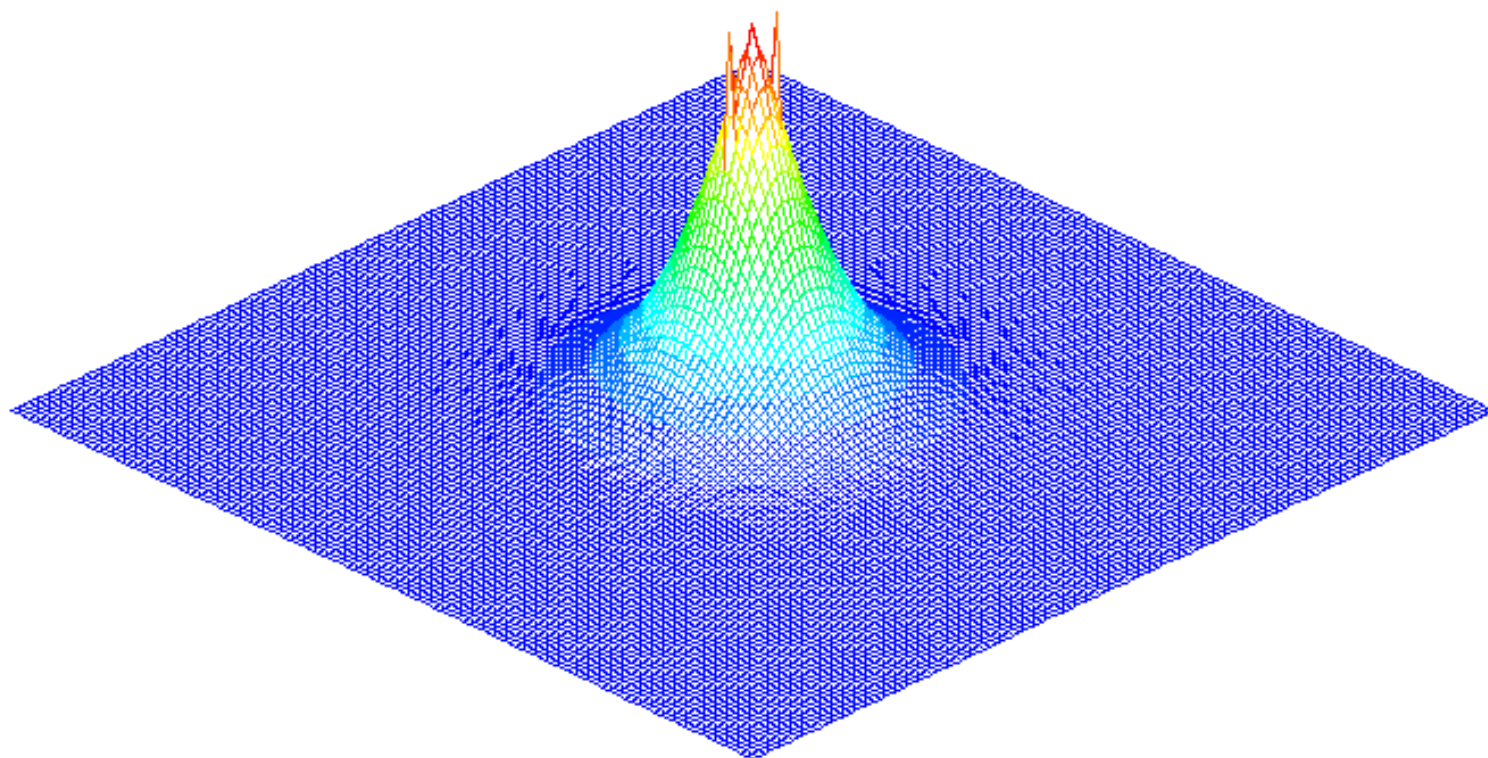
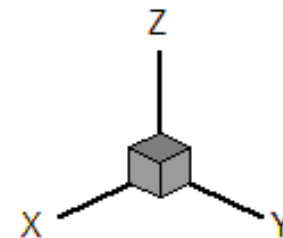
- For NLSE with external potential perturbed

- n=1: velocity density
- N=3: velocity density

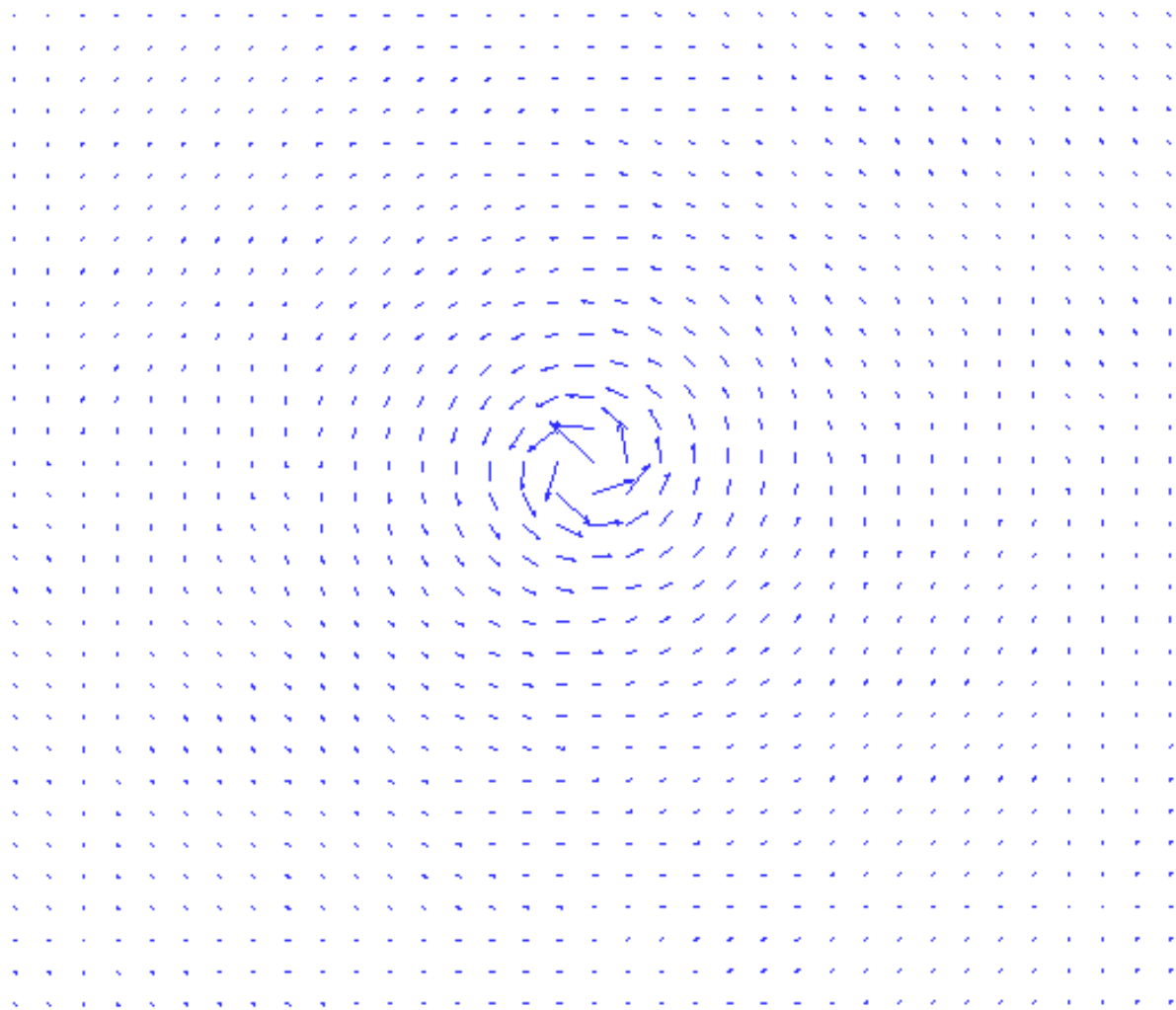
- Results



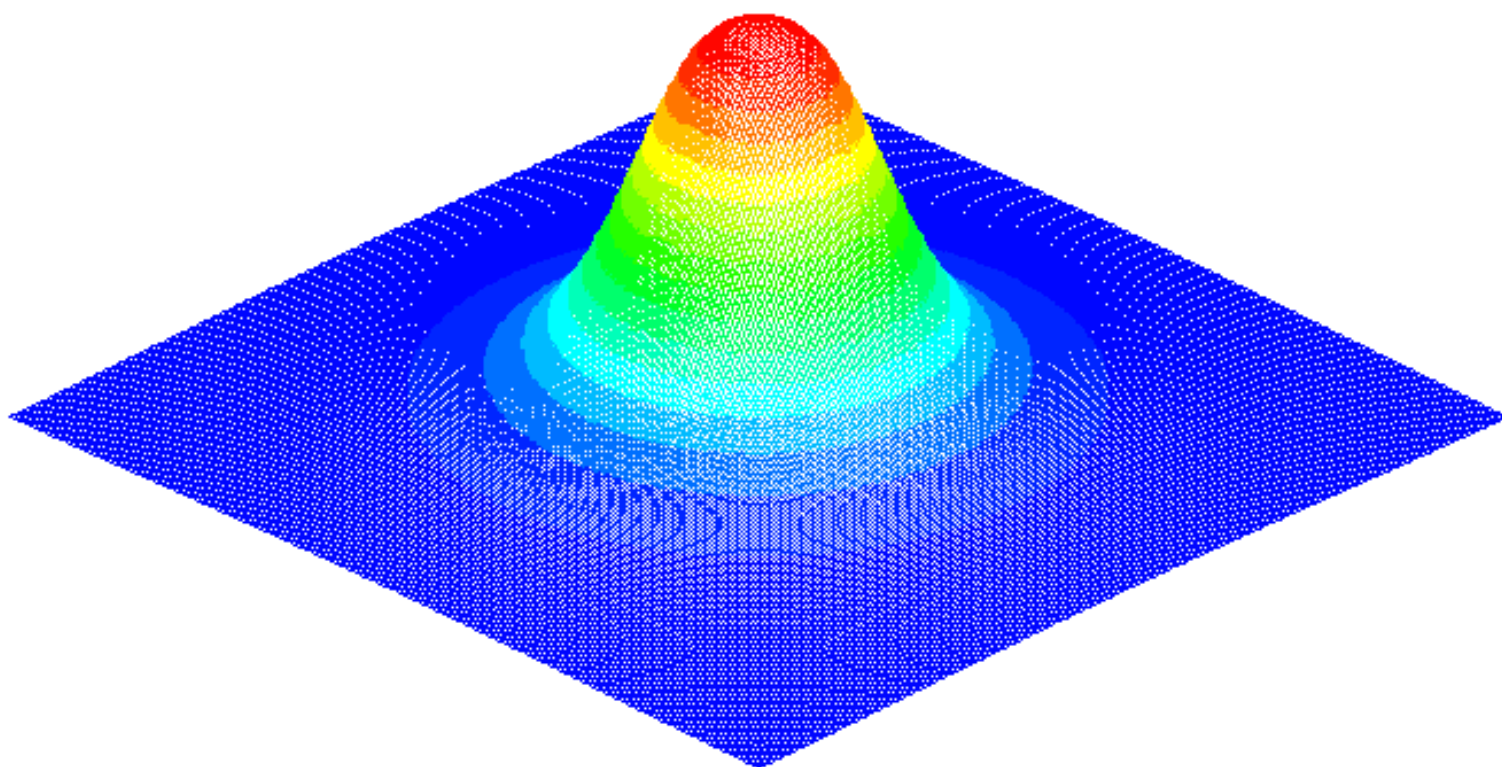
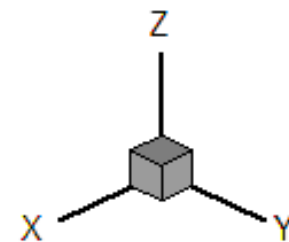
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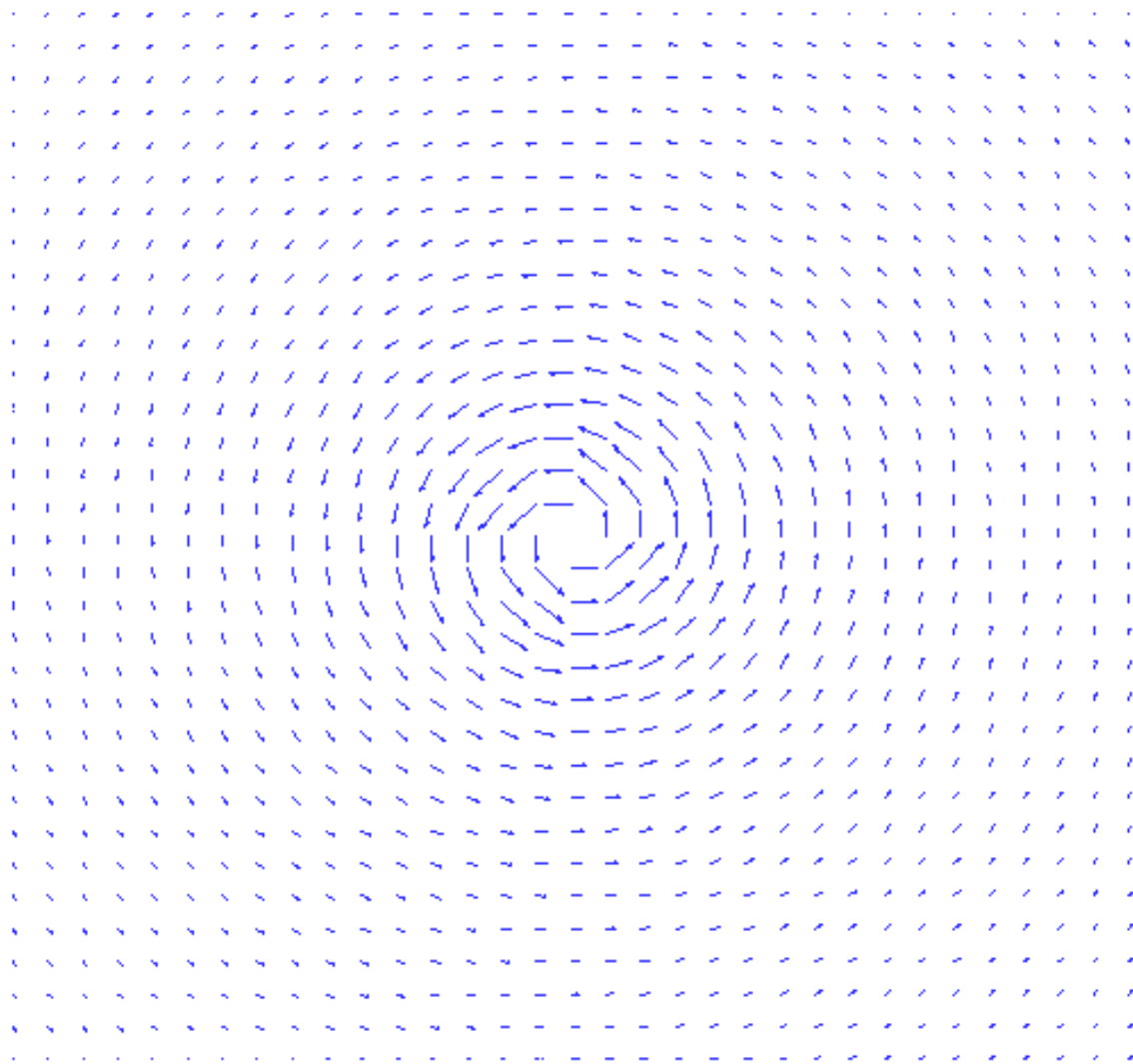
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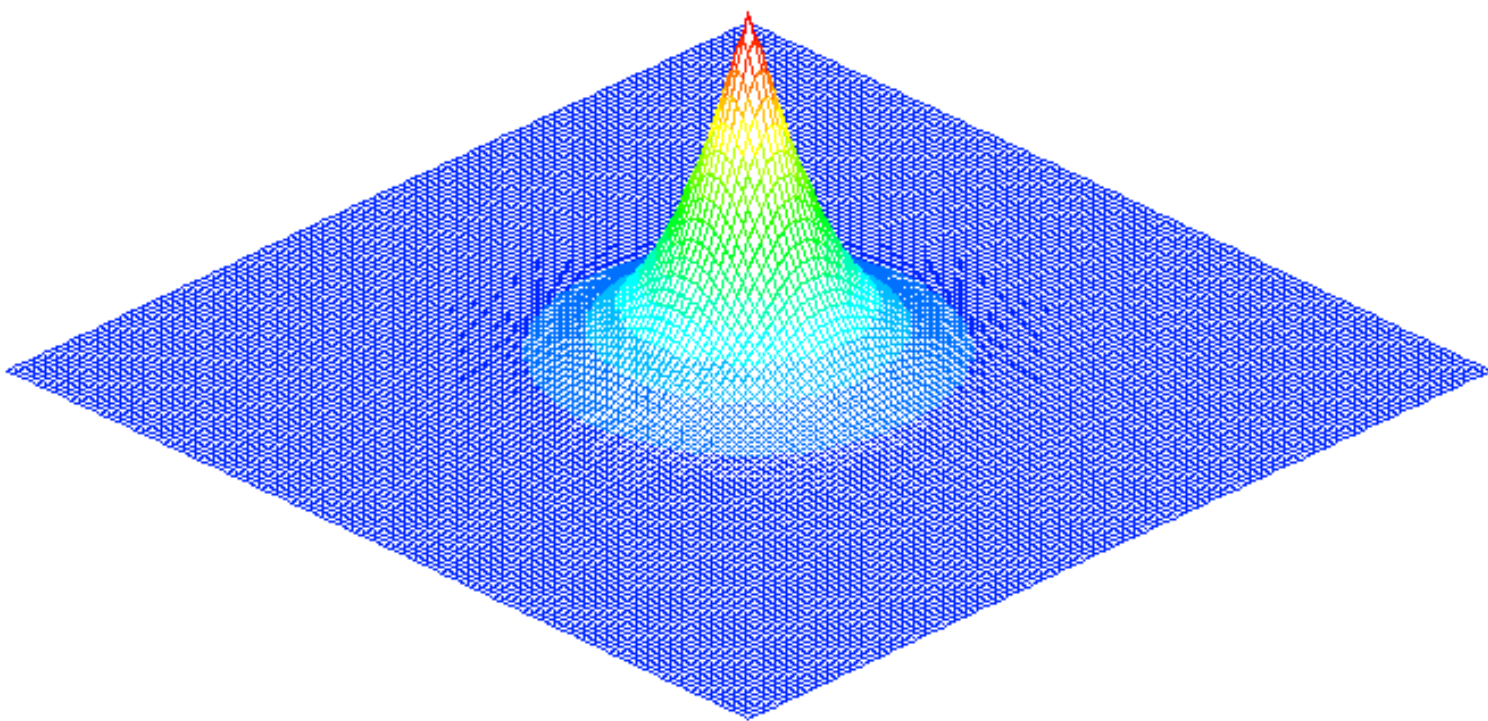
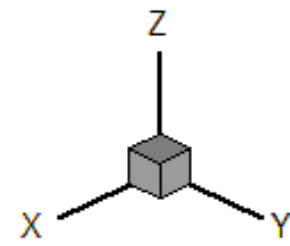
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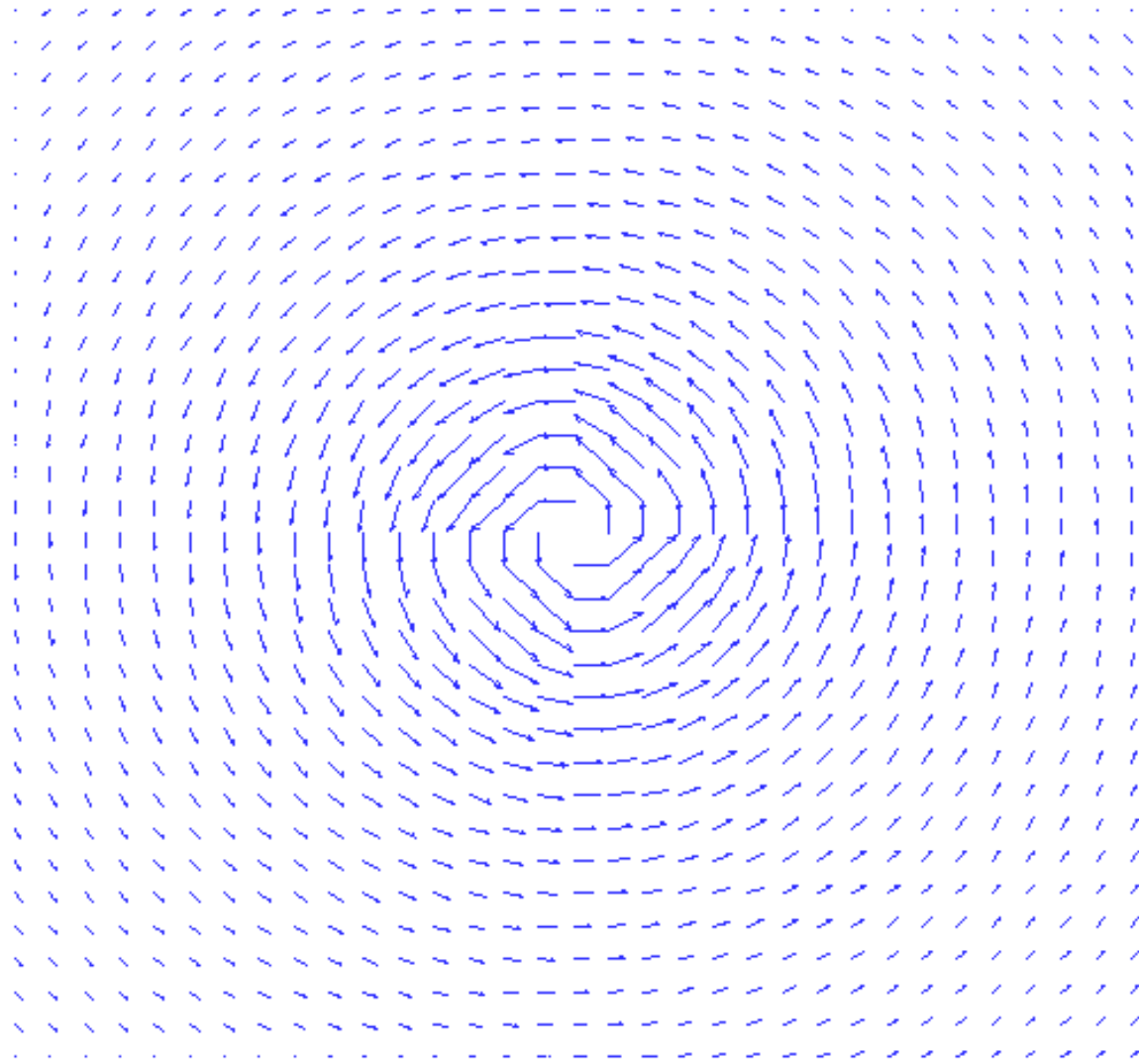
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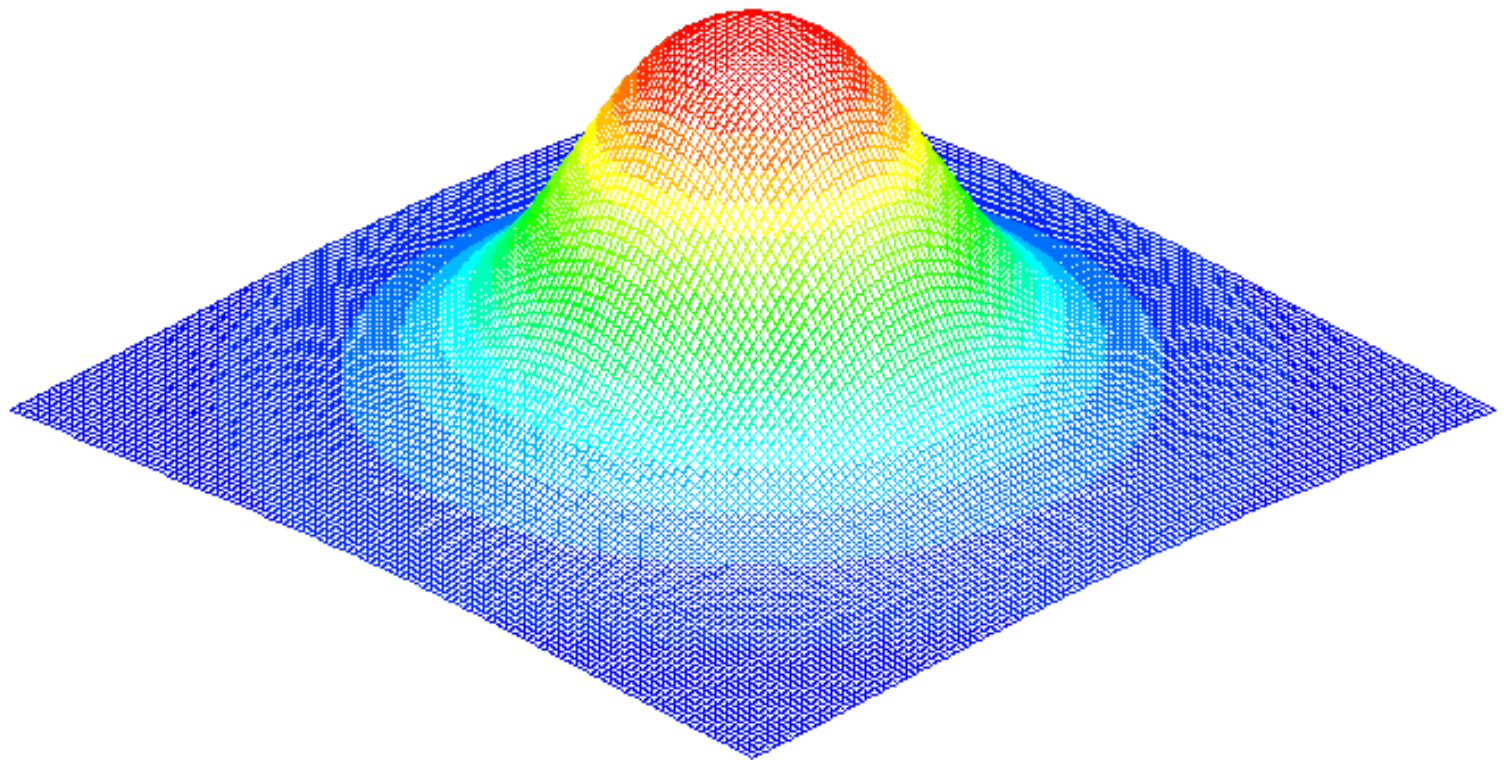
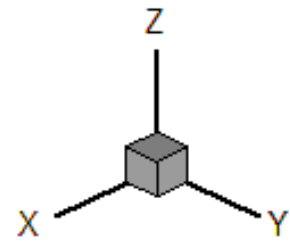
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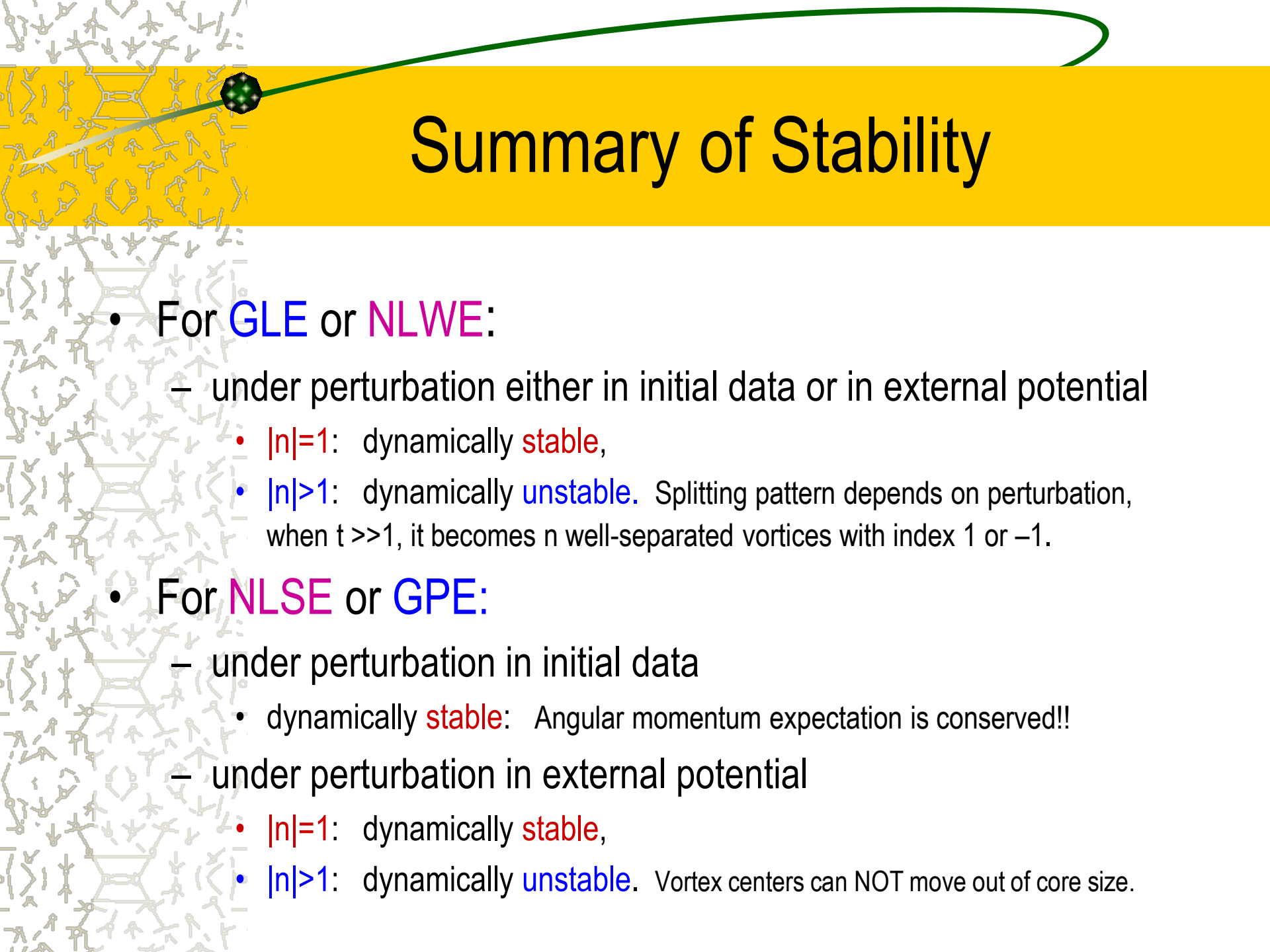
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Summary of Stability

