Eulerian dynamics in multi-dimensions with radial symmetry

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Emergent Phenomena: from Kinetic Models to Social Hydrodynamics
IMS, NUS, Singapore
2019.12.19

The compressible Euler equations

$$\begin{cases} \partial_t \rho + div\,(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + div\,(\rho \mathbf{u} \otimes \mathbf{u}) + div\,\mathbb{P} = \rho \mathbf{F}. \end{cases}$$

ρ : the density of the fluid.

$\mathbf{u}$: the macroscopic velocity ($\rho\mathbf{u}$ is the momentum).

$\mathbb{P}$: the pressure tensor.

$\mathbf{F}$: the force.

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$\mathbf{F}$: the force. nonlocal interaction forces

A multiscale framework

Agent-based model
Microscopic scale

$$\begin{aligned}\dot{x}_i &= v_i\\ \dot{v}_i &= F_i\end{aligned}$$

$N \to \infty$

Kinetic model
Microscopic scale

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [\mathbf{F}(f) f] = 0$$

Hydrodynamics

Fluid model
Macroscopic scale

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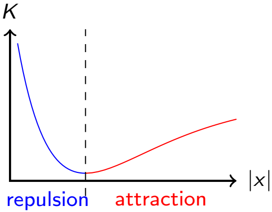
Interaction forces

- Attraction-repulsion force through an interaction potential

$$\mathbf{F} = -\int \nabla K(x-y)\rho(y,t)dy.$$



A typical example: when K is Newtonian.

The Euler-Poisson equation $\quad -\Delta K = \kappa\delta_0.$

- Alignment force

The Euler-alignment equation
(Cucker-Smale alignment interaction)

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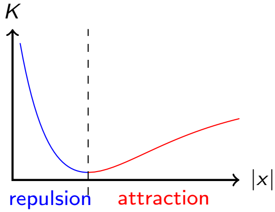

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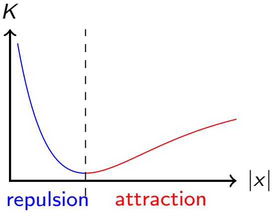

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Global regularity

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Goal: Understand the global wellposedness (or finite time singularity formations) of the system.

Global wellposedness is useful in understanding the emergent phenomena.

- Smoothness of the macroscopic system is often required to obtain a rigorous derivation of hydrodynamic limits.
- The Euler-alignment systems: strong solutions must flock.

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Pressureless Eulerian dynamics

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The dynamics of $\mathbf{u}$ (at least when $\rho > 0$) is the Burgers equation with force

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F}.$$

The convection term $(\mathbf{u} \cdot \nabla)\mathbf{u}$ is known to generates shock discontinuities in finite time.

Key to ensure the smoothness of the solution: boundedness of $\nabla \mathbf{u}$.

$$\int_0^T \|\nabla \mathbf{u}(\cdot, t)\|_{L^\infty} dt < +\infty.$$

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One-dimensional Burgers equation

$$\partial_t u + u\partial_x u = F.$$

Let $d = \partial_x u$. Differentiate in x at get

$$d' = -d^2 + \partial_x F,$$

where $' = \partial_t + u\partial_x$ denotes the derivative along the characterstic paths.

- Inviscid Burgers equation $\quad \partial_x F \equiv 0 \quad d_0 \geq 0.$
- Damped Burgers equation $\quad \partial_x F = -\nu d \quad d_0 \geq -\nu.$
- Euler-Poisson equation $\quad \partial_x F = \kappa\rho \quad d_0 \geq -\sqrt{2\kappa\rho_0}.$
  [Engelberg-Liu-Tadmor '01]
- Euler-alignment equation $\quad d_0 \geq -\psi * \rho_0.$
  [Carrillo-Choi-Tadmor-T. '16]

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Multi-dimensional Burgers equations

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{F}.$$

The quantity $\nabla\mathbf{u}$ is an n-by-n matrix which satisfies

$$(\nabla\mathbf{u})' = -(\nabla\mathbf{u})^2 + \nabla\mathbf{F}.$$

Question: what scaler quantity has the Recatti-type structure like in 1D?

