On Stability of Peakons to Nonlocal Integrable Equations

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Outline

- Introduction
- Integrability
- Analytic preliminaries
- Peaked solitons
- Stability of peakons
- Two-component quadratic μ-CH system
- Two-component cubic CH system
- Conclusions and remarks

1. Introduction

◇ The μ-Camassa-Holm (CH) equation

$$m_t + 2mu_x + um_x = 0, \quad t > 0, \quad x \in R, \tag{1}$$

where $m = \mu(u) - u_{xx}$, $\mu(u) = \int_S u(t,x)dx$. If $\mu(u) = 0$, which implies $\mu(u_t) = 0$, then this equation reduces to the Hunter-Saxton (HS) equation, which is a short wave limit of the CH equation. (Khesin, Lenells, Misiolek, 2008, Math. Ann.)

• It is a dynamical equation for rotators in liquid crystals with external magnetic field and self-intersection

• It is an integrable equation and admits peaked solitons

• The μ-CH equation describes the geodesic flow on $\mathcal{D}^s(S)$ with the right-invariant metric given by the inner product

$$< u, v >= \mu(u)\mu(v) + \int_S u_x v_x dx.$$

◇ The CH equation

$$m_t + 2u_x m + u m_x + \gamma u_x = 0, \quad m = u - u_{xx}.$$

(Camassa-Holm, 1993; Fokas-Fuchssteiner 1981)

◇ The HS equation

$$m_t + 2u_x m + u m_x + \gamma u_x = 0, \quad m = -u_{xx}.$$

(Hunter-Saxton, 1996)

- Integrability, 2×2 spectral problem
- Existence of peakons
- Water waves
- Wave breaking
- Geometric formulations
- Quadratic nonlinearities
- H^1-weak solution

◇ The Degasperis-Procesi equation

$$m_t + 3u_x m + u m_x + \gamma u_x = 0, \quad m = u - u_{xx}.$$

(Degasperis-Procesi, 1998)

- Intergrability, 3×3 spectral problem
- Existence of peakons
- Shock peakons
- Water waves
- Wave breaking
- Quadratic nonlinearities

◇ The μ-DP equation

$$m_t + 3mu_x + um_x = 0, \quad t > 0, \quad x \in R,$$

(Lenells, Misiolek, Tiğlay, 2010, CMP) where $m = \mu(u) - u_{xx}$. Setting $\mu(u) = 0$, this equation becomes the short wave limit of the DP equation or the Burgers equation. Geometrically, it describes an affine surface (Fu, Liu, Qu, J. Funct. Anal., 2012)

◇ The modified CH equation (with cubic nonlinear terms)

$$m_t + \left((u^2 - u_x^2)m\right)_x + \gamma u_x = 0, \quad m = u - u_{xx}.$$

(Olver, Rosenau, 1996; Fuchssteiner, 1996; Qiao, 2006)

◇ The short pulse equation

$$-u_{xt} + (u^2 u_x)_x + \gamma u = 0.$$

(Schäfer-Wayne, 2004) The short-pulse equation is the model for the propagation of ultra-short optical pulse approximation in nonlinear Maxwell's equations, where u is the magnitude of the electric field.

◇ The modified μ-CH equation

$$m_t + ((2\mu(u)u - u_x^2)m)_x = 0, \quad m = \mu(u) - u_{xx}.$$

(Qu, Fu, Liu, J. Func. Anal., 2014; Liu, Qu, Zhang, Phys. D, 2013)

Remark 1.1 Applying the tri-Hamiltonian duality approach (Olver, Rosenau, Fuchssteiner, 1995,1996) to the KdV and the mKdV equation yields the CH equation and the modified CH equation, respectively.

◇ A generalized Camassa-Holm equation with cubic and quadratic nonlinearities

$$m_t + k_1 \left((u^2 - u_x^2)m\right)_x + k_2 \left(2mu_x + um_x\right) + \gamma u_x = 0, \quad m = u - u_{xx}.$$

(Fokas, 1995; Fuchssteiner 1996; Qiao, Xia, Li, 2012; Liu, Liu, Olver, Qu, 2014)

◇ The generalzied μ-CH equation with cubic and quadratic nonlinearities

$$m_t + k_1((2\mu(u)u - u_x^2)m)_x + k_2(2u_x m + um_x) + \gamma u_x = 0, \quad (2)$$

where $m = \mu(u) - u_{xx}$. (Qu, Fu, Liu, Comm. Math. Phys. 2014; Qu, Liu, Liu, Zhang, Arch. Rat. Mech. Anal., 2014)

◇ The two-component Camassa-Holm system

$$m_t + um_x + 2mu_x \pm \rho\rho_x = 0,$$
$$\rho_t + (\rho u)_x = 0, \quad x \in R,$$

where $m = u - u_{xx}$.
(Olver, Rosenau, 1996 Chen, Liu, Zhang, 2006; Constatin, Ivanov, 2012)

Remark 1.1. The above system does not admit the peaked solitions.