- For **GLE** or **NLWE**:
 - under perturbation either in initial data or in external potential
 - $|n|=1$: dynamically **stable**,
 - $|n|>1$: dynamically **unstable**. Splitting pattern depends on perturbation, when $t \gg 1$, it becomes n well-separated vortices with index 1 or -1 .
- For **NLSE** or **GPE**:
 - under perturbation in initial data
 - dynamically **stable**: Angular momentum expectation is conserved!!
 - under perturbation in external potential
 - $|n|=1$: dynamically **stable**,
 - $|n|>1$: dynamically **unstable**. Vortex centers can NOT move out of core size.

Vortex interaction

- For 'bright-tail' vortex:

$$\psi_0(\vec{x}) = \prod_{j=1}^N \phi_{m_j}(\vec{x} - \vec{x}_j^0) = \prod_{j=1}^N \phi_{m_j}(x - x_j^0, y - y_j^0)$$

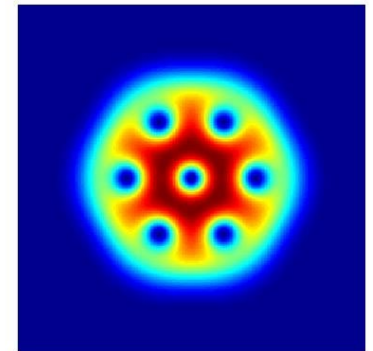
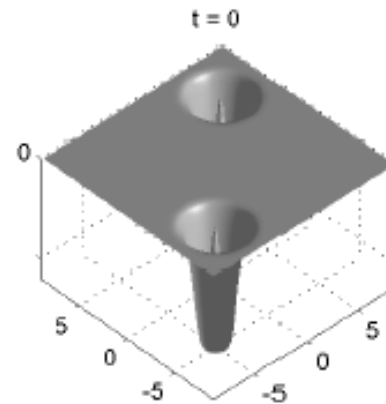
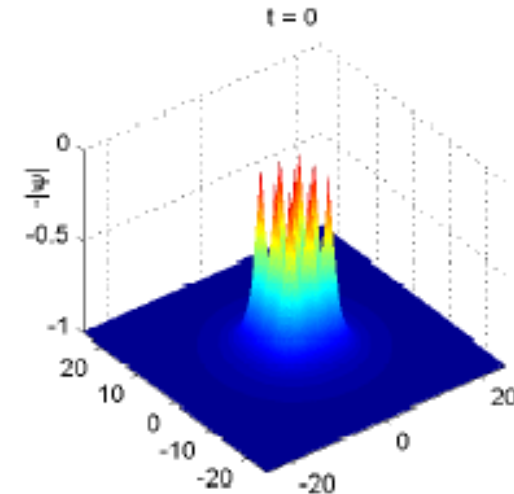
- For 'dark-tail' vortex

– Well-separate

$$\psi_0(x, y) = \sum_{j=1}^N \phi_{n_j}(x - x_j, y - y_j) \quad / \|\bullet\|$$

– Overlapped

$$\psi(\mathbf{r}, t=0) = \alpha \psi_{\text{gs}}(\mathbf{r}) \prod_{j=1}^n p_{q_j}(\mathbf{r} - \mathbf{r}_j),$$



Reduced dynamic laws

- For 'bright-tail' vortex

- Take ansatz

$$\psi(\vec{x}, t) = \prod_{j=1}^N \phi_{m_j}(\vec{x} - \vec{x}_j(t)) + \text{high order terms}$$

$$= \prod_{j=1}^N \phi_{m_j}(x - x_j(t), y - y_j(t)) + \text{high order terms}$$

- Plug into GLE or GPE or NLWE

- Using matched asymptotic techniques

- Prove rigorously

Reduced dynamic laws

- For 'bright-tail' vortex

– For GLE

$$\kappa \vec{v}_j(t) := \kappa \frac{d \vec{x}_j(t)}{dt} = 2m_j \sum_{l=1, l \neq j}^N m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{|\vec{x}_j(t) - \vec{x}_l(t)|^2}, \quad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \quad 1 \leq j \leq N.$$

– For GPE

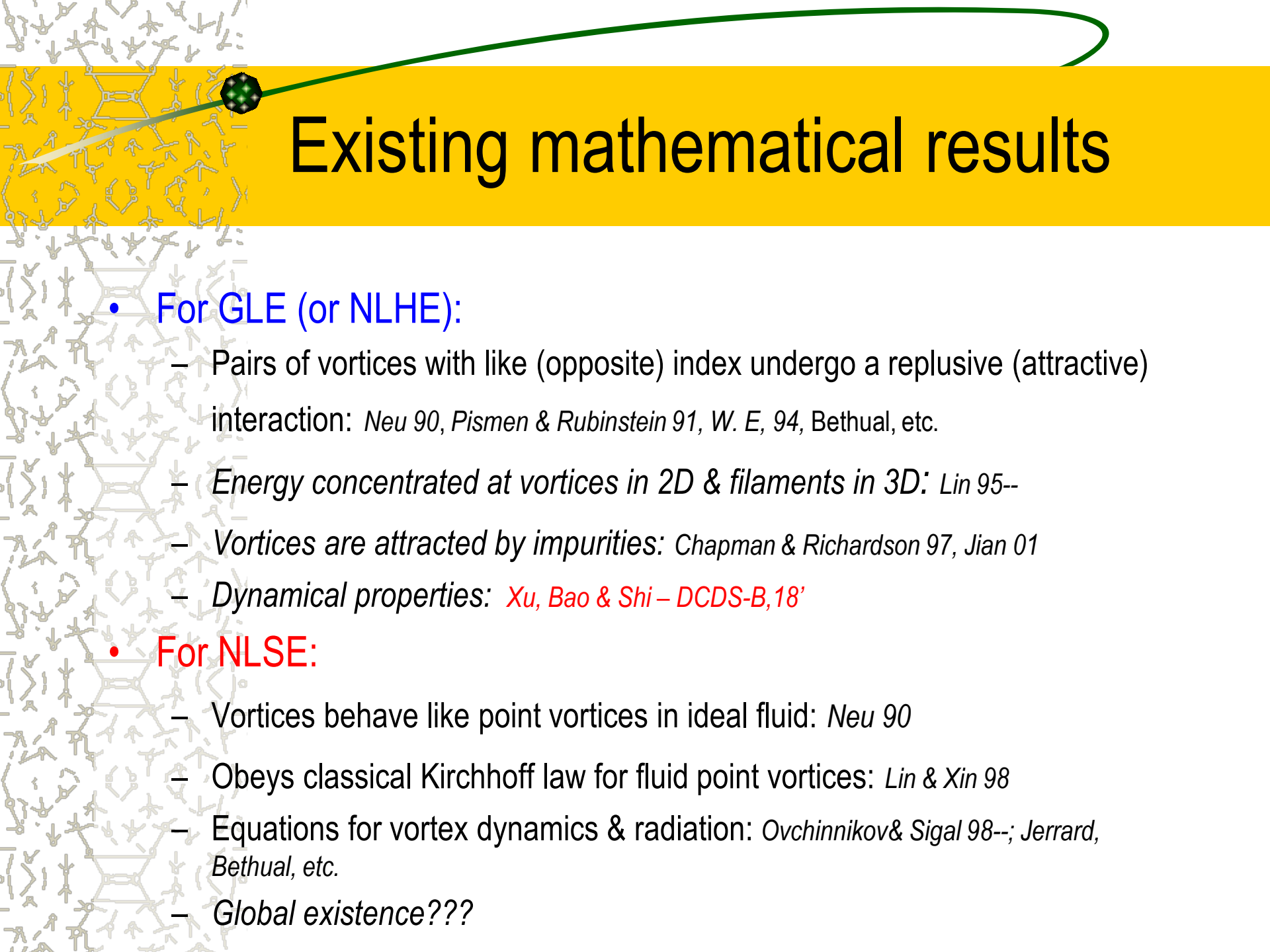
$$\vec{v}_j(t) := \frac{d \vec{x}_j(t)}{dt} = 2 \sum_{l=1, l \neq j}^N m_l \frac{J(\vec{x}_j(t) - \vec{x}_l(t))}{|\vec{x}_j(t) - \vec{x}_l(t)|^2} \quad t \geq 0,$$

$$\vec{x}_j(0) = \vec{x}_j^0 \quad 1 \leq j \leq N; \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

– For NLWE

$$\kappa \frac{d^2 \vec{x}_j(t)}{dt^2} = 2m_j \sum_{l=1, l \neq j}^N m_l \frac{\vec{x}_j(t) - \vec{x}_l(t)}{|\vec{x}_j(t) - \vec{x}_l(t)|^2}, \quad t \geq 0$$

$$\vec{x}_j(0) = \vec{x}_j^0, \quad \vec{x}_j'(0) = \vec{v}_j^0, \quad 1 \leq j \leq N.$$



Existing mathematical results

- For GLE (or NLHE):

- Pairs of vortices with like (opposite) index undergo a repulsive (attractive) interaction: *Neu 90, Pismen & Rubinstein 91, W. E, 94, Bethual, etc.*
- Energy concentrated at vortices in 2D & filaments in 3D: *Lin 95--*
- Vortices are attracted by impurities: *Chapman & Richardson 97, Jian 01*
- Dynamical properties: *Xu, Bao & Shi – DCDS-B, 18'*

- For NLSE:

- Vortices behave like point vortices in ideal fluid: *Neu 90*
- Obeys classical Kirchhoff law for fluid point vortices: *Lin & Xin 98*
- Equations for vortex dynamics & radiation: *Ovchinnikov & Sigal 98--; Jerrard, Bethual, etc.*
- Global existence???

Conservation laws

- Mass center

$$\bar{x}(t) := \frac{1}{N} \sum_{j=1}^N \vec{x}_j(t)$$

Lemma The mass center of the N vortices for GLE is conserved

$$\bar{x}(t) := \frac{1}{N} \sum_{j=1}^N \vec{x}_j(t) \equiv \bar{x}(0) := \frac{1}{N} \sum_{j=1}^N \vec{x}_j(0) = \frac{1}{N} \sum_{j=1}^N \vec{x}_j^0$$

Lemma The mass center of the N vortices for NLWE is conserved if initial velocity is zero and it moves linearly if initial velocity is nonzero

$$\bar{x}(t) := \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j(t) = \bar{x}(0) + t \bar{v}(0) = \frac{1}{N} \sum_{j=1}^N \mathbf{x}_j^0 + t \left(\frac{1}{N} \sum_{j=1}^N \mathbf{x}_j^1 \right), \quad t \geq 0,$$

Conservation laws

- Signed mass center

$$\tilde{x}(t) := \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j(t)$$

Lemma The signed mass center of the N vortices for NLSE is conserved

$$\tilde{x}(t) := \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j(t) \equiv \tilde{x}(0) := \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j(0) = \frac{1}{N} \sum_{j=1}^N m_j \vec{x}_j^0$$

Analytical Solutions

- $N (\geq 2)$ like vortices on a **circle** (Bao, Du & Zhang, SIAP, 07')

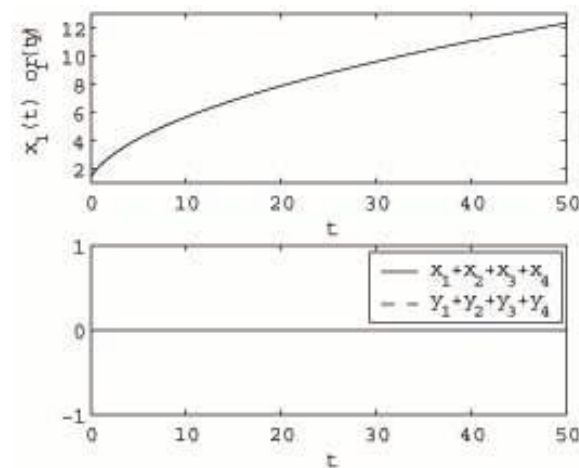
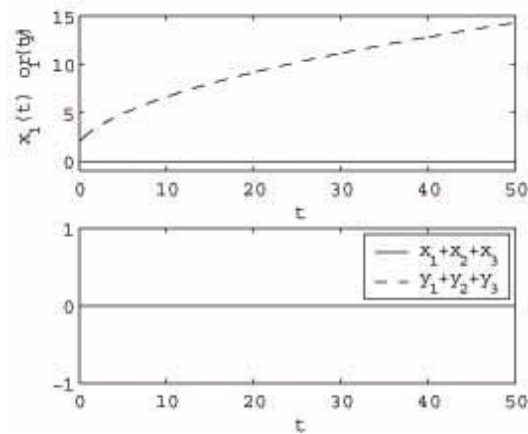
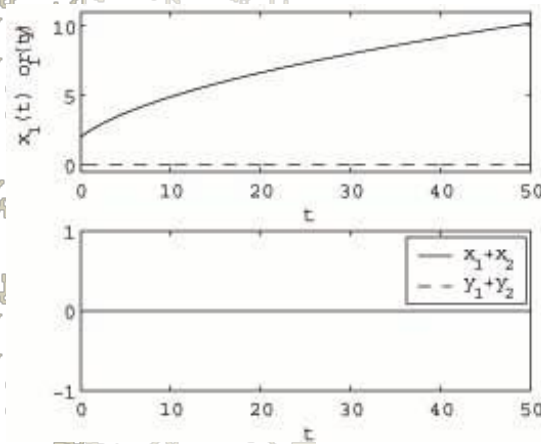
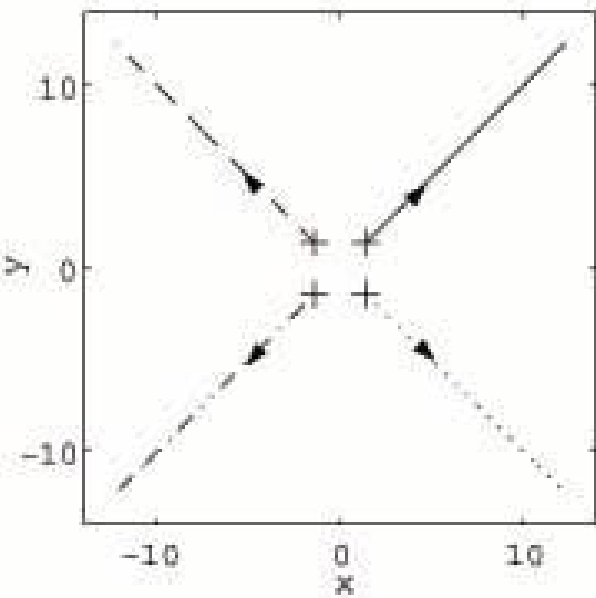
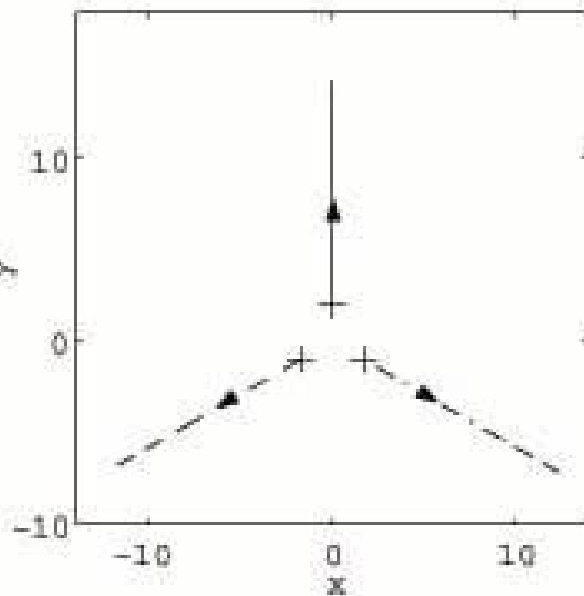
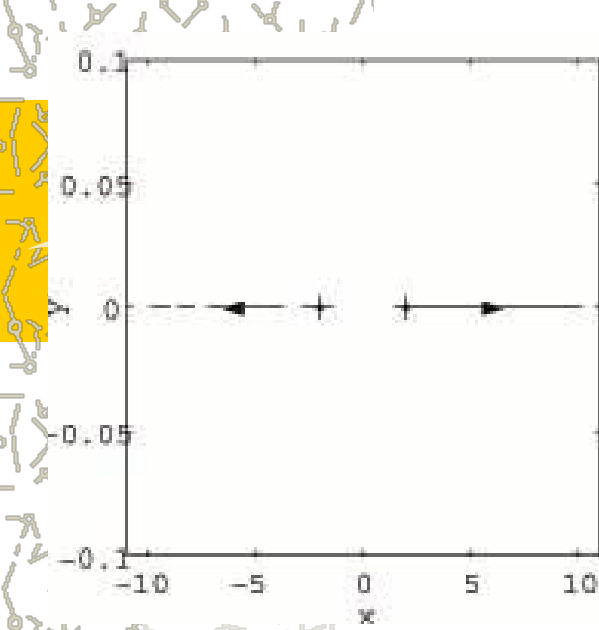
$$\vec{x}_j^0 = a \left(\cos\left(\frac{2j\pi}{N}\right), \sin\left(\frac{2j\pi}{N}\right) \right), \quad m_j = m_0 = \pm 1, \quad 1 \leq j \leq N$$

