On a kinetic Elo rating model for players with dynamical strength

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Joint work with

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Interacting many-agent systems in socio-economics

Examples: wealth distribution in an economy, opinion formation, crowd dynamics, ...

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$\rightsquigarrow$ mathematical tools from kinetic theory

Kinetic models for socio-economic systems

Conceptual approach (e.g. [Pareschi&Toscani, 2015], ...):

- describe dynamics of system by microscopic interactions among agents
- perform many interactions (analytically or numerically)
- observe emergent behaviour, patterns in macroscopic distribution of agents
- derive partial differential equations (Boltzmann, Fokker-Planck-type) which (approximatively) govern the time-evolution of the density

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Benefits:

- more (analytically and numerically) tractable model
- understanding role of parameters in the microscopic interactions for emergent behaviour
- PDE: nonlinear, anisotropic, nonlocal, degenerate

Elo rating for zero-sum games

- rating system developed by physicist Arpad Elo to determine relative skill levels of players in zero-sum games
- originally used for chess
- also for online gaming, table tennis, ...
- multiplayer games: football, basketball, ..
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- 2018: FIFA world ranking to use Elo system
- each player assigned rating number which may change as games played
- difference in rating between two players should predict outcome of a game
- players with same rating who play each other should have same probability of winning/loosing
- difference between ratings determines number of points gained or lost after a game

Elo rating: Junca & Jabin (2015)

continuous variables: strength ρ (fixed, unobservable)
rating R (variable, observable)

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Following each game, ratings are adjusted

$$R_i^* = R_i + \gamma(S_{ij} - b(R_i - R_j))$$
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- random variable $S_{ij} \in \{-1, 1\}$: score result of the game
- function b moderates extreme differences, e.g. $b(z) = \tanh(cz)$ with some $c > 0$
- assume mean score $\langle S_{ij} \rangle = b(\rho_i - \rho_j)$
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Effect:

- player with high rating wins against player with a low rating $\rightsquigarrow$ ratings change little
- player with low rating wins against highly rated player $\rightsquigarrow$ ratings are strongly adjusted

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Question: Is the rating fair, i.e. do '$R_i \to \rho_i$' over time?

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Kinetic equation: distribution of players $f = f(r,t)$ with respect to ratings satisfies

$$\partial_t f(r,t) + \partial_r (a(f)f) = 0$$

with $a(f) = \int_{\mathbb{R}^2} w(r-r')(b(\rho-\rho') - b(r-r'))f(t,r',\rho')d\rho' dr'$

and given interaction rate function $w(r-r')$

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Long time behaviour:

- $w = 1$ ('all-play-all' tournament): ratings converge exponentially fast to intrinsic strengths
- w with local interactions: ratings may not converge to intrinsic strengths, rating fails to give a fair representation of the player's strength distribution

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and players learn

$$\rho_i^* = \rho_i + \gamma h(\rho_j - \rho_i) + \eta$$
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where $\eta, \tilde{\eta}$ are random variables with mean zero.

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Choice of learning mechanism

We consider two main effects:

- learning by interaction: we assume each player learns in a game, however players with lower strength benefit more. Possible choice $h_1(\rho_j - \rho_i) = 1 + b(\rho_j - \rho_i)$
- gain/loss of self-confidence: assume gain/loss of stronger player is the same as that of the weaker one, e.g. $h_2(\rho_j - \rho_i) = S_{ij}[1 - \tanh^2(\rho_j - \rho_i)]$

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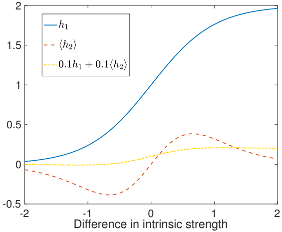
With parameters α, β we have in summary

$$h(\rho_j - \rho_i) = \alpha h_1(\rho_j - \rho_i) + \beta h_2(\rho_j - \rho_i)$$



Some properties of the interaction

Preservation of total value of the rating pointwise and in mean,

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$\rightsquigarrow$ constant increase of strength of population