Option #1: eigenvalues of $\nabla\mathbf{u}$. Let $\{\lambda_i\}_{i=1}^N$ be the eigenvalues of $\nabla\mathbf{u}$, with corresponding left and right eigenvectors as $\{\mathbf{l}_i, \mathbf{r}_i\}_{i=1}^N$. Then,

$$\lambda_i' = -\lambda_i^2 + \mathbf{l}_i^T(\nabla\mathbf{F})\mathbf{r}_i.$$

- Inviscid Burgers equation $\quad \mathbf{l}_i^T(\nabla\mathbf{F})\mathbf{r}_i \equiv 0$

$$\lambda_i(0) \notin \{a+bi \mid a<0, b=0\}.$$

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The spectral gap

$$(\nabla \mathbf{u})' = -(\nabla \mathbf{u})^2 + \nabla \mathbf{F}.$$

Question: what scaler quantity has the Recatti-type structure like in 1D?

Option #2: The divergence *div* $\mathbf{u}$.

$$(div\, \mathbf{u})' = -\mathrm{tr}(\nabla \mathbf{u})^2 + div\, \mathbf{F}.$$

Advantage: easier to handle the force term.

- Euler-Poisson equation $div\, \mathbf{F} = \kappa\rho.$
- Euler-Alignment equation $div\, \mathbf{F} = -(\psi * \rho)' - (\psi * \rho) div\, \mathbf{u}.$

Main difficulty: $\mathrm{tr}(\nabla \mathbf{u})^2 \neq (div\, \mathbf{u})^2$. The difference between the two quantities is related to *the spectral gap*. A lot of effort has been made in order to control the spectral gap.

- Restricted Euler-Poisson equation [Liu-Tadmor '02, ...]
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Radially symmetric solution

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To have a better understanding of the spectral gap, we focus on the solution which is radially symmetric.

$$\rho(\mathbf{x}) = \rho(r), \quad \mathbf{u}(\mathbf{x}) = \frac{\mathbf{x}}{r} u(r), \qquad r = |\mathbf{x}|.$$

Under this setup, the system is reduced to 1D. But the effect of the spectral gap persists.

$$\mathrm{tr}(\nabla \mathbf{u})^2 - (div\,\mathbf{u})^2 = -(n-1)\left[\frac{2uu_r}{r} - \frac{(n-2)u^2}{r^2}\right]$$

It can not be easily controled by $div\,\mathbf{u}$.

A new option

The spectral gap can be controled by u_r and $\frac{u}{r}$.

$$\mathrm{tr}(\nabla \mathbf{u})^2 - (div\,\mathbf{u})^2 = -(n-1)\left[\frac{2uu_r}{r} - \frac{(n-2)u^2}{r^2}\right].$$

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$$div\,\mathbf{u} = u_r + (n-1)\frac{u}{r}.$$

For the inviscid Burgers equation, the dynamics of $(u_r, \frac{u}{r})$ reads

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The damped Burgers equation

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$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nu \mathbf{u}.$$

The dynamics of $(u_r, \frac{u}{r})$ along characterstic paths:

$$\begin{cases} u_r' = -u_r^2 - \nu u_r \\ \left(\frac{u}{r}\right)' = -\left(\frac{u}{r}\right)^2 - \nu \frac{u}{r} \end{cases}$$

Solution is globally regular if and only if

$$u_r(0) \geq -\nu \quad \text{and} \quad \frac{u(0)}{r} \geq -\nu.$$

Remark: The condition $div\, \mathbf{u}_0 \geq -\nu$ is neither sufficient nor necessary for global regularity.