- Analytical solutions for **GLE** [figure](#)

$$\vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-1)}{\kappa}t} \left(\cos\left(\frac{2j\pi}{N}\right), \sin\left(\frac{2j\pi}{N}\right) \right)$$

- Analytical solutions for **NLSE** [figure](#) [next](#)

$$\vec{x}_j(t) = a \left(\cos\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right), \sin\left(\frac{2j\pi}{N} + \frac{N-1}{a^2}t\right) \right)$$

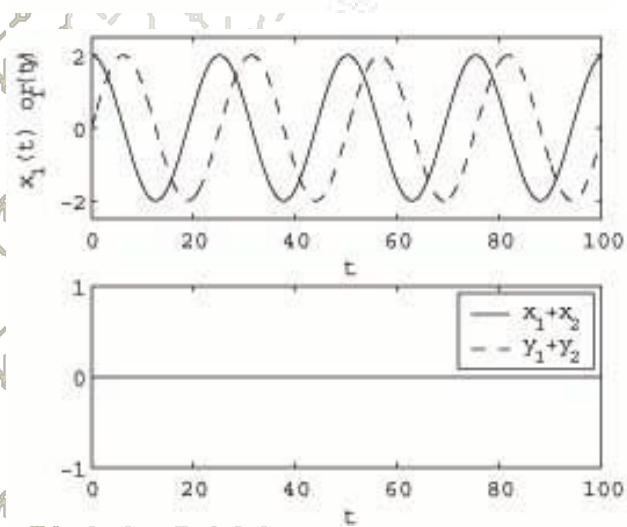
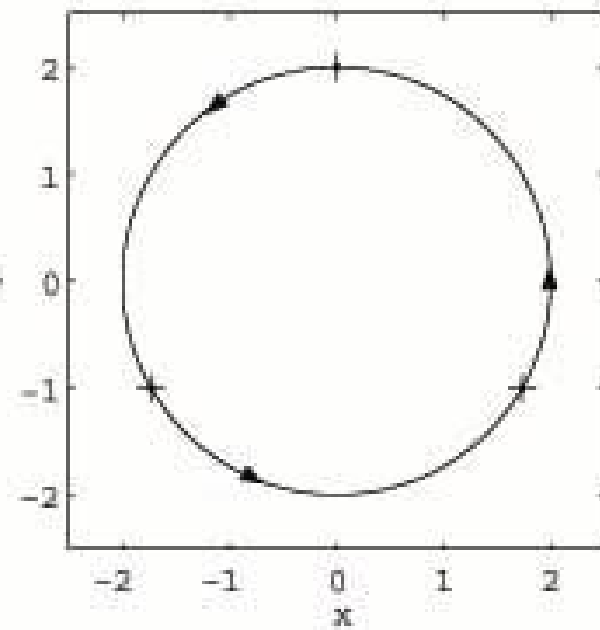
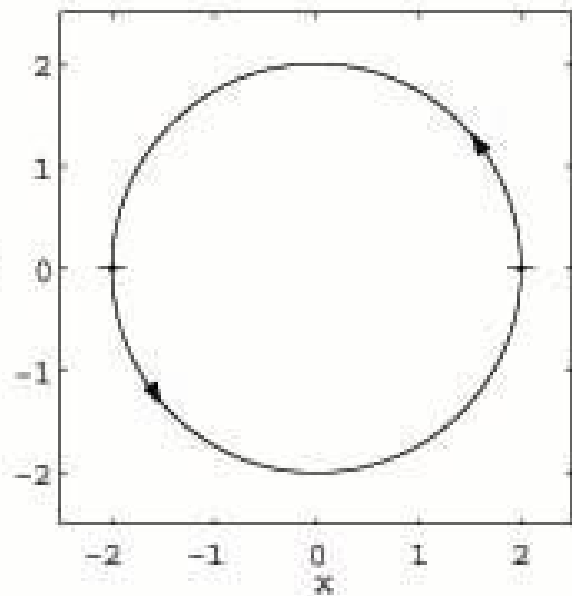


$N=2$

$N=3$

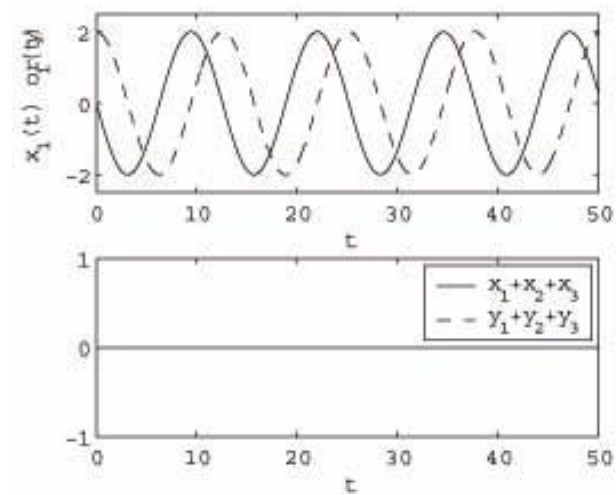
$N=4$

[back](#)



N=2

[back](#)



N=3

Analytical Solutions

- $N (\geq 3)$ like vortices on a circle and its center (Bao, Du & Zhang, SIAP, 07')

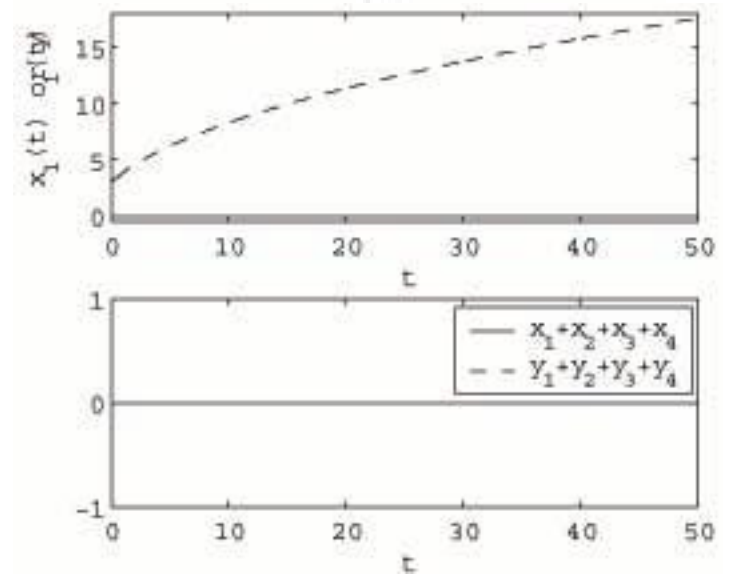
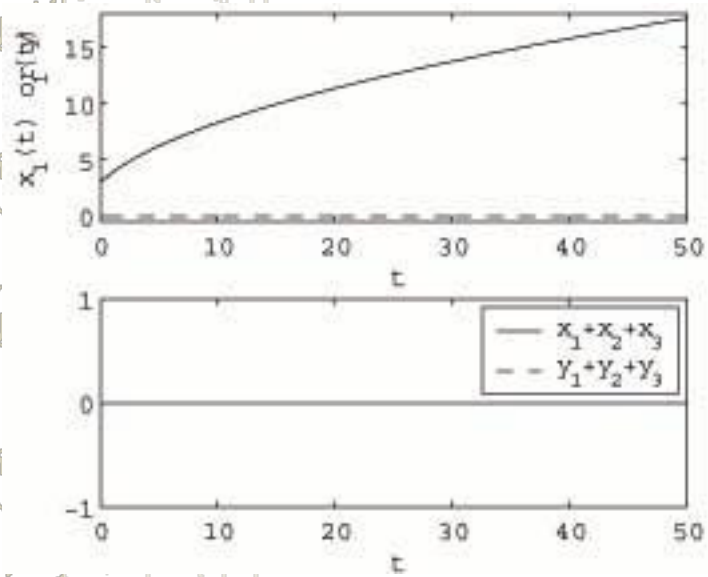
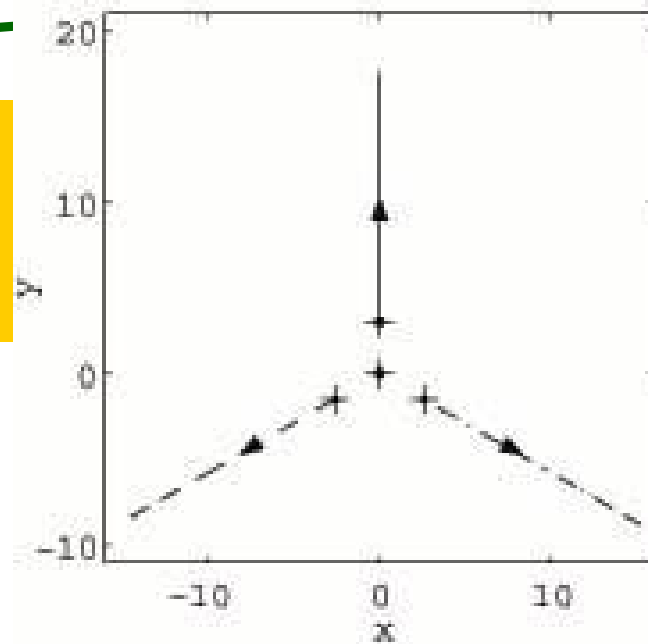
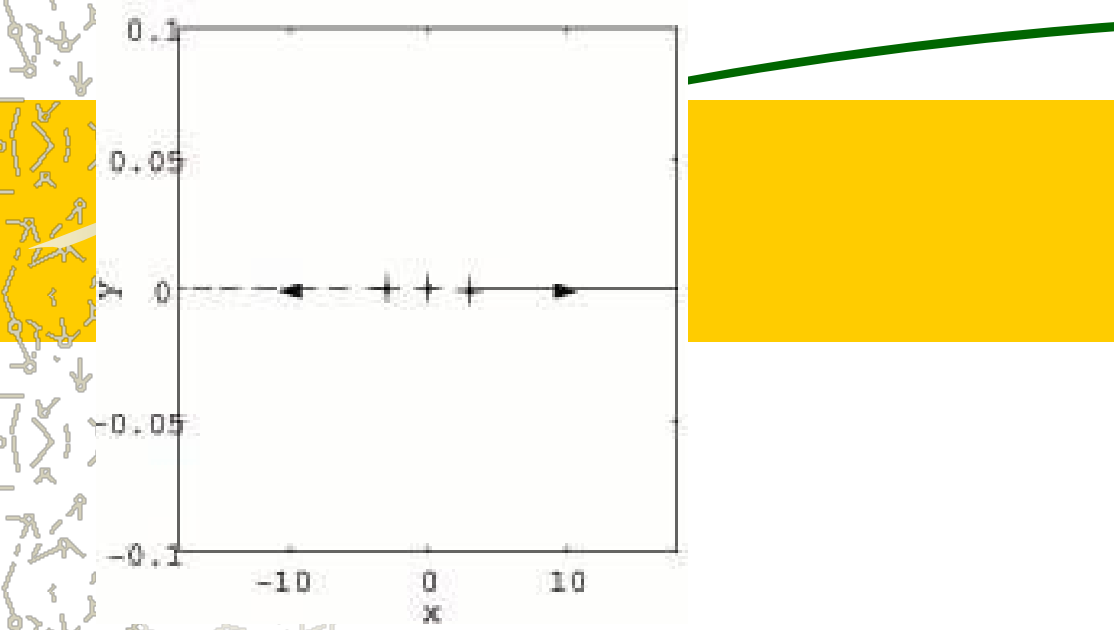
$$\vec{x}_N^0 = \vec{0}; \quad \vec{x}_j^0 = a \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for GLE figure

$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = \sqrt{a^2 + \frac{2N}{\kappa}t} \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for NLSE figure next

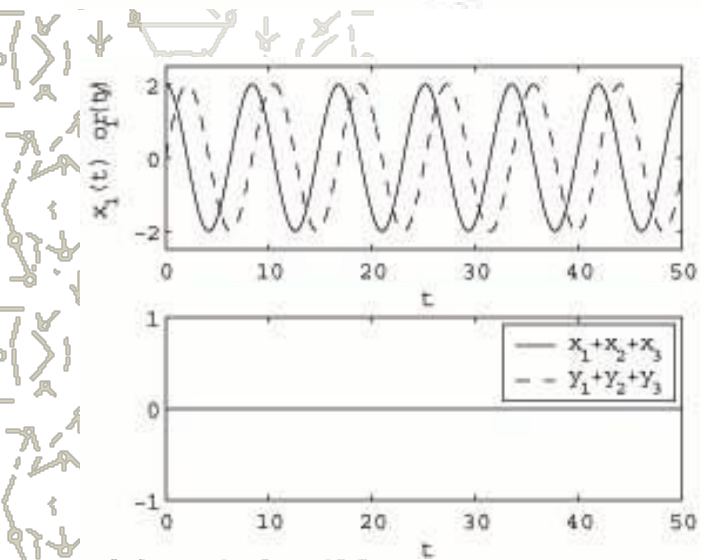
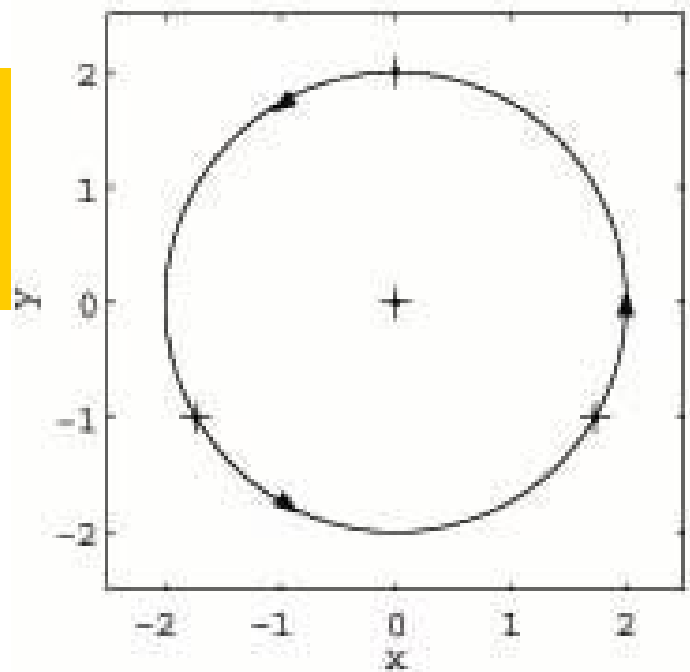
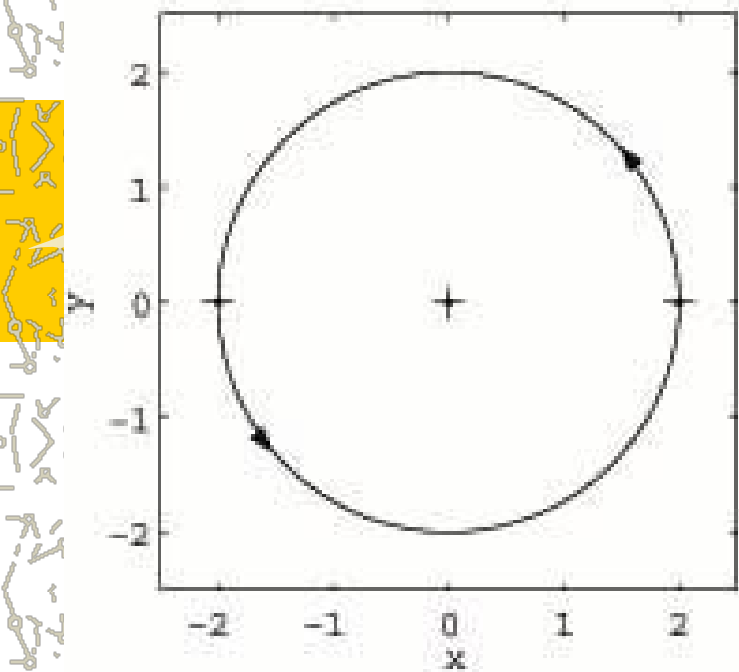
$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = a \left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right), \sin\left(\frac{2j\pi}{N-1} + \frac{N-2}{a^2}t\right) \right)$$



N=3

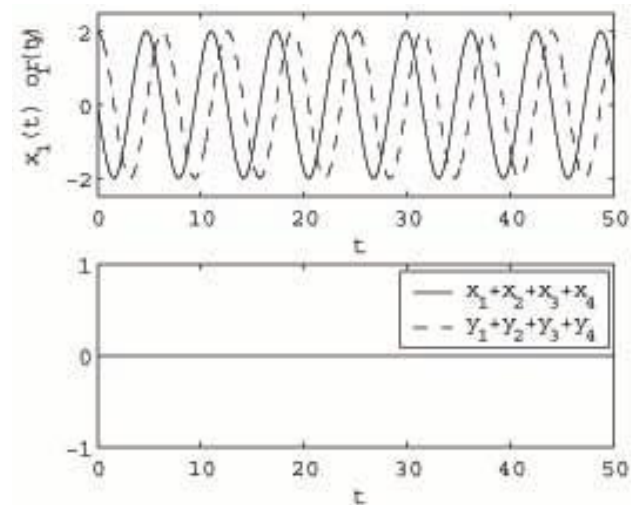
[back](#)

N=4



$N=3$

[back](#)



$N=4$

Analytical Solutions

- Two **opposite** vortices (Bao, Du & Zhang, SIAP, 07')

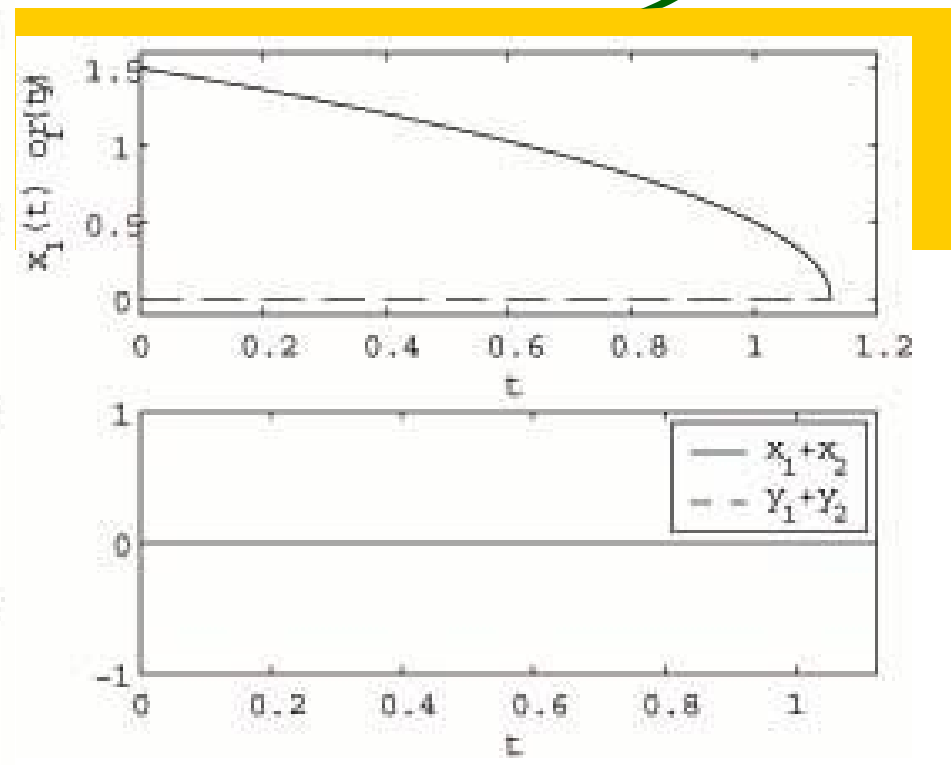
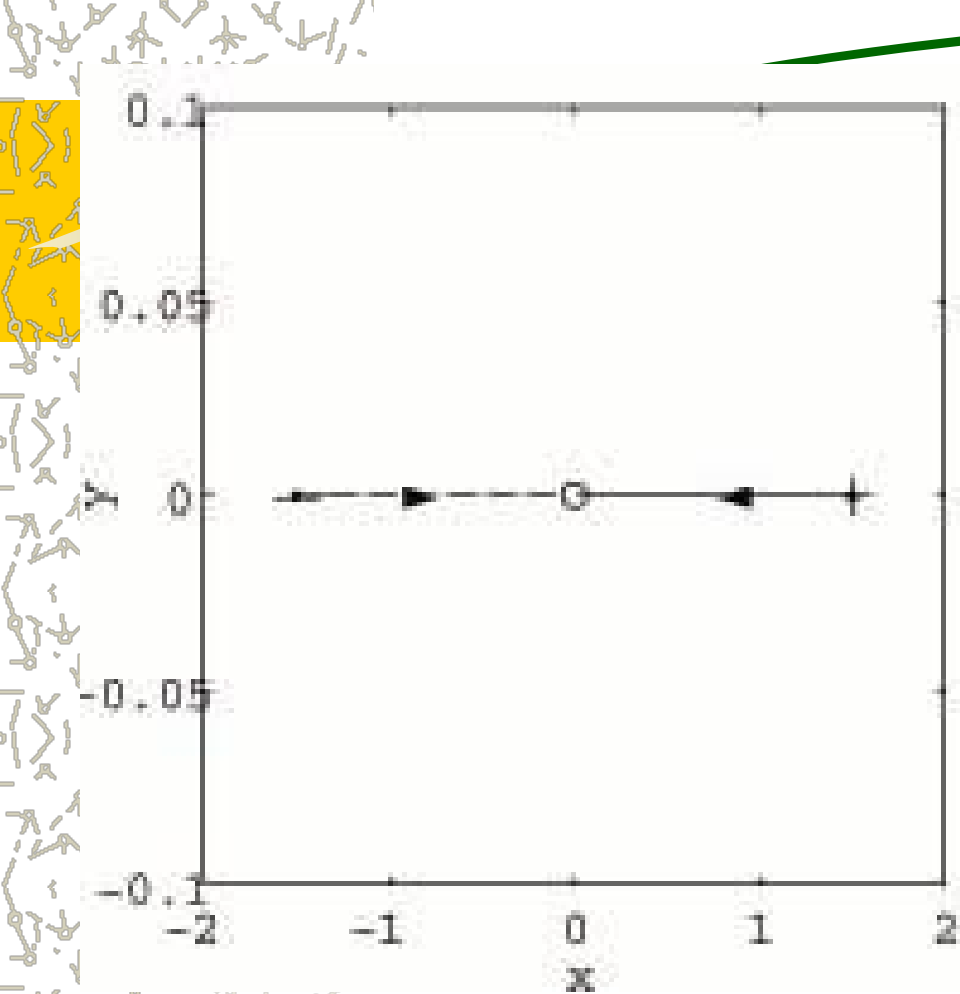
$$\vec{x}_1^0 = -\vec{x}_2^0 = a \left(\cos(\theta_0) \quad , \quad \sin(\theta_0) \right), \quad m_1 = -m_2 = 1$$