◇ A two-component Camassa-Holm system

$$m_t + 2mu_x + m_x u + (mv)_x + nv_x = 0,$$
$$n_t + 2nv_x + n_x v + (nu)_x + mu_x = 0,$$

where $m = u - u_{xx}$, $n = v - v_{xx}$.
(Qu, Fu, 2009)
This system is equivalent to the following two-component CH system

$$\xi_t + \sigma\xi_x + 2\xi\sigma_x + \eta\bar{\eta}_x = 0, \ \ \xi = (1 - \partial_x^2)\sigma,$$
$$\eta_t + (\eta\sigma)_x = 0, \ \ \eta = (1 - \partial_x^2)\bar{\eta}, \quad x \in R,$$

via the linear change of variables $\xi = m + n, \quad \eta = m - n$, which was derived by (Holm et al, 1996) from the Euler-Poincare equation.

Question: Are there two-component μ-CH systems which admit peaked solutions and H^1-conservation law?

◇ A two-component μ-CH system

$$\begin{aligned} m_t + 2mu_x + m_x u + (mv)_x + nv_x = 0,\\ n_t + 2nv_x + n_x v + (nu)_x + mu_x = 0, \end{aligned} \tag{3}$$

where $m = \mu(u) - u_{xx}$, $n = \mu(v) - v_{xx}$.
(Li, Fu, Qu, 2019)
This system is equivalent to the following two-component μ-CH system

$$\begin{aligned} \xi_t + \sigma\xi_x + 2\xi\sigma_x + \eta\bar{\eta}_x = 0, \quad \xi = (\mu - \partial_x^2)\sigma,\\ \eta_t + (\eta\sigma)_x = 0, \quad \eta = (\mu - \partial_x^2)\bar{\eta}, \qquad x \in R, \end{aligned}$$

via the linear change of variables $\xi = m + n, \quad \eta = m - n$.
This system can also be obtained from the Euler-Poincare equation with the Lagragian

$$L = \frac{1}{2}(\mu^2(\sigma) + \mu^2(\bar{\eta}) + \|\sigma_x\|_{L^2}^2 + \|\bar{\eta}_x\|_{L^2}^2).$$

◇ The general two-component μ-CH system

$$m_{k,t} = \sum_{i,j=1}^{2} a_{i,j}^{k} m_i u_{j,x} + \sum_{i,j=1}^{2} b_{i,j}^{k} m_{i,x} u_j, \quad k = 1, 2, \tag{4}$$

where $u_k(t, x)$ is a function of time t and a single spatial variable x, and

$$m_k = \mu(u_k) - u_{k,xx}, \qquad \mu(u_k) = \int_S u_k(t, x)dx,$$

with $S = R/Z$ which denotes the unit circle on the plane.

◇ A two-component modified CH system

$$m_{i,t} = \frac{1}{2} \sum_{j=1}^{n} \Big[(u_j^2 - u_{j,x}^2) m_i\Big]_x - \sum_{j=1}^{n} (u_i u_{j,x} - u_j u_{i,x}) m_j,$$

where $m_i = u_i - u_{i,xx}$, $1 \le i \le n$. (Qu, Song, Yao, 2013, SIGMA)

◇ A two-component modified μ-CH system

$$m_{i,t} = \frac{1}{2} \sum_{j=1}^{n} \Big[(2\mu(u_j) u_j - u_{j,x}^2) m_i\Big]_x - \sum_{j=1}^{n} (u_i u_{j,x} - u_j u_{i,x}) m_j,$$

where $m_i = \mu(u_i) - u_{i,xx}$, $1 \le i \le n$. (Qu, Song, Yao, 2013, SIGMA)