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$$\begin{cases} \partial_t \rho + div\,(\rho \mathbf{u}) = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\kappa \nabla \phi, \\ -\Delta \phi = \rho - c. \end{cases}$$

κ : strength of the interaction: $\kappa > 0$ repulsion, $\kappa < 0$ attraction.
c : background charge: typical choices $c = 0$ or $c = \bar{\rho}$.

1D dynamics: a closed system of $d = \partial_x u$ and ρ along characterstic paths

$$\begin{cases} d' = -d^2 + \kappa(\rho - c), \\ \rho' = -\rho d, \end{cases}$$

Sharp threshold condition: global regularity if and only if

$$(\partial_x u_0(x), \rho_0(x)) \in \Sigma, \quad \forall\, x \in \mathbb{R}.$$

where Σ is the collection of all (d_0, ρ_0) such that the dynamics with the initial data is bounded globally in time.

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Euler-Poisson equation in multi-D with radial symmetry

The dynamics of $(p = u_r, q = \frac{u}{r}, \rho)$ versus $(d = div\, \mathbf{u}, \rho)$

$$\begin{cases} p' = -p^2 - \kappa\phi_{rr}, \\ q' = -q^2 - \kappa\frac{\phi_r}{r}, \\ \rho' = -\rho(p + (n-1)q). \end{cases} \qquad \begin{cases} d' = -(p^2 + (n-1)q^2) + \kappa(\rho - c), \\ \rho' = -\rho d. \end{cases}$$

Note that $\Delta\phi = \phi_{rr} + (n-1)\frac{\phi_r}{r} = -(\rho - c)$.

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The extra term $\frac{\phi_r}{r}$ needs to be controled to close the system.

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$$\begin{cases} d' = -(p^2 + (n-1)q^2) + \kappa(\rho - c), \\ \rho' = -\rho d. \end{cases}$$

Note that $\Delta\phi = \phi_{rr} + (n-1)\frac{\phi_r}{r} = -(\rho - c)$.

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The extra term $\frac{\phi_r}{r}$ needs to be controled to close the system.

Control the term $\frac{\phi_r}{r}$

To control the term $\frac{\phi_r}{r}$, we start with the density dynamics

$$\rho_t + (\rho u)_r = -(n-1)\rho\frac{u}{r}.$$

To absorb the right hand side, we scale ρ as

$$(r^{n-1}\rho)_t + (r^{n-1}\rho u)_r = 0.$$

Its premitive e satisfies

$$e_t + ue_r = 0.$$

Note that $\Delta\phi = r^{1-n}(r^{n-1}\phi_r)_r = -(\rho - c)$.

Let $s = -\frac{\phi_r}{r} = er^{-n}$, it satisfies

$$s' = e'r^{-n} - nr^{-n-1}r'e = -c\frac{u}{r} - nu\frac{s}{r} = -(c+ns)q.$$

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$$\partial_t \left(r^{n-1}(\rho - c) \right) + \partial_r \left(r^{n-1}(\rho - c) u \right) = -\partial_r \left(c r^{n-1} u \right).$$

Its premitive $e = -r^{n-1}\phi_r$ satisfies

$$e_t + u e_r = -c r^{n-1} u.$$

Note that $\Delta\phi = r^{1-n}(r^{n-1}\phi_r)_r = -(\rho - c)$.

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A sharp threshold condition

A closed ODE system for $\left(p = \partial_x u, q = \frac{u}{r}, s = -\frac{\phi_r}{r}, \rho\right)$

$$\begin{cases} p' = -p^2 + \kappa(\rho - c - (n-1)s), \\ q' = -q^2 + \kappa s, \\ s' = -(ns + c)q, \\ \rho' = -\rho(p + (n-1)q). \end{cases}$$

Theorem (Sharp critical threshold)

Consider the Euler-Poisson equations with radial symmetry. It admits a global smooth solution if and only if

$$\left(\partial_r u_0(r), \frac{u_0(r)}{r}, -\frac{\partial_r \phi_0(r)}{r}, \rho_0(r)\right) \in \Sigma, \quad \forall\ r \geq 0.$$

where $\Sigma \in \mathbb{R}^4$ be the set such that: $(p_0, q_0, s_0, \rho_0) \in \Sigma$ if and only if the ODE system with initial data (p_0, q_0, s_0, ρ_0) is bounded globally in time.