- Analytical solutions for **GLE** [figure](#)

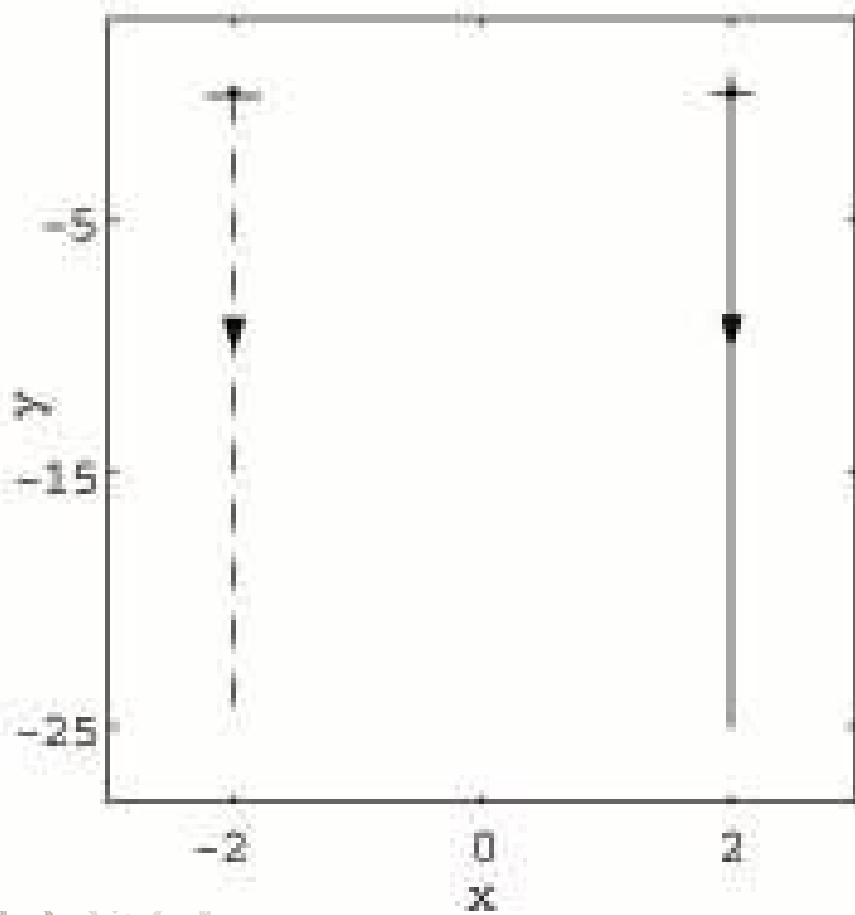
$$\vec{x}_1(t) = -\vec{x}_2(t) = \sqrt{a^2 - \frac{2}{\kappa}t} \left(\cos(\theta_0) \quad , \quad \sin(\theta_0) \right)$$

- Analytical solutions for **NLSE** [figure](#) [next](#)

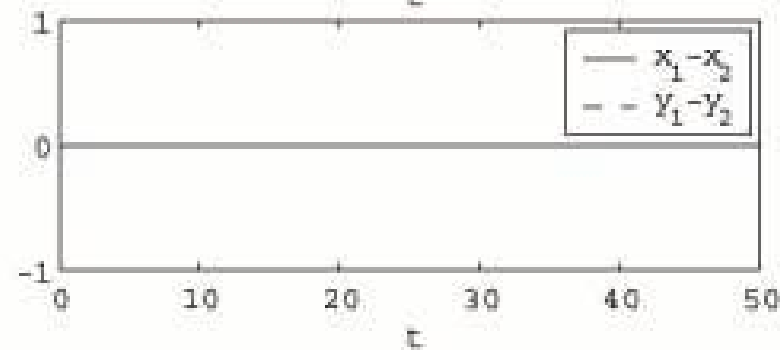
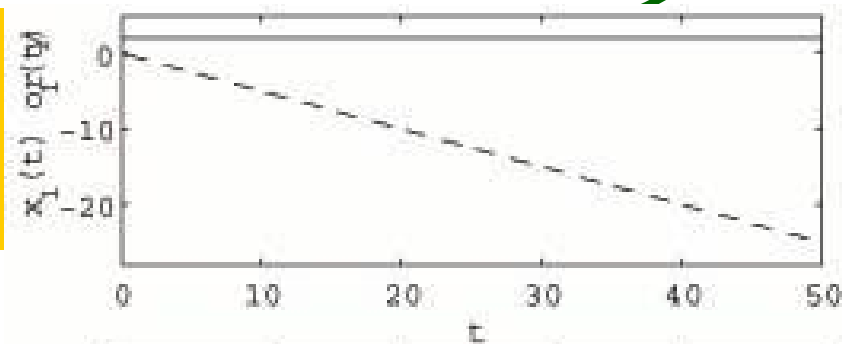
$$\vec{x}_j(t) = \vec{x}_j^0 + \frac{t}{a} \left(-\sin(\theta_0) \quad , \quad \cos(\theta_0) \right), \quad j = 1, 2$$



[back](#)



$x_1(t)$ or $y_1(t)$



[back](#)

Analytical Solutions

- $N (\geq 3)$ opposite vortices on a **circle and its center**

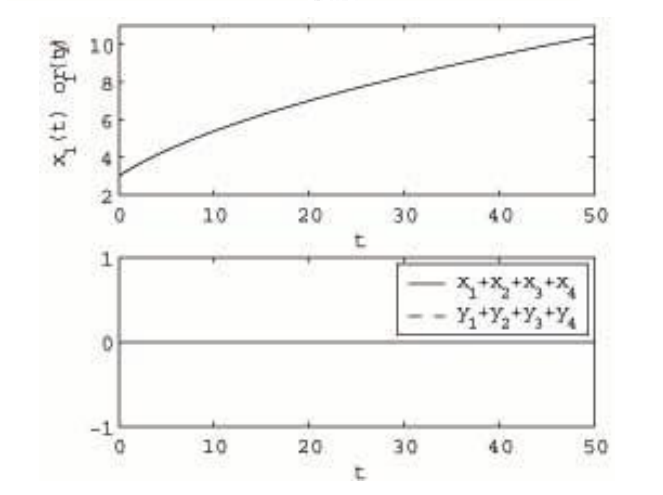
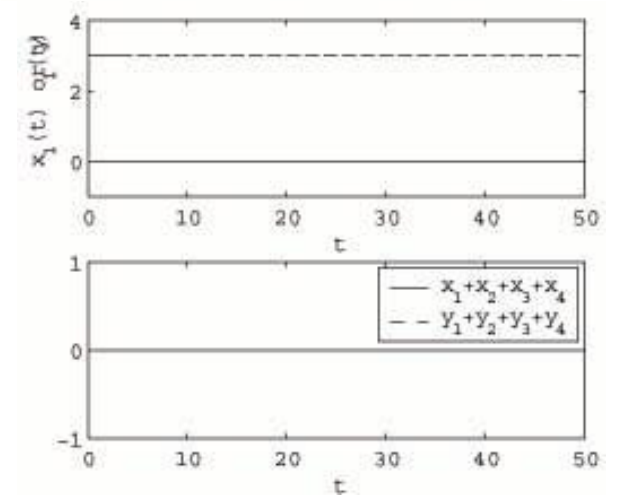
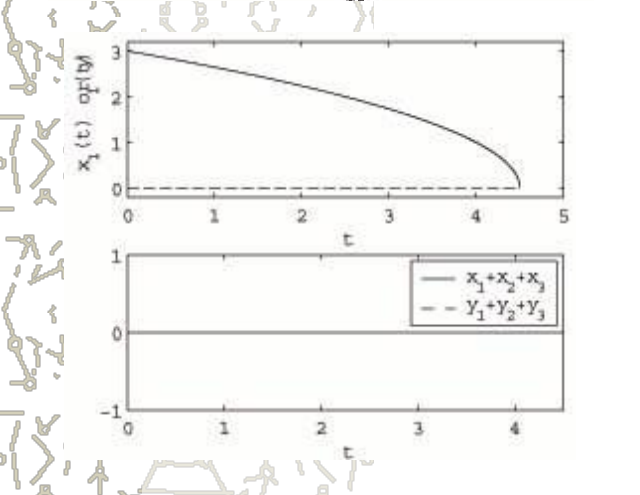
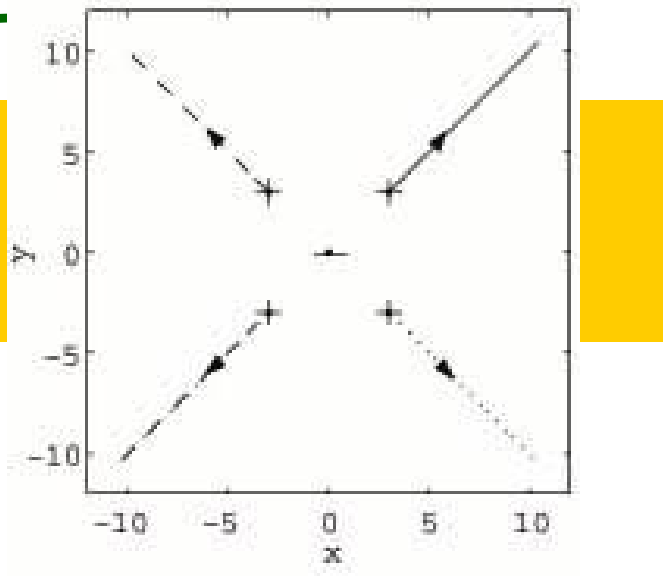
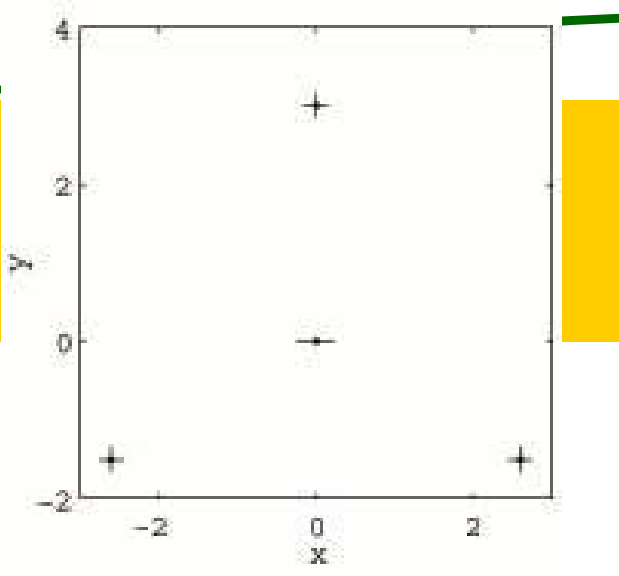
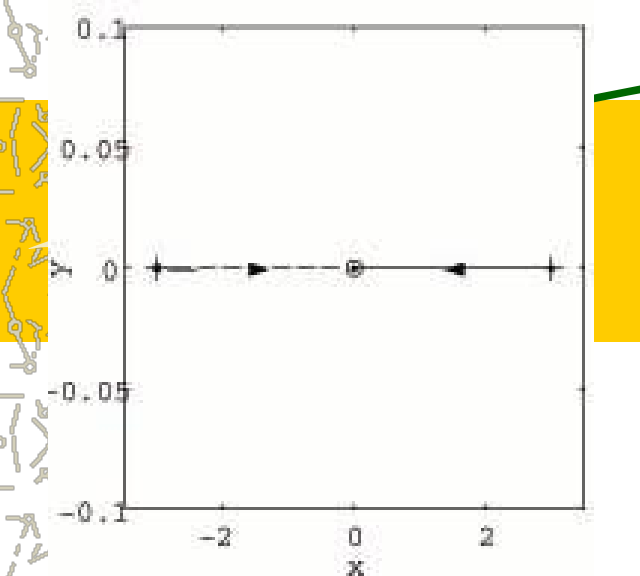
$$\vec{x}_N^0 = \vec{0} \quad (-); \quad \vec{x}_j^0 = a \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right) \quad (+), \quad 1 \leq j \leq N-1$$

- Analytical solutions for **GLE** [figure](#)

$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = \sqrt{a^2 + \frac{2(N-4)}{\kappa}t} \left(\cos\left(\frac{2j\pi}{N-1}\right), \sin\left(\frac{2j\pi}{N-1}\right) \right), \quad 1 \leq j \leq N-1$$

- Analytical solutions for **NLSE** [figure](#) [next](#)

$$\vec{x}_N(t) = \vec{0}; \quad \vec{x}_j(t) = a \left(\cos\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right), \sin\left(\frac{2j\pi}{N-1} + \frac{N-4}{a^2}t\right) \right)$$

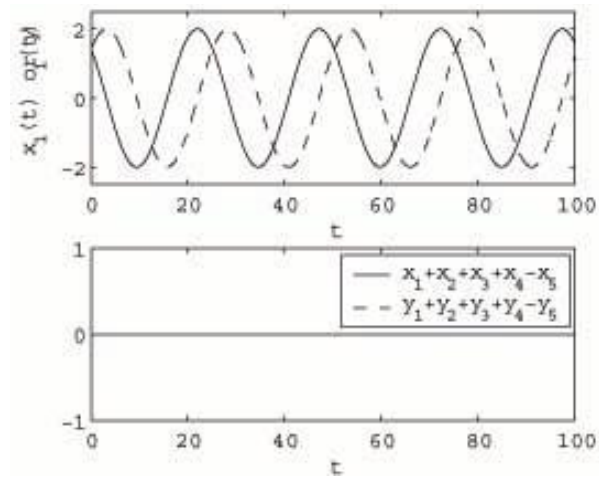
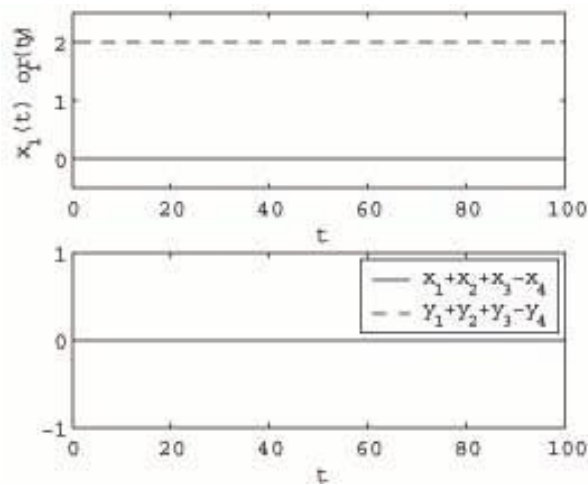
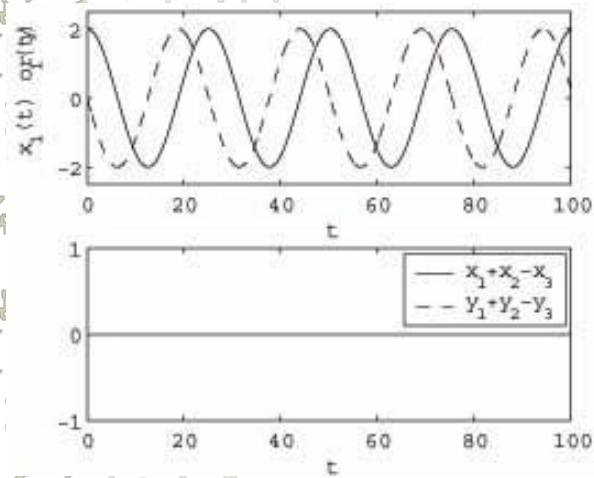
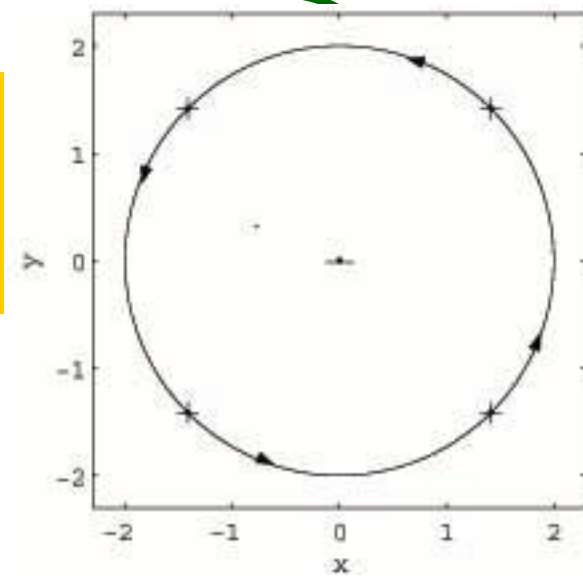
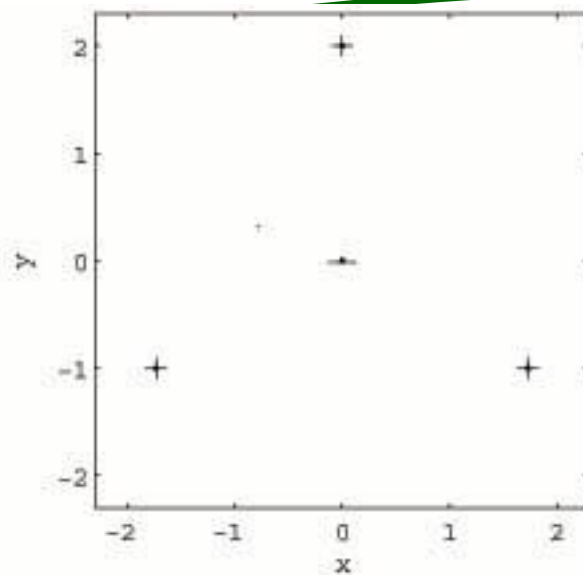
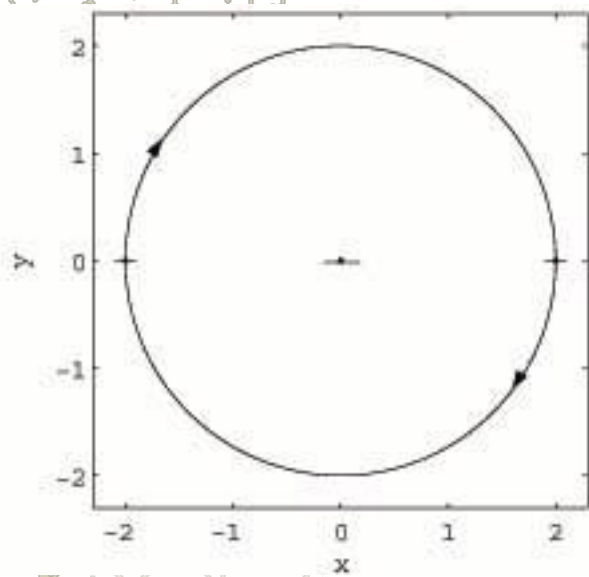


N=3

N=4

N=5

[back](#)