Boltzmann type equation

Distribution function $f_\gamma = f_\gamma(\rho, R, t)$ satisfies

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \phi(\rho_i, R_j) f_\gamma(\rho_i, R_i, t) d\rho_i dR_i \\
&= \frac{1}{2}\Bigg\langle \int_\Omega \int_\Omega \Big(\phi(\rho_i^*, R_j^*) + \phi(\rho_j^*, R_j^*) - \phi(\rho_i, R_i) - \phi(\rho_j, R_j)\Big) \\
&\qquad \times\ w(R_i - R_j) f_\gamma(\rho_i, R_i, t) f_\gamma(\rho_j, R_j, t)\, d\rho_j dR_j d\rho_i dR_i \Bigg\rangle
\end{aligned}
$$

where $\phi(\cdot)$ is a (smooth) test function

Fokker-Planck limit

Rescaling $t' = \gamma t$, in the quasi-invariant limit

$$\gamma \to 0,\ \sigma_\eta \to 0 \text{ such that } \frac{\sigma_\eta^2}{\gamma} =: \sigma^2 \text{ is fixed}$$

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we obtain the Fokker-Planck equation

$$\frac{\partial f(\rho,R,t)}{\partial t} + \frac{\partial}{\partial R}\left(a[f]f(\rho,R,t)\right) + \frac{\partial}{\partial \rho}\left(c[f]f(\rho,R,t)\right) - \frac{\sigma^2}{2}d[f]\frac{\partial^2}{\partial \rho^2}f(\rho,R,t) = 0$$

where

$$a[f] = \int_{\mathbb{R}^2} w(R-R_j)(b(\rho-\rho_j) - b(R-R_j))f(\rho_j,R_j,t)\,d\rho_j dR_j$$

$$c[f] = \int_{\mathbb{R}^2} w(R-R_j)\big(\alpha h_1(\rho_j-\rho) + \beta\langle h_2(\rho_j-\rho)\rangle\big)f(\rho_j,R_j,t)\,d\rho_j dR_j$$

$$d[f] = \int_{\mathbb{R}^2} w(R-R_j)f(\rho_j,R_j,t)\,d\rho_j dR_j$$

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Consider $g(\rho, R, t) = f(\rho + H(\rho, R, t), R, t)$ where H is given by

$$\frac{\partial H(\rho, R, t)}{\partial t} = \int_{\mathbb{R}^2} \alpha w(R - R_j) f(\rho_j, R_j, t)\, d\rho_j dR_j.$$

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The evolution equation for $g(\rho, R, t)$ is

$$\frac{\partial g}{\partial t} + \frac{\partial}{\partial R}(a[g]g) + \frac{\partial}{\partial \rho}(\tilde{c}[g]g) - \frac{\sigma^2}{2} d[g] \frac{\partial^2}{\partial \rho^2} g = 0,$$

where

$$\tilde{c}[g] = \int_{\mathbb{R}^2} \big(\alpha b(\rho_j - \rho) + \beta \langle h_2(\rho_j - \rho) \rangle\big) w(R - R_j) g(\rho_j, R_j, t)\, d\rho_j dR_j.$$

Shifted Fokker-Planck equation

We consider the following problem on a bounded domain $\Omega \subset \mathbb{R}^2$, with no-flux boundary condition

$$
\begin{aligned}
\frac{\partial g}{\partial t} + \frac{\partial}{\partial R}(a[g]g) + \frac{\partial}{\partial \rho}(\tilde{c}[g]g) - \frac{\sigma^2}{2} d[g] \frac{\partial^2}{\partial \rho^2} g = 0, &\quad \text{in } \Omega \times (0,T), \\
\frac{\partial}{\partial \nu} g = 0 &\quad \text{on } \partial\Omega, \\
g(\rho, R, 0) = g_0(\rho, R) &\quad \text{in } \Omega.
\end{aligned}
$$

FP equation: existence of weak solutions

Let $\Omega \subset \mathbb{R}^2$ bounded Lipschitz domain.

Theorem

Let $g_0 \in H^1(\Omega)$ and $0 \leq g_0 \leq M_0$ for some $M_0 > 0$ and assume h_1, $\langle h_2 \rangle$, $b \in L^\infty(\Omega) \cap C^2(\Omega)$. Then there exists a weak solution $g \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$, satisfying $0 \leq g \leq M_0 e^{\lambda t}$ for all $(\rho, R) \in \Omega$, $t > 0$, with a constant $\lambda > 0$ depending on the functions $h_1, \langle h_2 \rangle, b$ and w.