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1D dynamics (p, ρ) : [Engelberg-Liu-Tadmor '01, ...]

Multi-D dynamics : the spectral gap is characterized by (q, s).

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Control the spectral gap: $c = 0$

$$\begin{cases} q' = -q^2 + \kappa s, \\ s' = -nsq, \end{cases}$$

Recall $s = er^{-n}$ and $e_r = r^{n-1}\rho \geq 0$. This implies $s \geq 0$.

For $\kappa > 0$, $q \geq 0$ is an invariant region. So the spectral gap is well-under control if $q_0 \geq 0$.

Sharp threshold given that $u_0(r) \geq 0$. [Wei-Tadmor-Bae '12]

What if the initial flow is not pointing out?

Theorem (Control the spectral gap)

For any initial data $q_0 < 0$ and $s_0 > 0$, the solution is globally bounded. Moreover, (q, s) converges to $(0, 0)$ as time approaches infinity.

Remark: the theorem holds for $n \geq 2$, but is false when $n < 2$.

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Theorem (Control the spectral gap)

Let $\kappa > 0$. For any initial data (q_0, s_0) such that $s_0 > -\frac{c}{n}$, solution is globally bounded.

Remark: unlike the zero-background case, the solution will travel around a closed orbit, and won't converge as time becomes infinity.

The Euler-alignment equation

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$$\begin{cases}\partial_t\rho + div\,(\rho\mathbf{u}) = 0,\\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \int_{\mathbb{R}^n}\psi(|\mathbf{x}-\mathbf{y}|)(\mathbf{u}(\mathbf{y})-\mathbf{u}(\mathbf{x}))\rho(\mathbf{y})d\mathbf{y}..\end{cases}$$

1D dynamics (auxiliary quantity) $d = u_x$ [Carrillo-Choi-Tadmor-T. '16]

$$(d+\mathcal{L}\rho)' = -d^2 - d\mathcal{L}\rho = -d(d+\mathcal{L}\rho).$$

Multi-D with radial symmetry: the dynamics of $(p = u_r, q = \frac{u}{r})$

$$\begin{cases}(p+\mathcal{L}\rho)' = -p(p+\mathcal{L}\rho) - (n-1)\zeta,\\ q' = -q(q+\mathcal{L}\rho) + \zeta.\end{cases}, \quad \text{where } \zeta(r)\mathbf{x} = \mathcal{L}(\rho\mathbf{u}).$$

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Idea of getting global wellposedness and finite time blowup :

1. Show that ζ is uniformly bounded.
2. Use comparison principle to get either wellposedness or blowup.

Theorem

There exists a set Σ_+ such that solutions are globally regular if the initial data

$$\left(\partial_r u_0(r), \frac{u_0(r)}{r}\right) \in \Sigma_+, \quad \forall\ r \geq 0.$$

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Summary

Eulerian dynamics in multi-dimensions

$$\begin{cases} \partial_t \rho + div\,(\rho \mathbf{u}) = 0, \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \mathbf{F}. \end{cases}$$

1. $\{\lambda_i\}_{i=1}^n$ eigenvalues of $\nabla \mathbf{u}$: not friendly to the force $\mathbf{F}$.
2. $div\, \mathbf{u}$: not friendly to the convection, spectral gap.
3. $(u_r, \frac{u}{r})$: a better choice in the radially symmetric case.

Ongoing and future work:

- Adding pressure (e.g. p-system). [Guo, Yang, ...]
- Radially symmetric solution with swirl. [Tadmor, Wei, ...]
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Thanks for your attention!