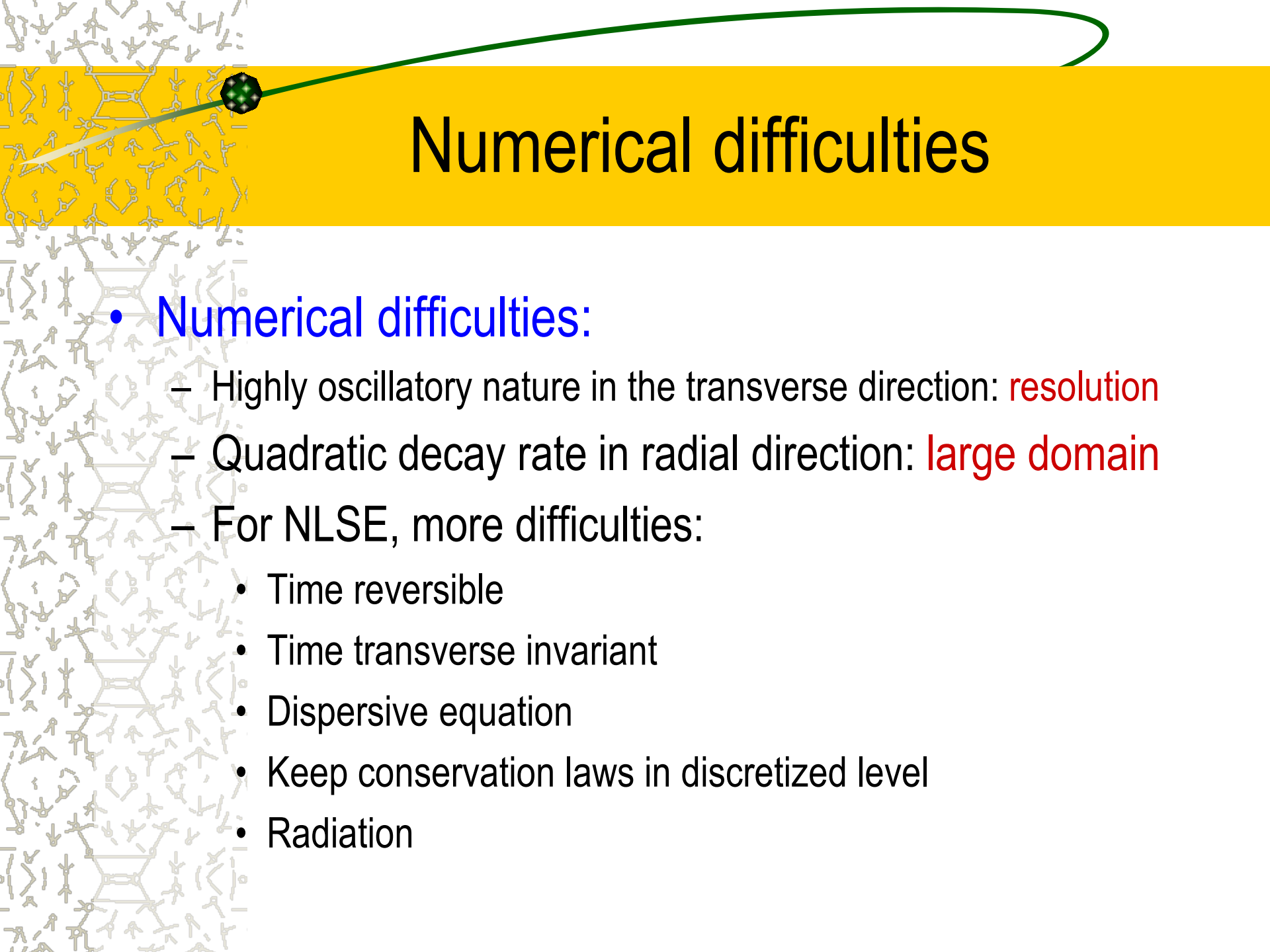


N=3

N=4

N=5

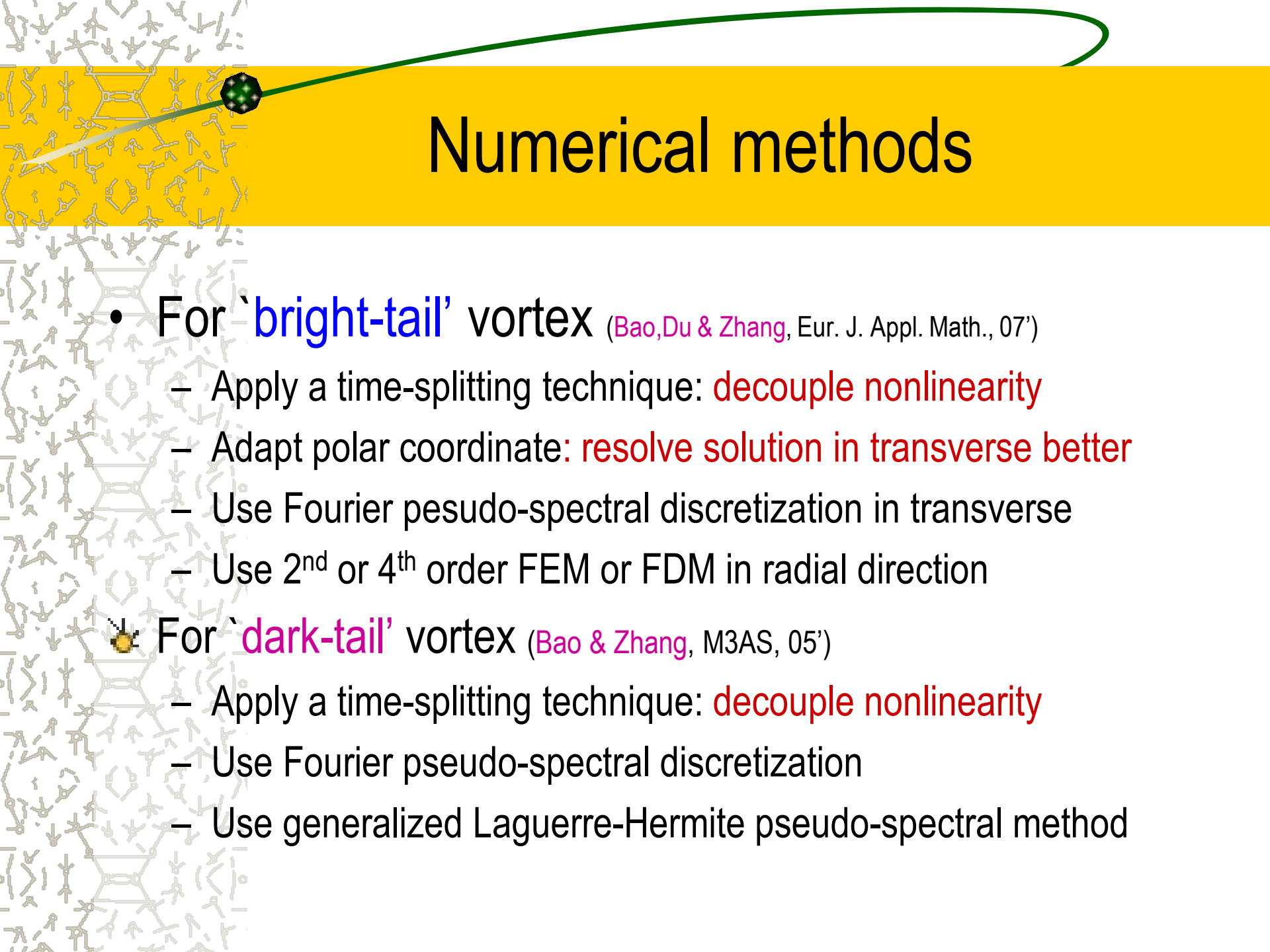
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Numerical difficulties

- Numerical difficulties:

- Highly oscillatory nature in the transverse direction: resolution
- Quadratic decay rate in radial direction: large domain
- For NLSE, more difficulties:
 - Time reversible
 - Time transverse invariant
 - Dispersive equation
 - Keep conservation laws in discretized level
 - Radiation



Numerical methods

- For 'bright-tail' vortex (Bao, Du & Zhang, Eur. J. Appl. Math., 07')
 - Apply a time-splitting technique: decouple nonlinearity
 - Adapt polar coordinate: resolve solution in transverse better
 - Use Fourier pseudo-spectral discretization in transverse
 - Use 2nd or 4th order FEM or FDM in radial direction
- For 'dark-tail' vortex (Bao & Zhang, M3AS, 05')
 - Apply a time-splitting technique: decouple nonlinearity
 - Use Fourier pseudo-spectral discretization
 - Use generalized Laguerre-Hermite pseudo-spectral method

Vortex dynamics & interaction

- Data chosen (Bao, Du & Zhang, SIAM, J. Appl. Math., 07')

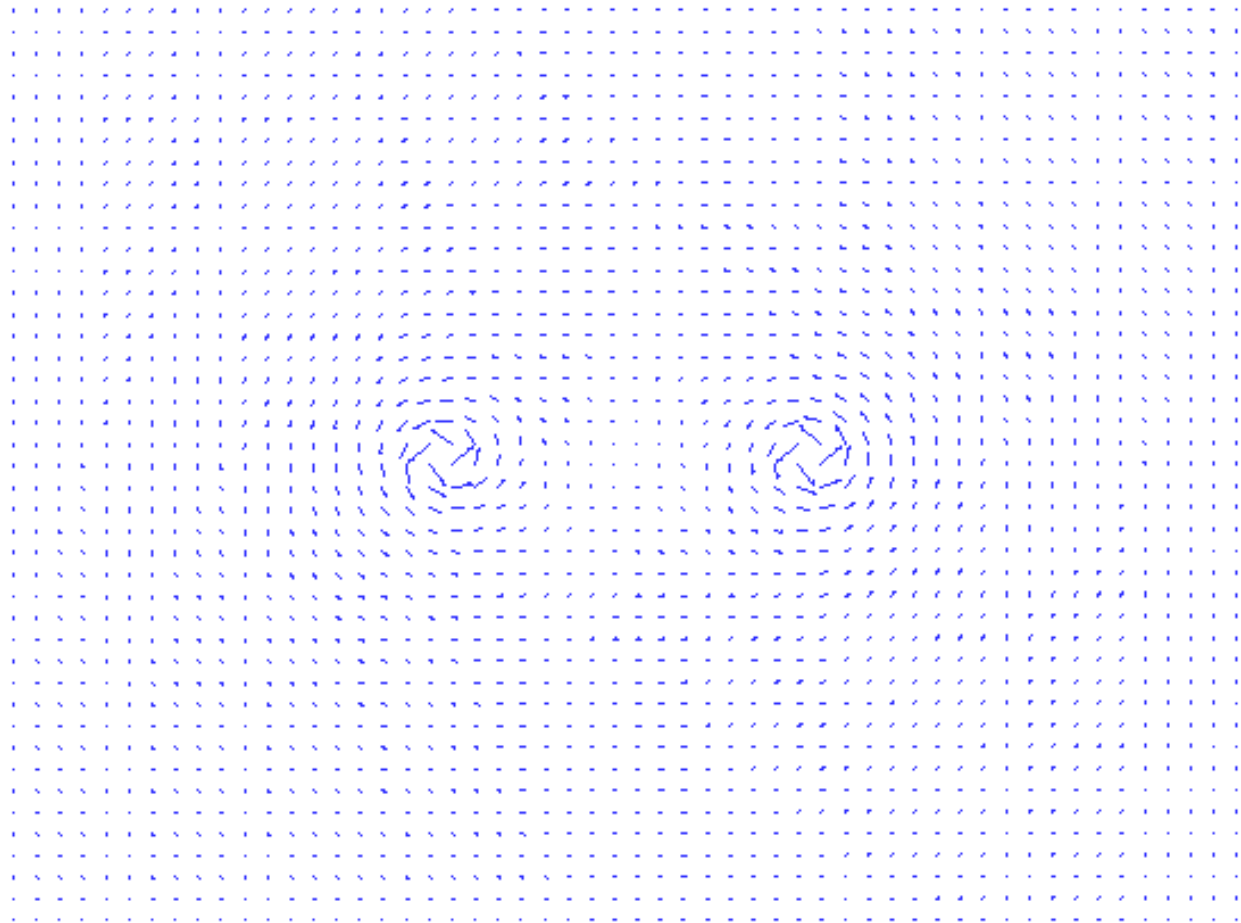
$$\varepsilon = 1, \quad V(\vec{x}, t) \equiv 1, \quad \psi_0(\vec{x}) = \prod_{j=1}^N \phi_{n_j}(\vec{x} - \vec{x}_j), \quad m = \sum_{j=1}^N n_j$$

- Two vortices with like winding numbers

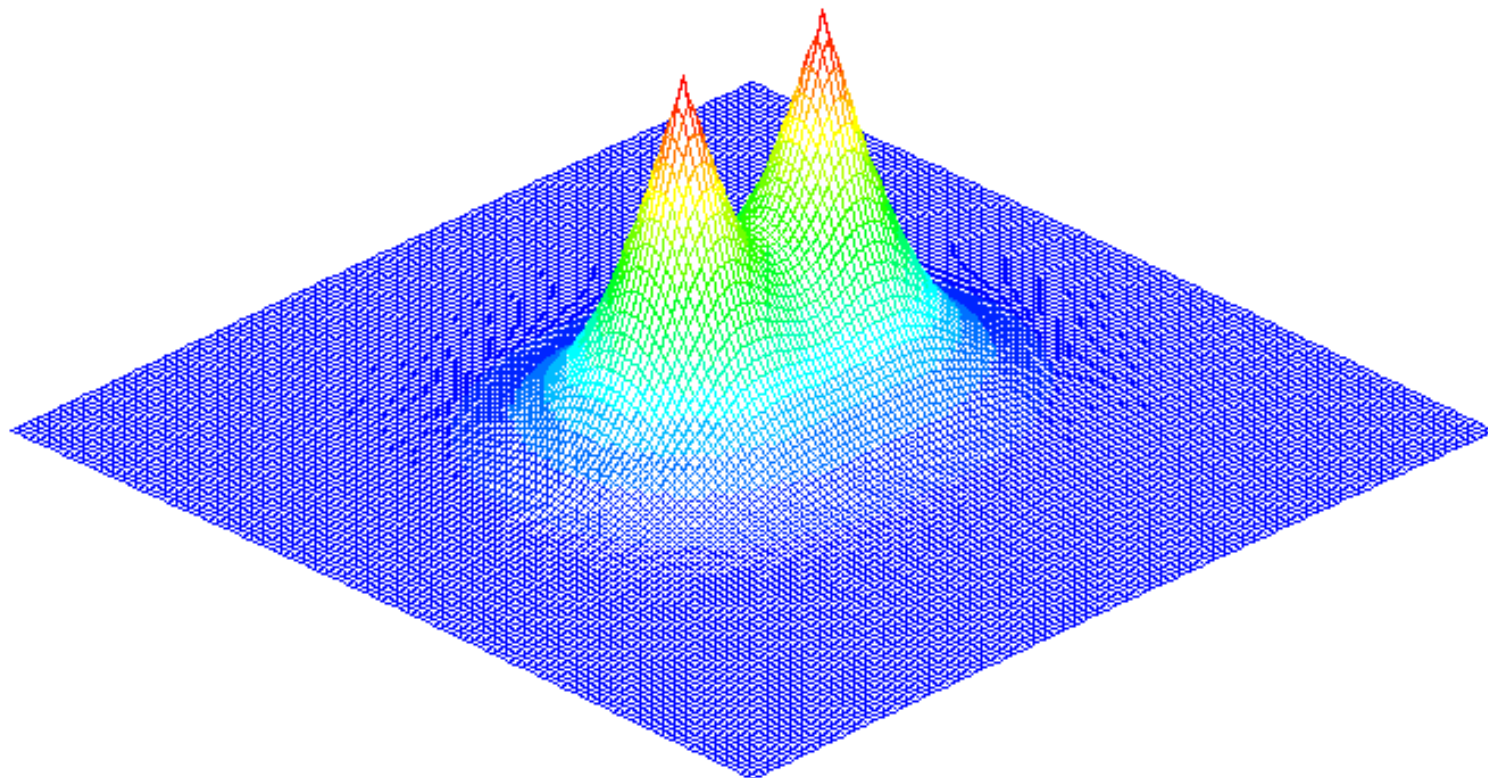
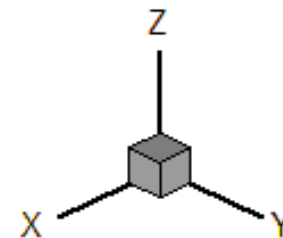
$$N = 2, \quad n_1 = 1, \quad n_2 = 1, \quad \vec{x}_1 = (a, 0), \quad \vec{x}_2 = (-a, 0)$$