Sketch of the proof

The proof involves several steps:

- Step 0: regularised, truncated problem, adding $\mu\Delta g(\rho, R, t)$, $\mu > 0$
- Step 1: solution of linearised, regularised problem; definition of fixed point operator
- Step 2: uniform L^∞ bounds and existence of fixed point (Leray-Schauder)
- Step 3: uniform H^1 bound (independent of μ)
- Step 4: limit $\mu \to 0$ (Aubin-Lions lemma)

Long-time behaviour of solutions

Define the energy $E_2(t) = \int_{\mathbb{R}^2} (\rho - R)^2 g(\rho, R, t)\, d\rho dR.$

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At least for $w = 1$ we can compute

$$\begin{aligned}
&\frac{d}{dt} E_2(t) \\
= &-\int_{\mathbb{R}^4} (R - R_j) b(R - R_j) g(\rho, R, t) g(\rho_j, R_j, t)\, d\rho_j dR_j d\rho dR \\
&-\int_{\mathbb{R}^4} (\rho - \rho_j) b(\rho - \rho_j) g(\rho, R, t) g(\rho_j, R_j, t)\, d\rho_j dR_j d\rho dR \\
&-\alpha \int_{\mathbb{R}^4} (\rho - \rho_j) b(\rho - \rho_j) g(\rho, R, t) g(\rho_j, R_j, t)\, d\rho_j dR_j d\rho dR \\
&-2\beta \int_{\mathbb{R}^4} (\rho - \rho_j) \langle h_2(\rho - \rho_j) \rangle g(\rho, R, t) g(\rho_j, R_j, t)\, d\rho_j dR_j d\rho dR \\
&+\sigma^2
\end{aligned}$$

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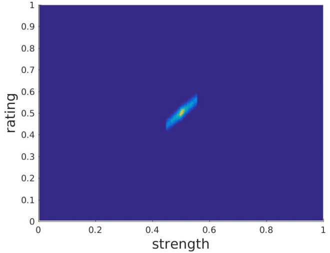
\end{aligned}
$$

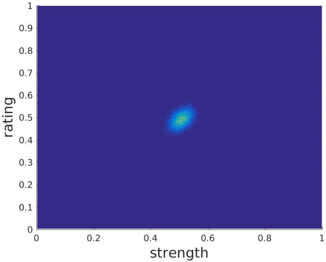
$\rightsquigarrow$ indicates concentration in neighbourhood of diagonal

Numerical results: all-play-all

Direct Monte Carlo simulation method: $N = 5000$ players



Steady state (top view) – no diffusion



Steady state (top view) – diffusion $\nu = 0.025$

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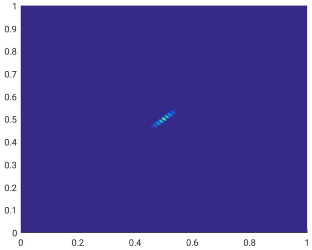
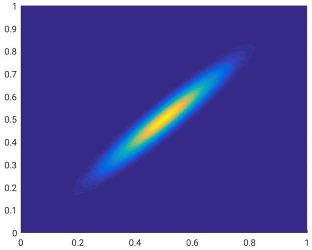
Numerical steady states for Fokker-Planck equation



no diffusion



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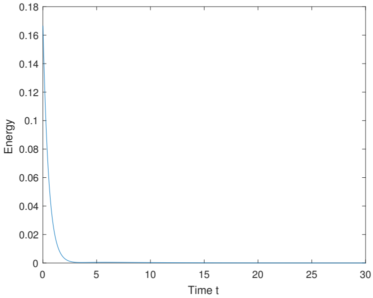
Energy decay for Fokker-Planck equation
$E_2(t) = \int_{\mathbb{R}^2} (\rho - R)^2 g(\rho, R, t)\, d\rho dR$



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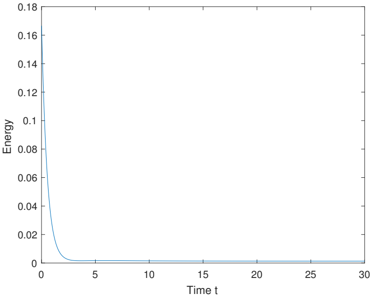

with diffusion

Numerical results: competition with similar rating

Consider two groups of players:

- first group is underrated, all players have rating $R = 0.2$, but $\rho \in \mathcal{N}(0.75, 0.1)$
- second group is overrated, with rating $R = 0.9$ and uniform distribution in ρ

Choose $\alpha = 0.1$ and $\beta = 0$.