- For GLE: velocity density
- For GPE: velocity density
- Trajectory

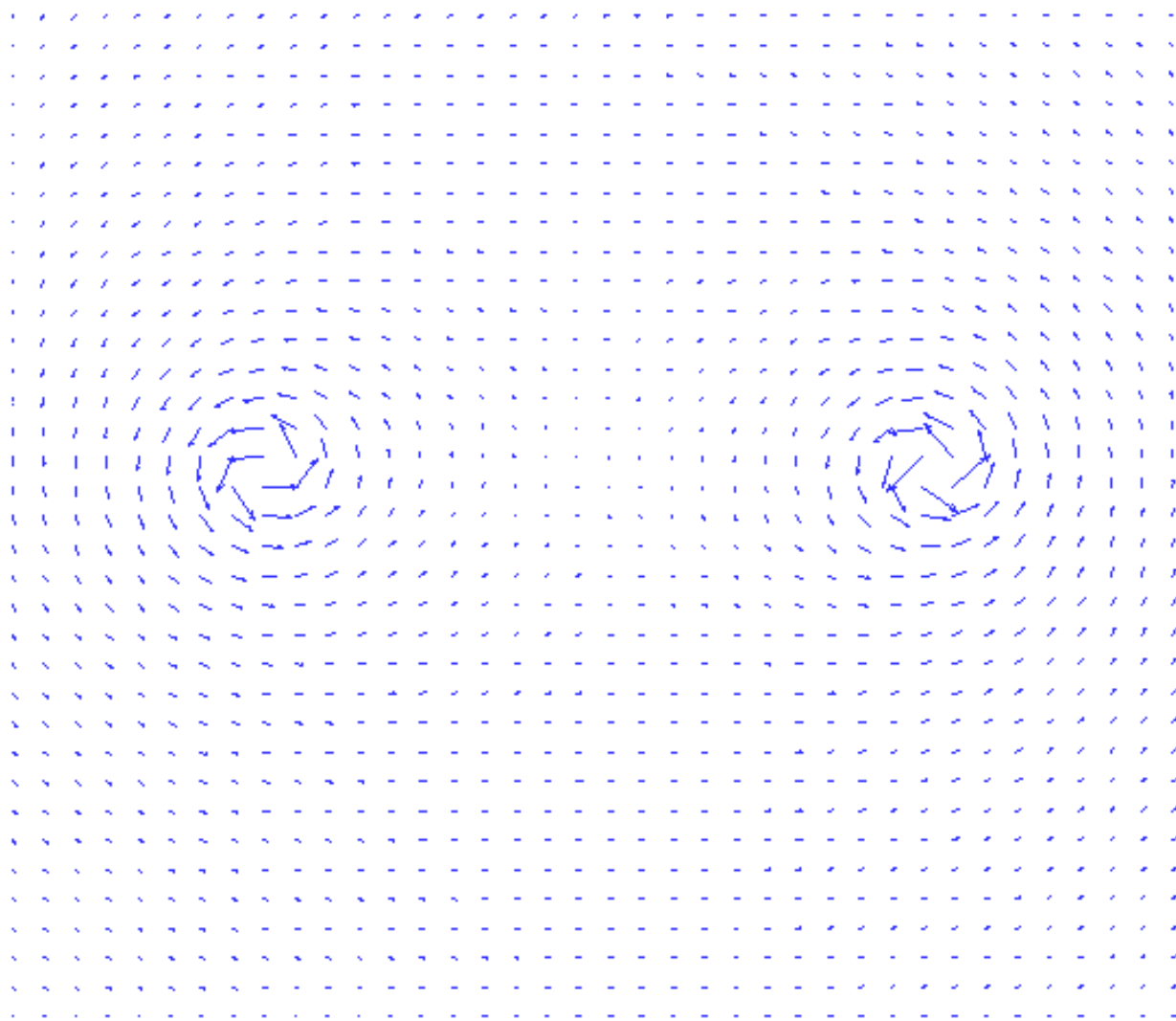
- Summary



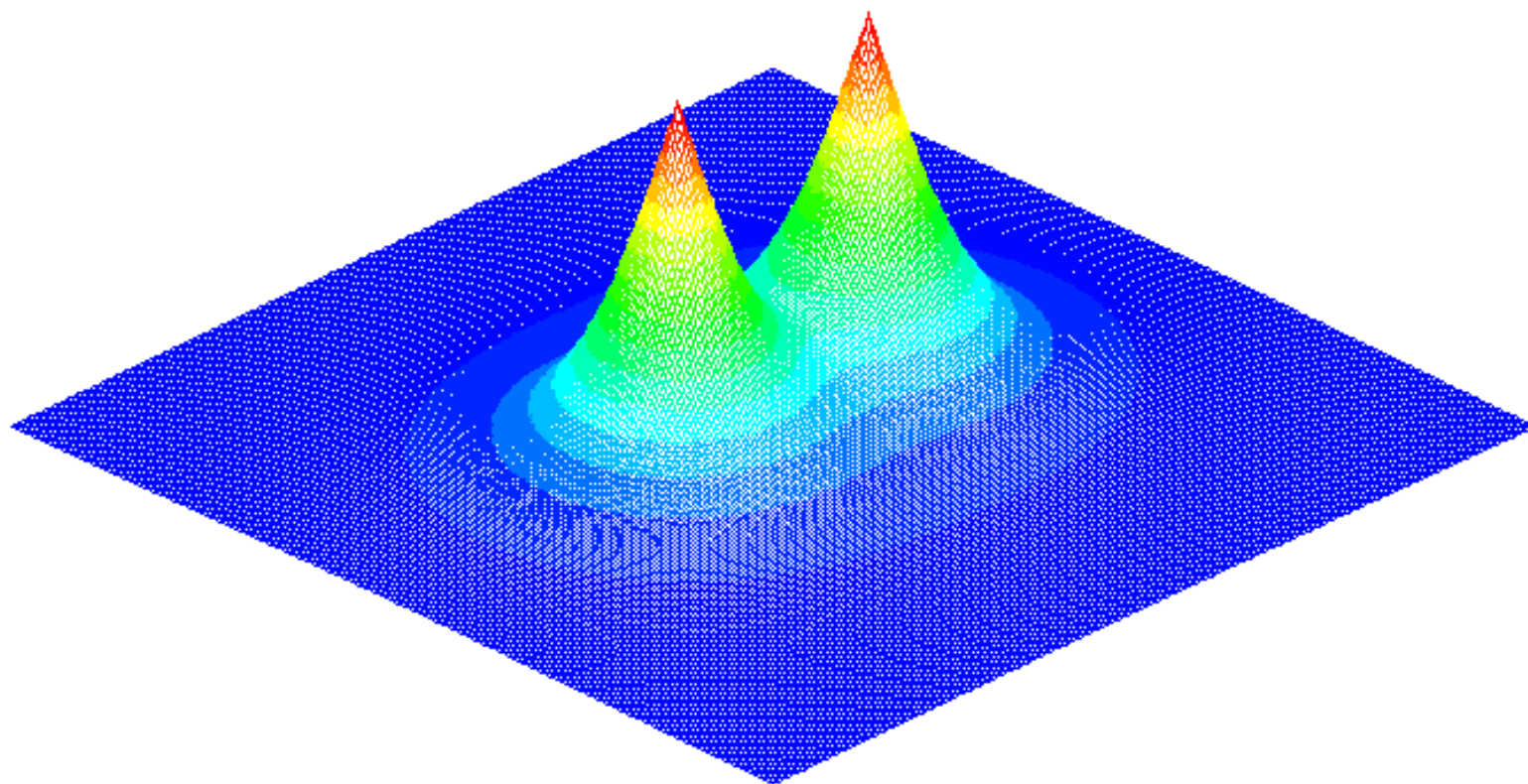
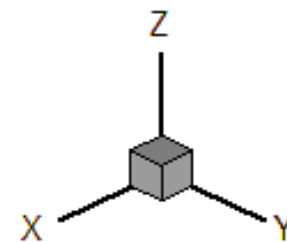
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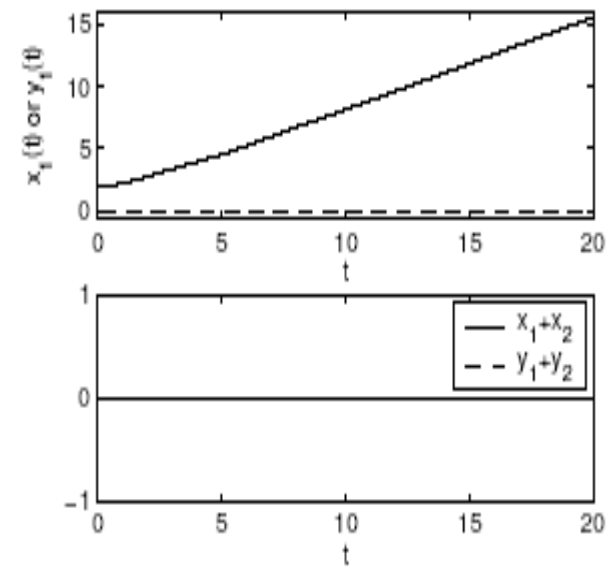
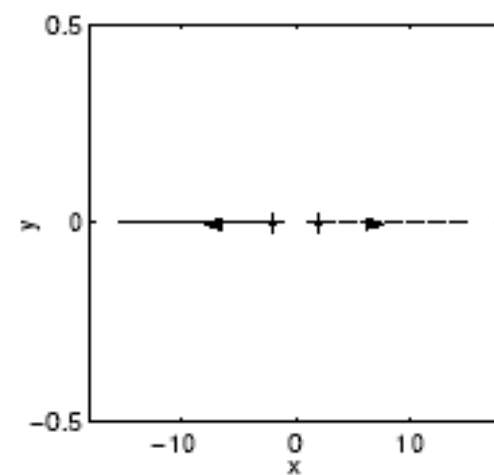
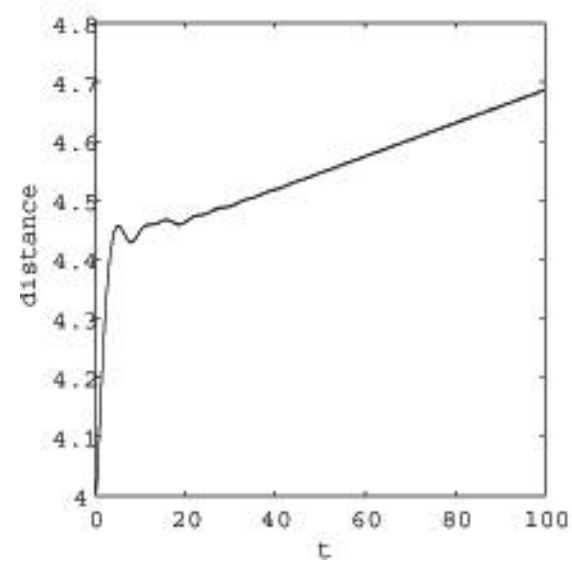
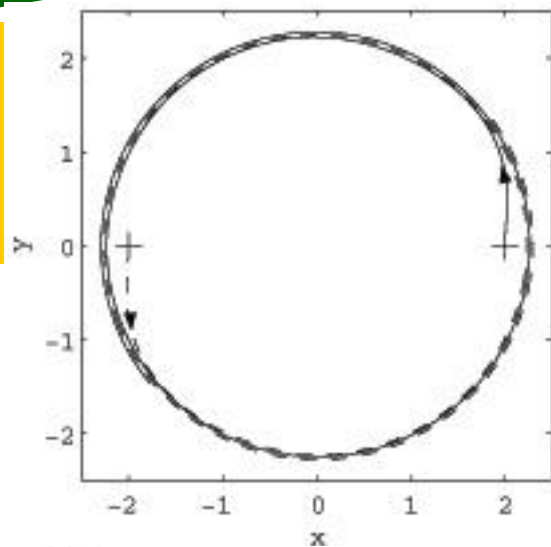
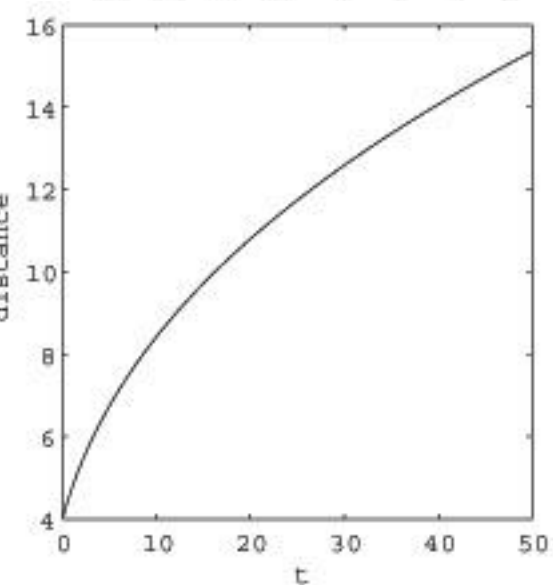
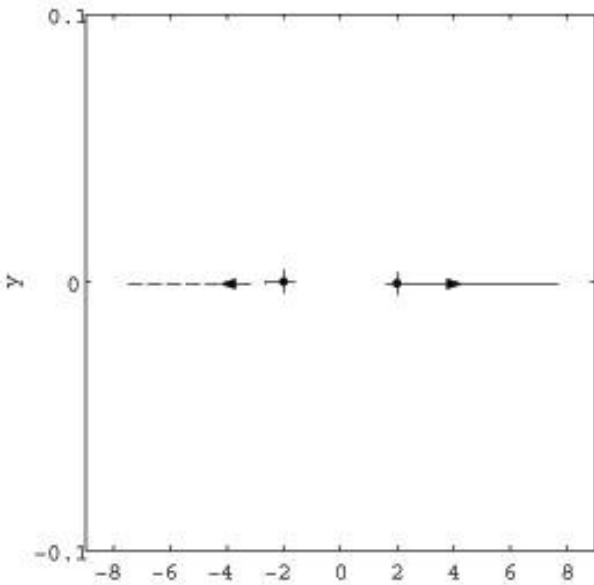
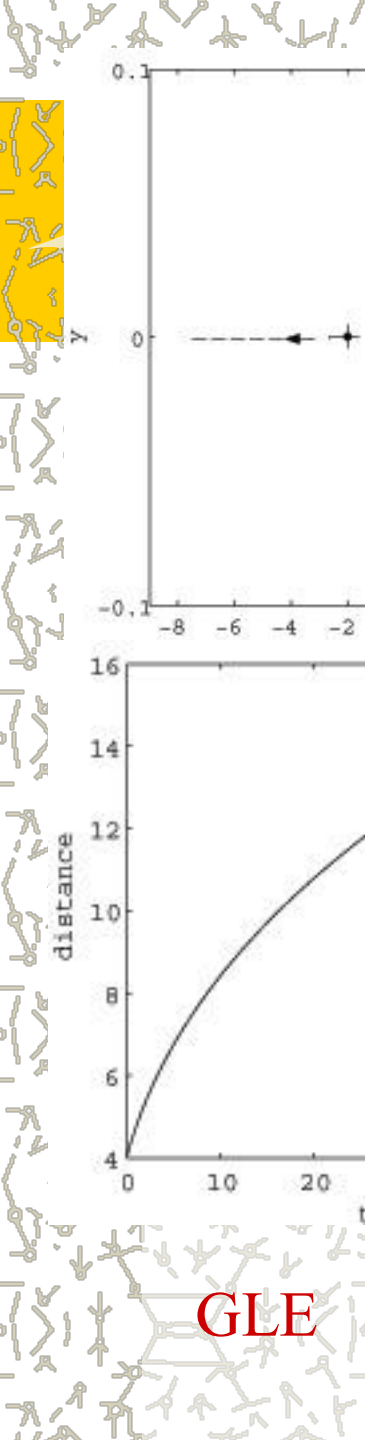
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GLE

GPE

NLWE

back

Summary of interaction

- **For GLE** (Bao, Du & Zhang, SIAP, 07')
 - Undergo a repulsive interaction
 - Core size of the two vortices will well separated
 - Energy decreasing: $E(\psi(\vec{x}, t)) \rightarrow 2E(\psi_1(\vec{x})), \quad t \rightarrow \infty$
- **For NLSE** (Bao, Du & Zhang, SIAP, 07')
 - Behave like point vortices in ideal fluid
 - The two vortex centers move almost along a circle
 - Energy conservation & radiation
- **For NLWE** (Bao & Zhang, Physica D 07')
 - Similar as GLE but with different speed

Vortex dynamics & interaction

- Two vortices with opposite winding numbers

$$N=2, \quad n_1=1, \quad n_2=-1, \quad \vec{x}_1=(a,0), \quad \vec{x}_2=(-a,0)$$

– For GLE:

velocity

density

– For NLSE:

- Case 1:

velocity

density

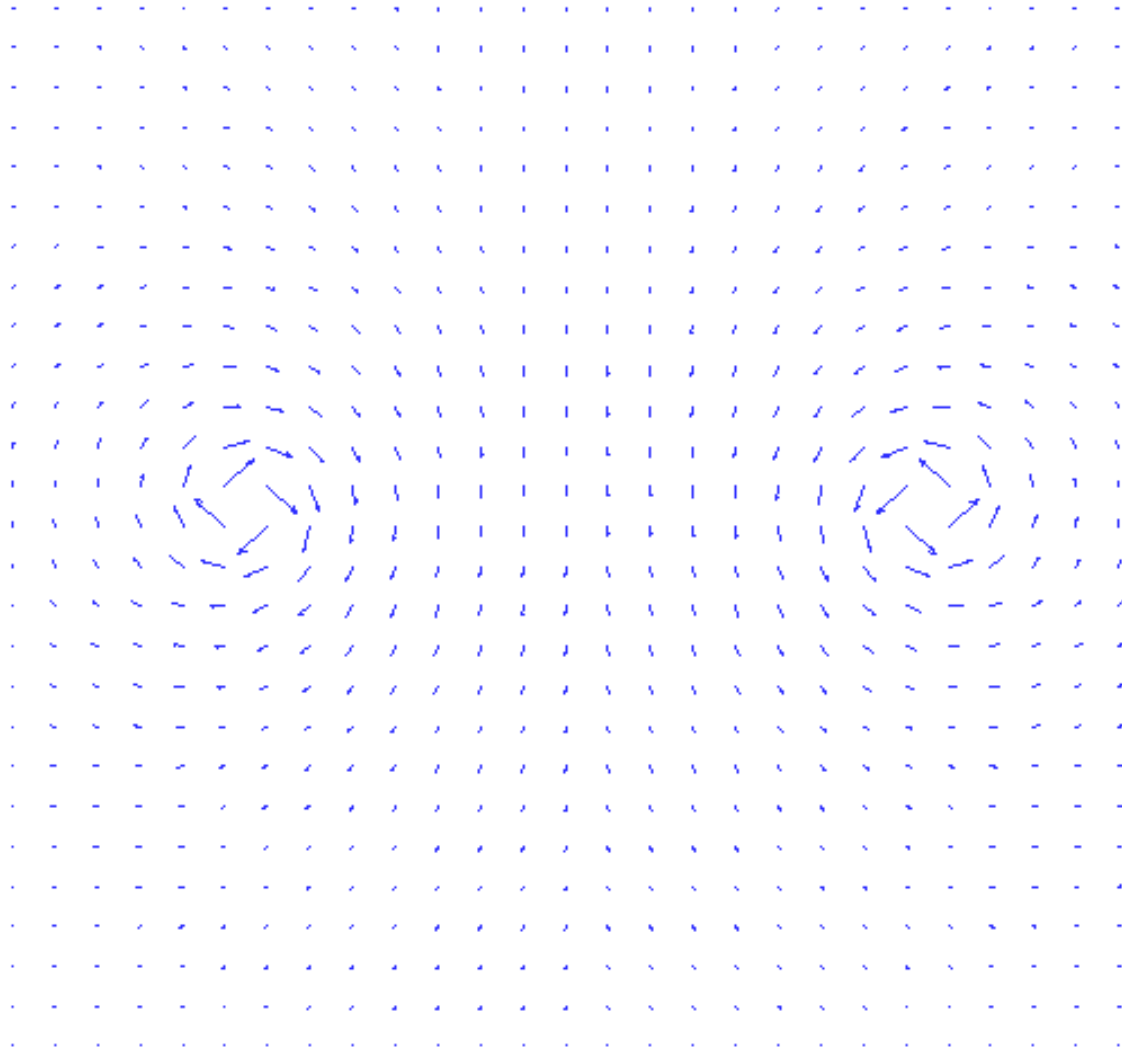
- Case 2:

velocity

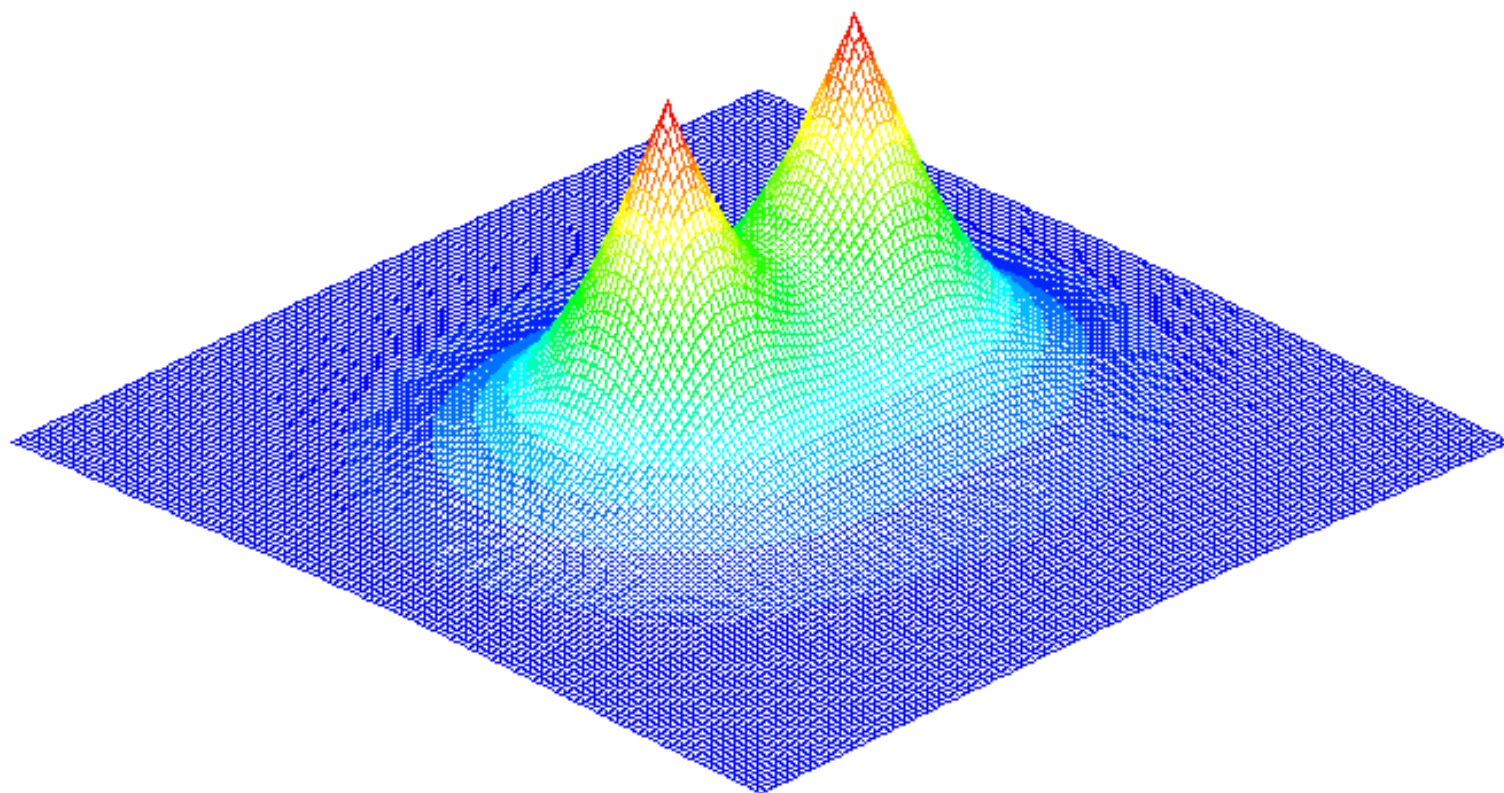
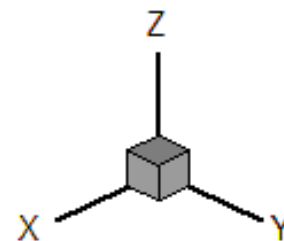
density

– Trajectory

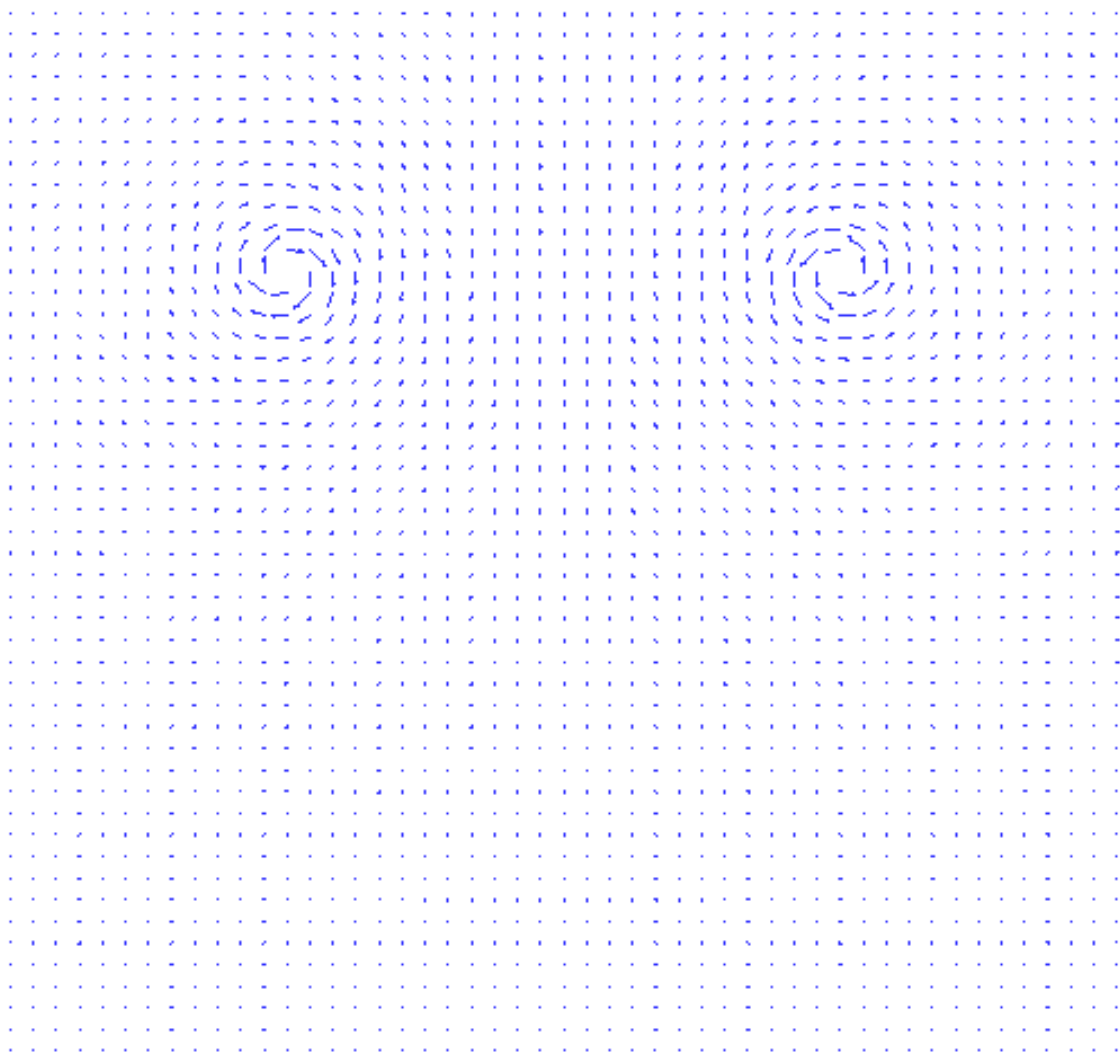
- Summary



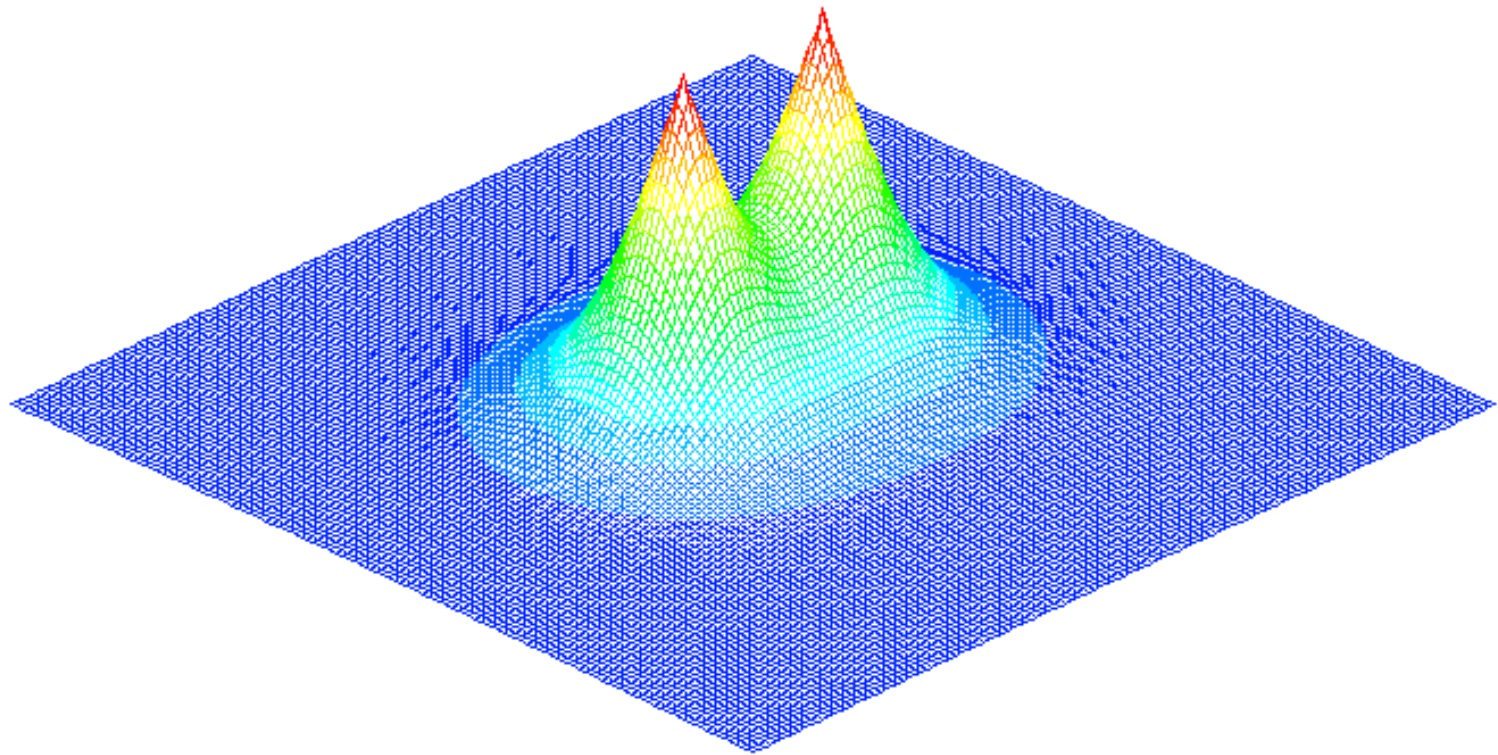
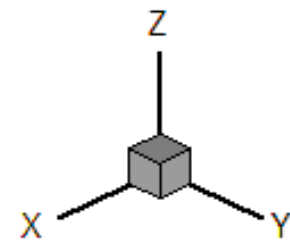
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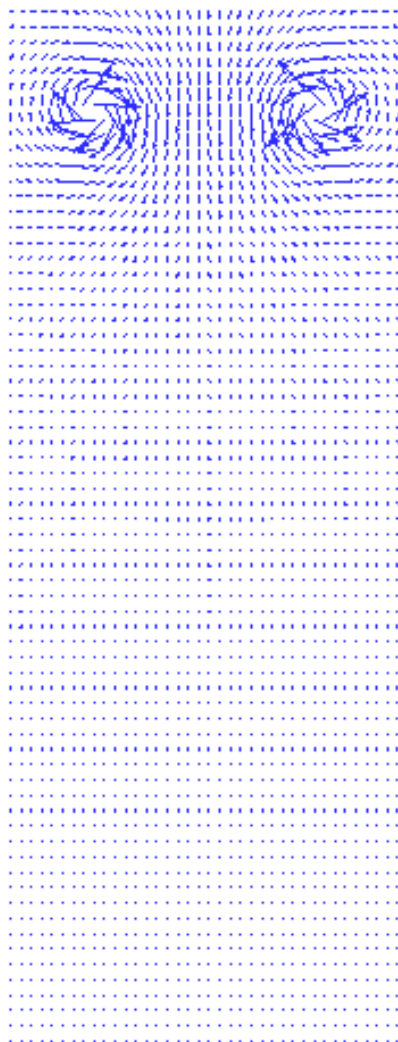
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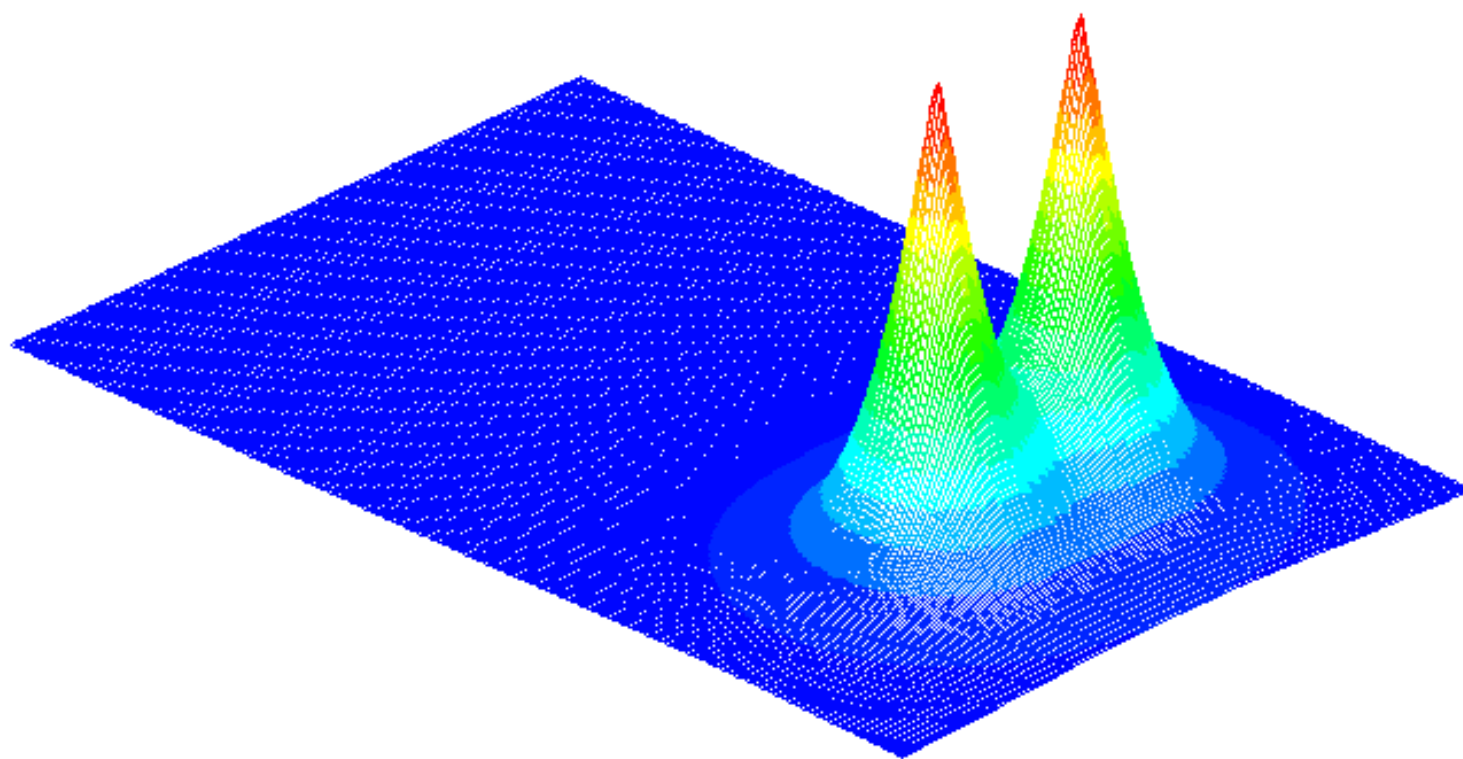
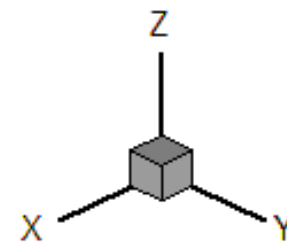
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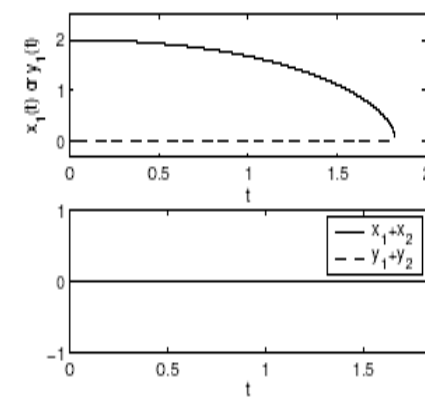
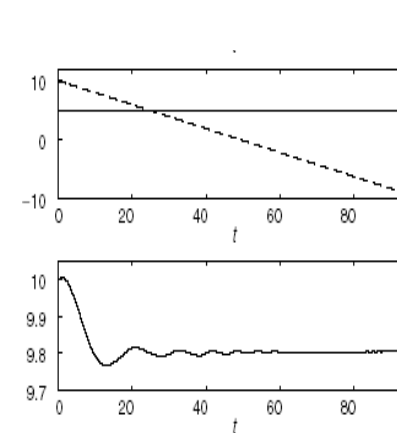
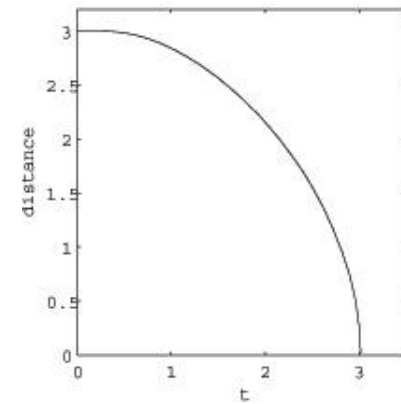
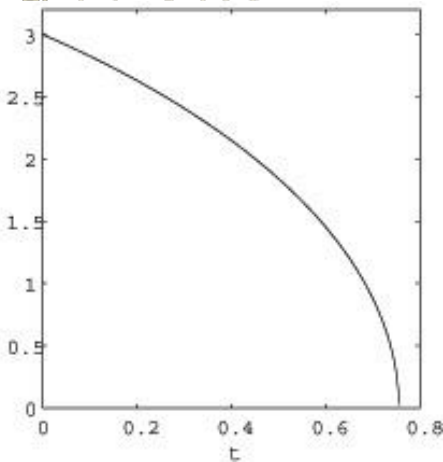
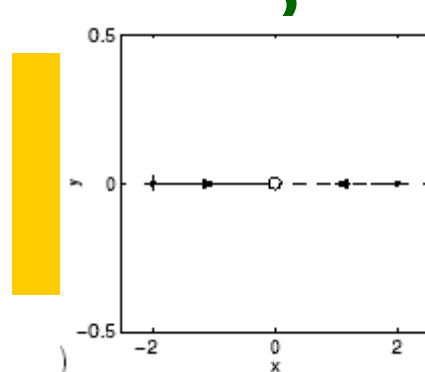
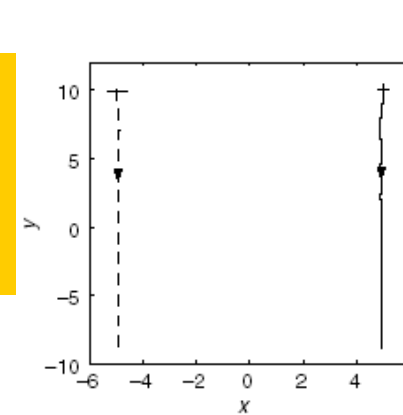
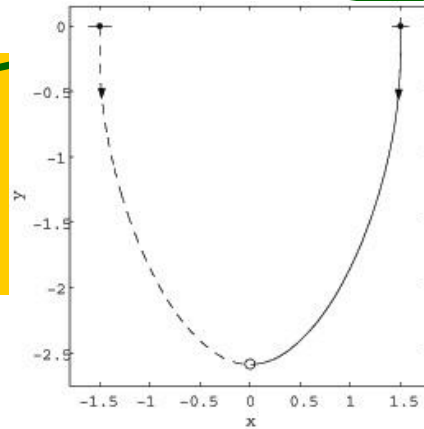
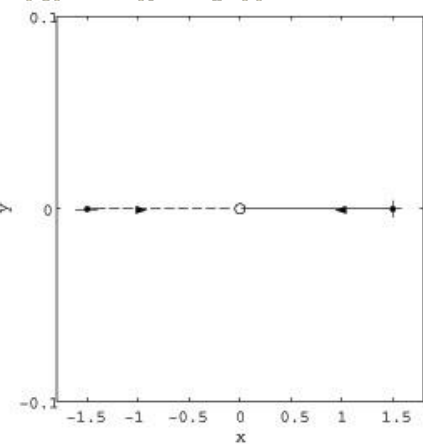
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GLE

NLSE

NLSE

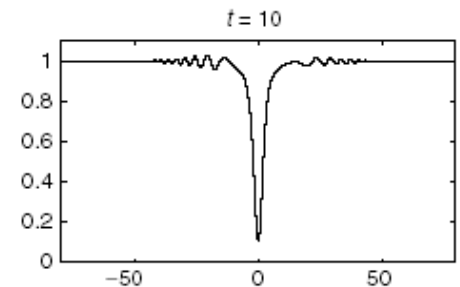
NLWE

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Summary of interaction

- For GLE (Bao, Du & Zhang, SIAP, 07')
 - Undergo an attractive interaction & merge at finite time
 - Energy decreasing: $E(\psi(\vec{x}, t)) \rightarrow 0, \quad t \rightarrow \infty$

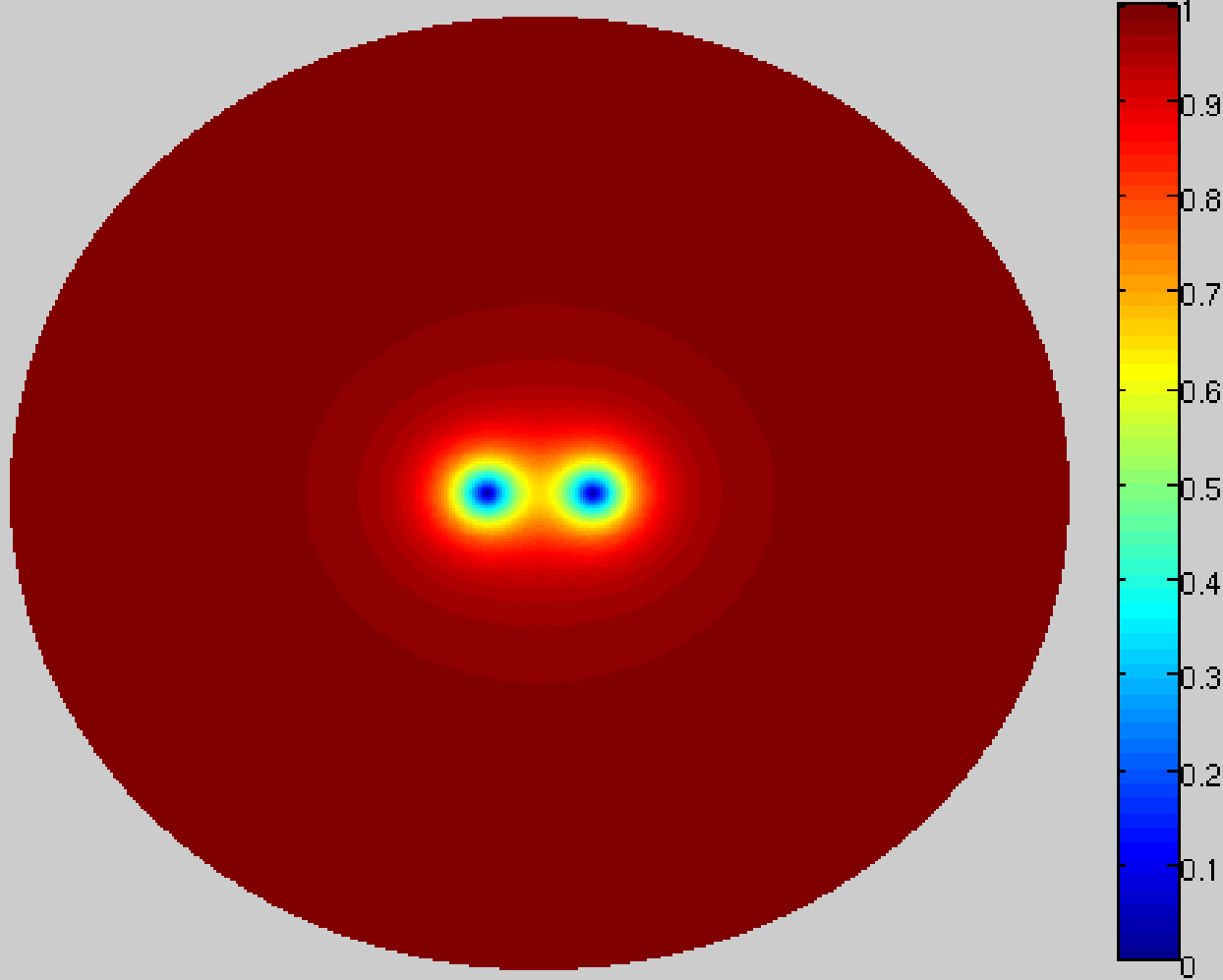
- For GPE (Bao, Du & Zhang, SIAP, 07')
 - depends on initial distance of the two centers
 - Small: merge at finite time & generate shock wave
 - Large: move almost paralleling & don't merge, solitary wave
 - Energy conservation & radiation
 - Sound wave generation



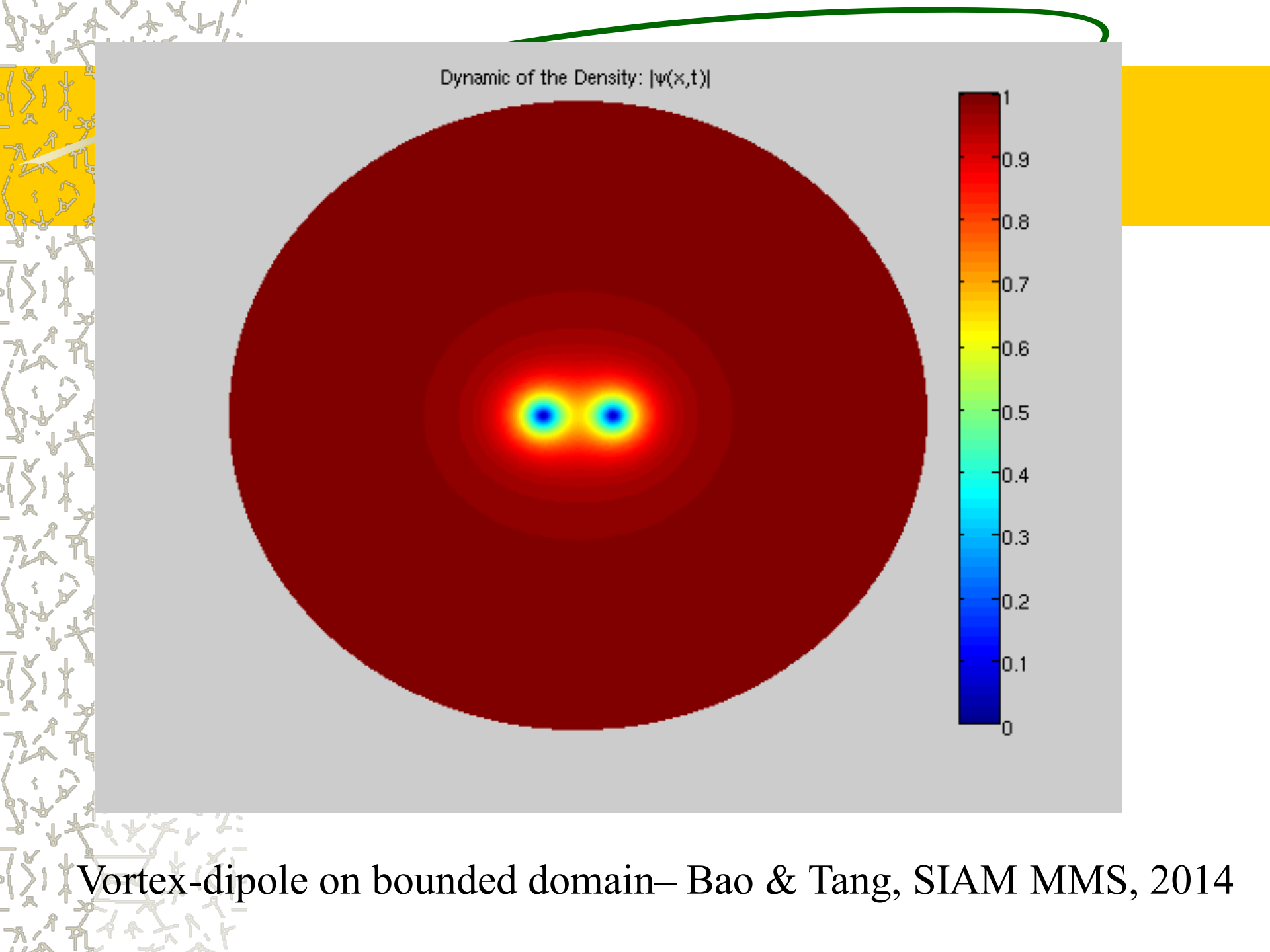
- For NLWE (Bao & Zhang, Physica D 07')

- Undergo an attractive interaction & merge at finite time

Dynamic of the Density: $|\psi(x,t)|$



Vortex-pair on bounded domain – Bao & Tang, SIAM MMS, 2014



Vortex-dipole on bounded domain– Bao & Tang, SIAM MMS, 2014

Dynamics of vortices in weakly interacting Bose-Einstein condensates

Alexander Klein and Dieter Jaksch

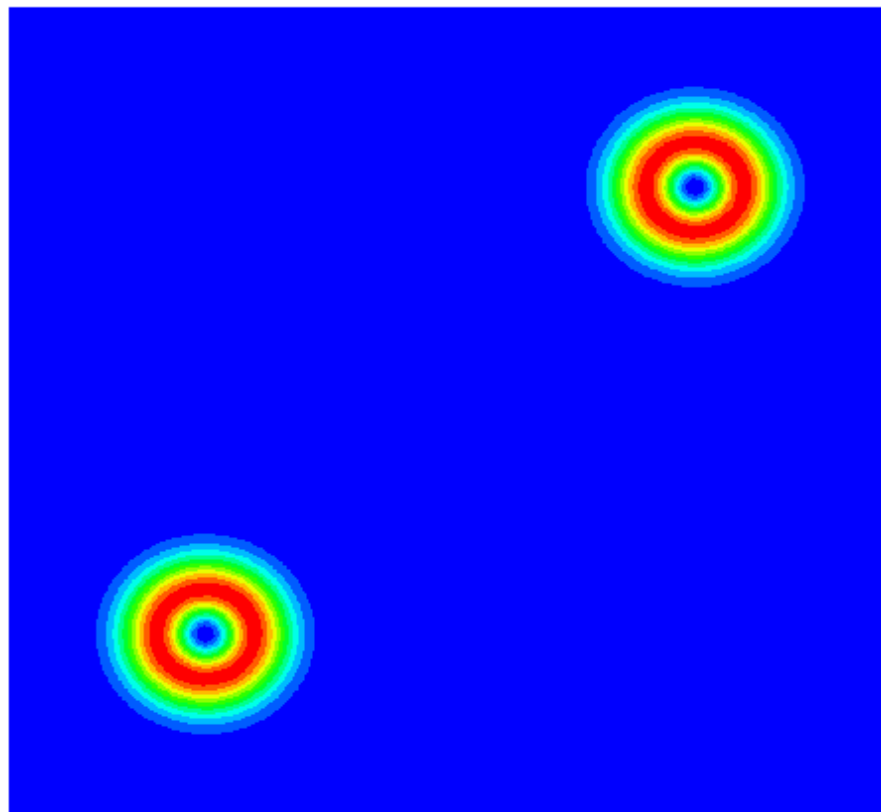
*Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom
and Keble College, Parks Road, Oxford OX1 3PG, United Kingdom*