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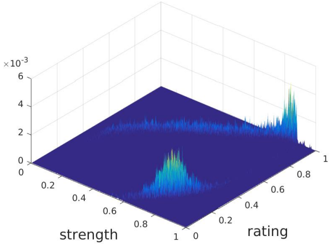

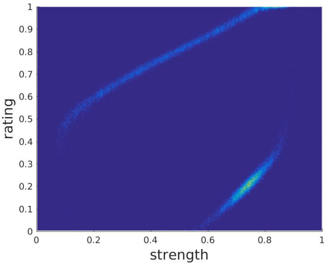

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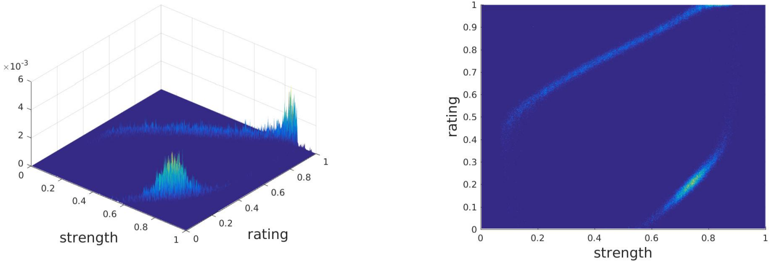

$\rightsquigarrow$ assigning initial ratings is a delicate issue

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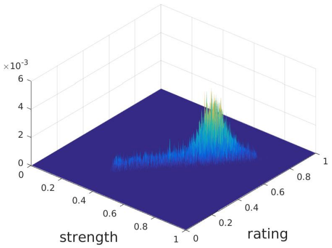

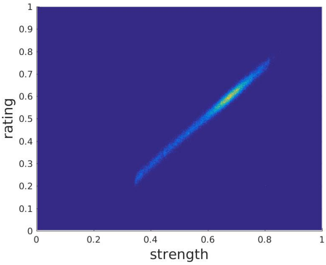

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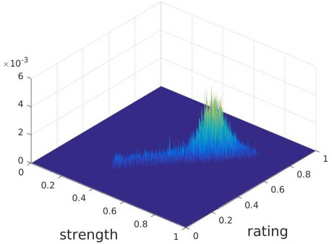

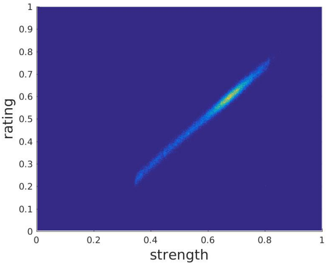

$\rightsquigarrow$ adapted learning mechanism leads to convergence of the ratings

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Assume one player is playing unfair, e.g. through cheating, doping or bribing of referees.

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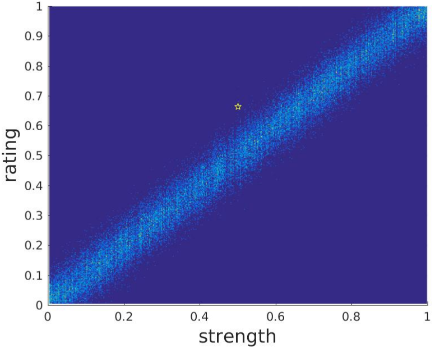

Ratings and strength of all players except the first one converge around diagonal.
The cheating player (indicated by a star) ends up with a higher rating.

Summary

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- well-posedness
- long-time behaviour: convergence to players' strength
- assigning initial ratings is delicate

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Boltzmann and Fokker-Planck equations modelling the Elo rating system with learning effects.
J. Nonlinear Sci. **29**(3) (2019), 1095-1128. (arXiv:1806.06648)

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  $\rightsquigarrow$ Boltzmann-type, Fokker-Planck-type limit equations
- well-posedness
- long-time behaviour: convergence to players' strength
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THANK YOU!