Yanzhi Zhang and Weizhu Bao

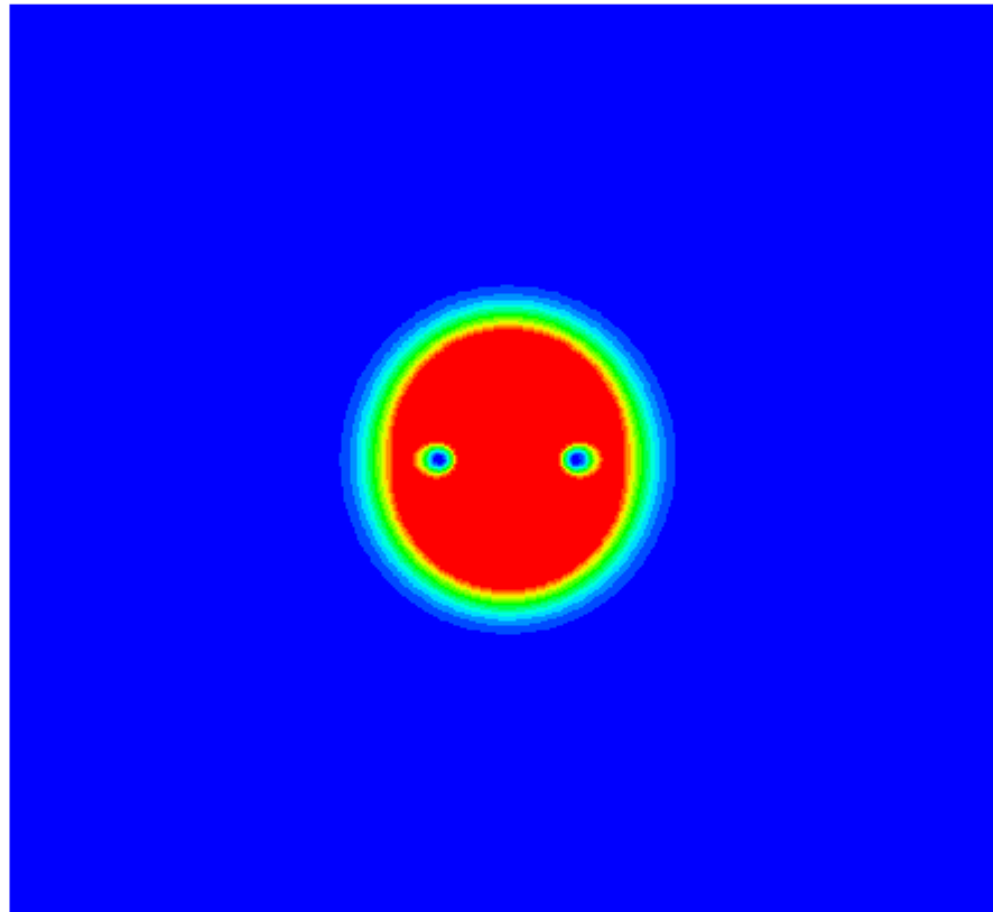
*Department of Mathematics and Center for Computational Science and Engineering, National University of Singapore,
Singapore 117543*

(Received 3 August 2007; published 2 October 2007)

Frame001 | 23 Nov 2005

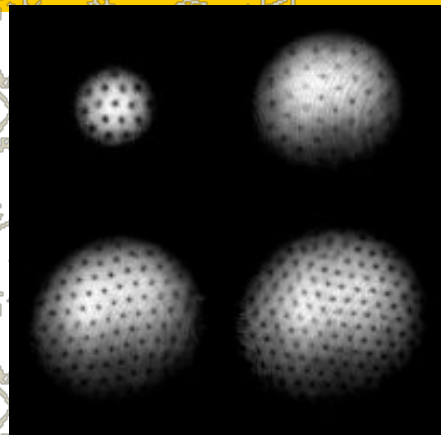


‘dark-tail’ vortex-pair in BEC – well-separate

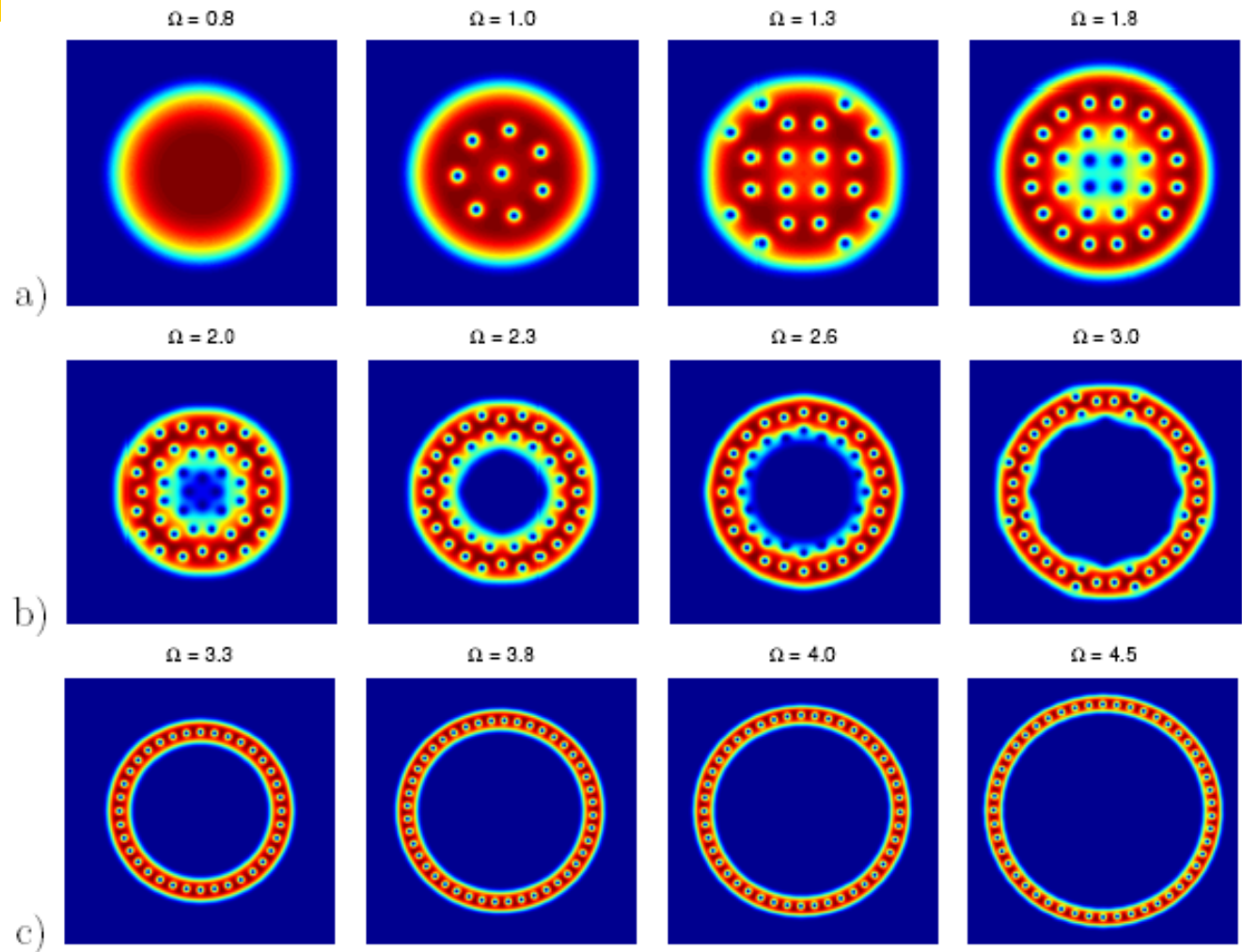
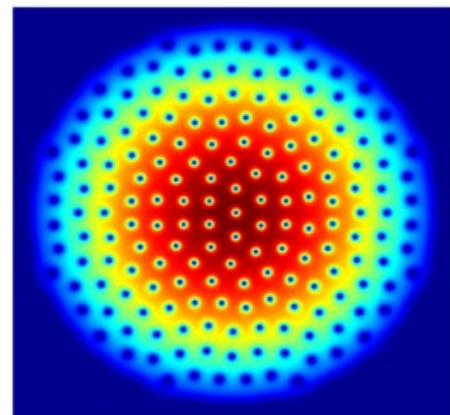


'dark-tail' vortex-dipole in BEC

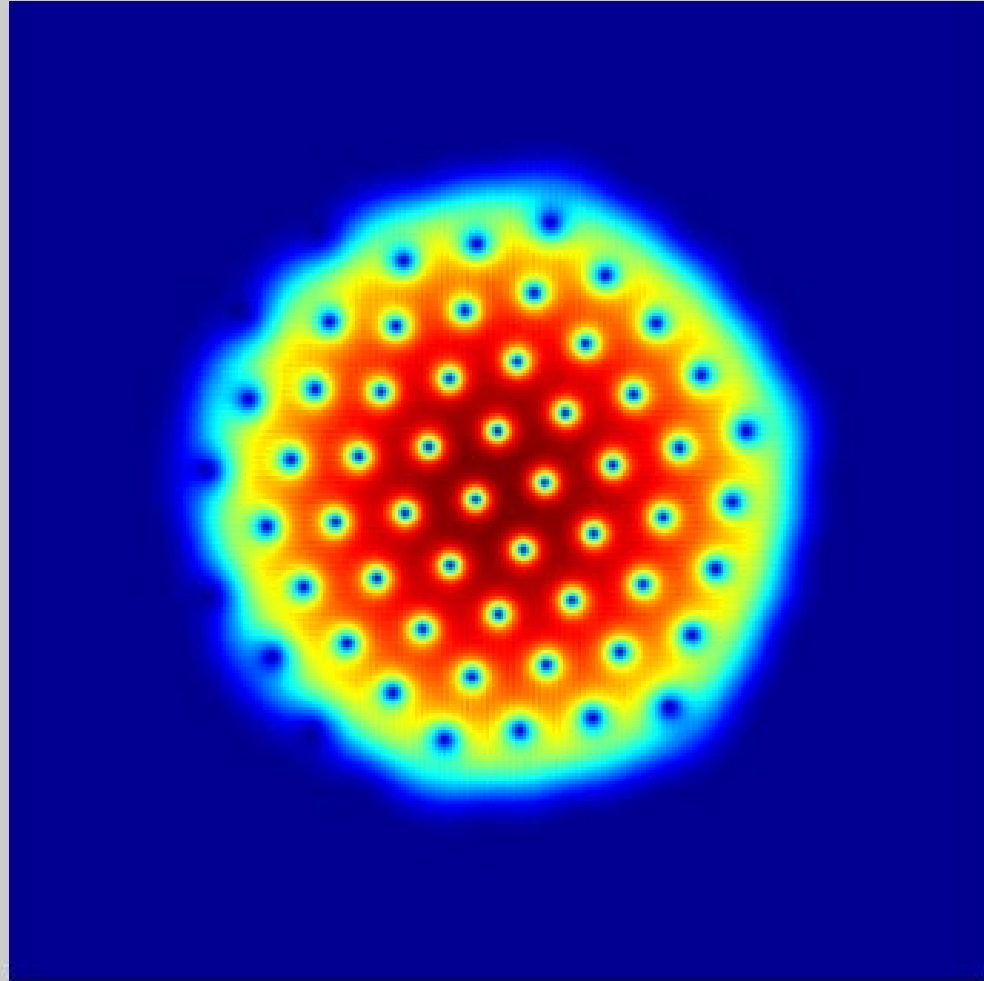
Ground states of rapid rotation



BEC@MIT



$t=0$



'dark-tail' vortex lattice in BEC

Impurities Bound by Vortex lattice

PRL 116, 240402 (2016)

PHYSICAL REVIEW LETTERS

week ending
17 JUNE 2016

Hubbard Model for Atomic Impurities Bound by the Vortex Lattice of a Rotating Bose-Einstein Condensate

T. H. Johnson,^{1,2,3} Y. Yuan,^{4,5,6} W. Bao,^{5,*} S. R. Clark,^{7,3,†} C. Foot,² and D. Jaksch^{2,1,3}

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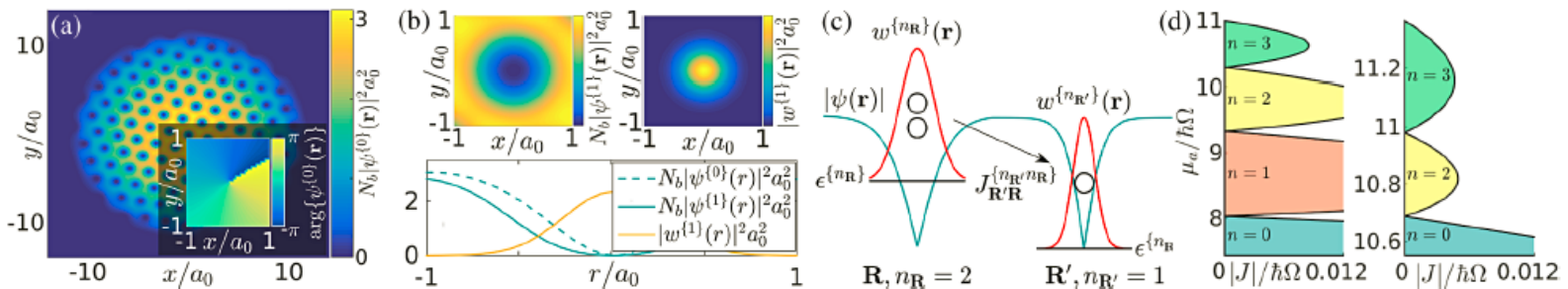
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We investigate cold bosonic impurity atoms trapped in a vortex lattice formed by condensed bosons of another species. We describe the dynamics of the impurities by a bosonic Hubbard model containing occupation-dependent parameters to capture the effects of strong impurity-impurity interactions. These



Vortex patterns and the critical rotational frequency in rotating dipolar Bose-Einstein condensates

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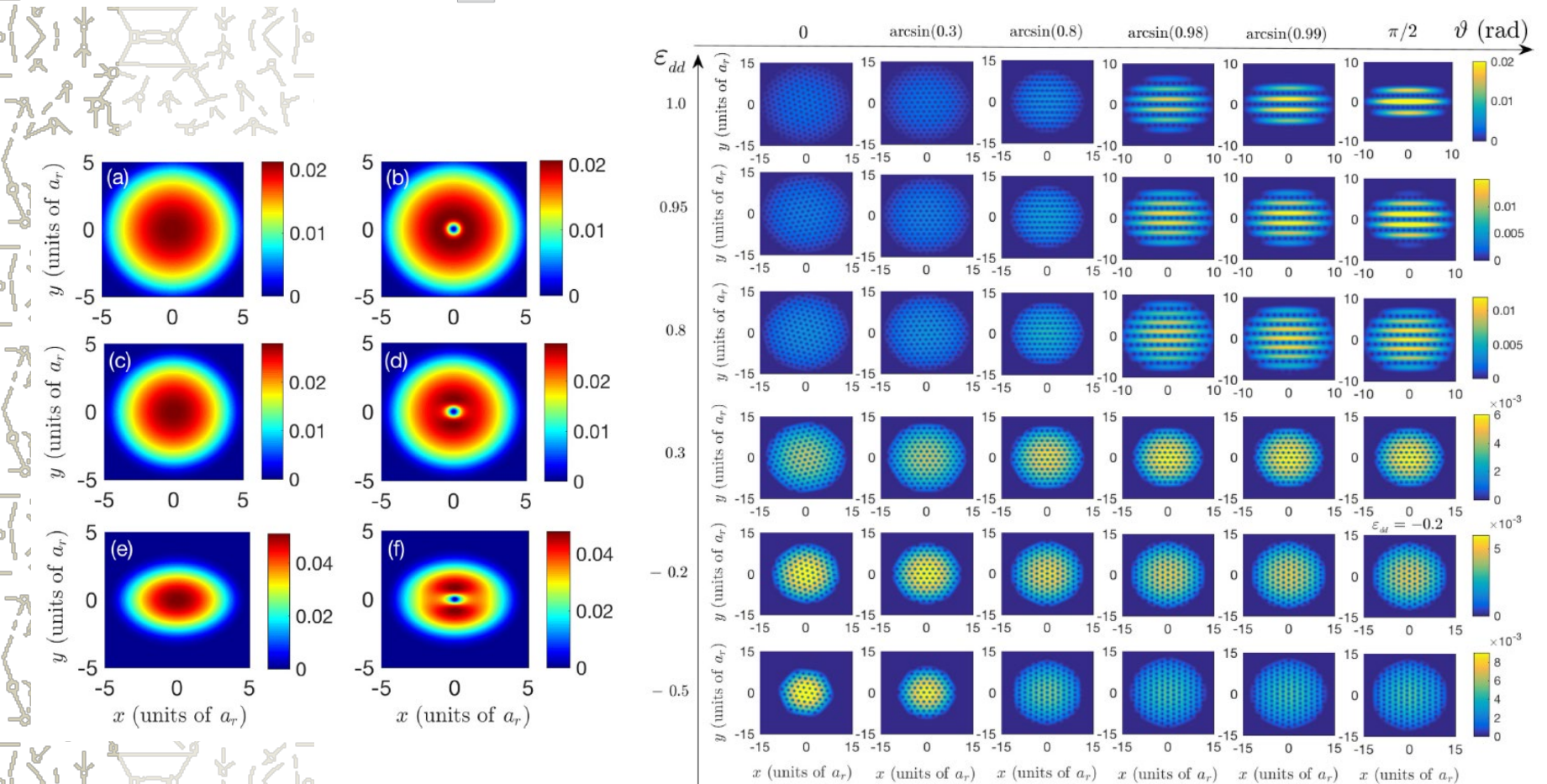
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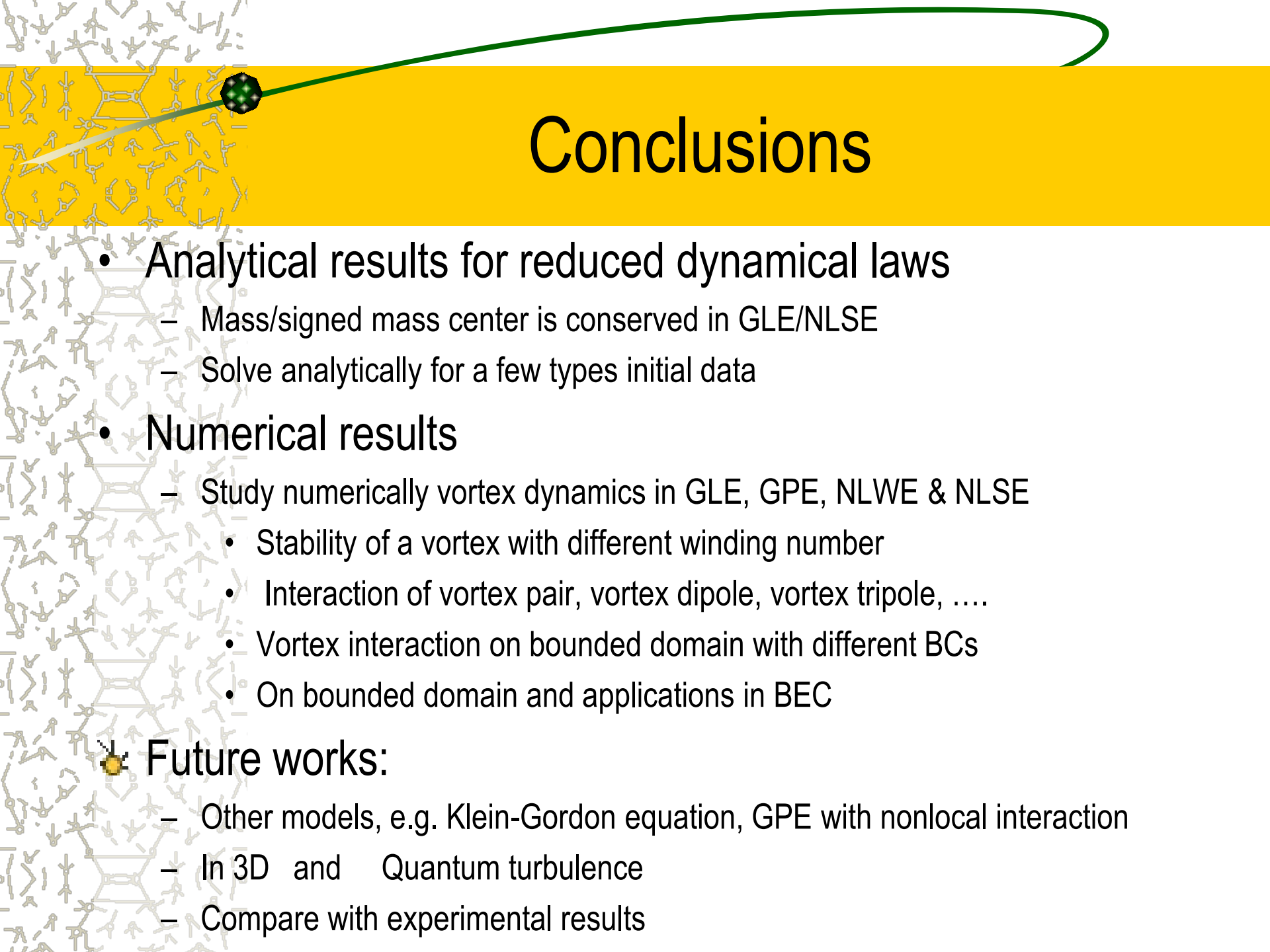
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Conclusions

- Analytical results for reduced dynamical laws
 - Mass/signed mass center is conserved in GLE/NLSE
 - Solve analytically for a few types initial data
- Numerical results
 - Study numerically vortex dynamics in GLE, GPE, NLWE & NLSE
 - Stability of a vortex with different winding number
 - Interaction of vortex pair, vortex dipole, vortex tripole,
 - Vortex interaction on bounded domain with different BCs
 - On bounded domain and applications in BEC
- ✶ Future works:
 - Other models, e.g. Klein-Gordon equation, GPE with nonlocal interaction
 - In 3D and Quantum turbulence
 - Compare with experimental results