2. Integrability

◇ Bi-Hamiltonian structure

The generalized μ-CH equation (2) admits the bi-Hamiltonian structure

$$\frac{\partial m}{\partial t} = J\frac{\delta H_1}{\delta m} = K\frac{\delta H_2}{\delta m},$$

where

$$J = -k_1\partial_x m\partial_x^{-1} m\partial_x - k_2(m\partial_x + \partial_x m) - \frac{1}{2}\gamma u_x, \quad K = -\partial A = \partial_x^3$$

are compatible Hamiltonian operators, while

$$H_1 = \frac{1}{2}\int_R umdx,$$

and

$$H_2 = k_1 \int_R (\mu^2(u)u^2 + \mu(u)uu_x^2 - \frac{1}{12}u_x^4 + 2\gamma u_x^2)dx$$
$$+k_2 \int_R (\mu(u)u + \frac{1}{2}uu_x^2)dx$$

are the corresponding Hamiltonian functionals.

◇ The Lax-pair

Equation (2) has the following Lax-pair

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x = U(m,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}, \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_t = V(m,u,\lambda)\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},$$

where U and V are given by, for $\gamma = 0$

$$U(m,\lambda) = \begin{pmatrix} 0 & \lambda m \\ -k_1\lambda m - k_2\lambda & 0 \end{pmatrix},$$

and

$$V(m,u,\lambda) = \begin{pmatrix} -\frac{1}{2}k_2 u_x & -\frac{\mu(u)}{2\lambda} - k_1\lambda(2\mu(u)u - u_x^2)m - k_2\lambda u m \\ G & \frac{1}{2}k_2 u_x \end{pmatrix},$$

with

$$\begin{aligned} G = {} & k_2(\frac{1}{2\lambda} + \lambda k_2 u) + \frac{1}{2\lambda}k_1\mu(u) + k_1^2\lambda(2\mu(u)u - u_x^2)m \\ & + k_1k_2\lambda(2\mu(u)u - u_x^2 - um) \end{aligned}$$

and for $\gamma \neq 0$

$$U(m,\lambda)=\begin{pmatrix} -\lambda\sqrt{-\frac{\gamma}{2}} & \lambda m \\ -k_1\lambda m-k_2\lambda & \lambda\sqrt{-\frac{\gamma}{2}} \end{pmatrix},\quad V(m,u,\lambda)=\begin{pmatrix} A & B \\ G & -A \end{pmatrix},$$

with

$$\begin{aligned} A &= \frac{1}{4}\sqrt{-2\gamma}\left(\lambda^{-1}+2k_1\lambda(2u\mu(u)-u_x^2)+2\lambda k_2 u\right)-\frac{1}{2}k_2u_x, \\ B &= -\frac{\mu}{2\lambda}+\frac{1}{2}\sqrt{-2\gamma}u_x-k_1\lambda(2u\mu(u)-u_x^2)m-\lambda k_2 um. \end{aligned}$$

◇ Conservation laws ($\gamma = 0$)

$$H_0 = \int_S u dx,$$
$$H_1 = \frac{1}{2}\int_S (\mu^2(u) + u_x^2)dx,$$
$$H_2 = k_1 \int_S \left(\mu^2(u)u^2 + \mu(u)uu_x^2 - \frac{1}{12}u_x^4\right) dx + k_2 \int_S (\mu(u)u + \frac{1}{2}uu_x^2)dx,$$
$$\cdots.$$

◇ Short capillary-gravity wave equation

Applying the scaling transformation

$$x \to \epsilon x, t \to \epsilon^{-1}t, u \to \epsilon^2 u$$

to (2) produces

$$(\epsilon^2\mu(u) - u_{xx})_t + k_1((2\epsilon^2 u\mu(u) - u_x^2)(\epsilon^2\mu(u) - u_{xx}))_x$$
$$+k_2(2u_x(\epsilon^2\mu(u) - u_{xx}) + u(\epsilon^2\mu(u) - u_{xx})_x) + \gamma u_x = 0.$$

Expanding

$$u(t,x) = u_0(t,x) + \epsilon\, u_1(t,x) + \epsilon^2\, u_2(t,x) + \cdots$$

in the small parameter ϵ, the leading order term $u_0(t,x)$ satisfies

$$-u_{0,xxt} + k_1(u_{0,x}^2 u_{0,xx})_x - k_2(u_{0,x}u_{0,xx} + u_0 u_{0,xxx}) + \gamma u_{0,x} = 0.$$

Then $v = u_{0,x}$ satisfies the integrable equation

$$v_{xt} - k_1 v_x^2 v_{xx} + k_2(v v_{xx} + \frac{1}{2} v_x^2) - \gamma v = 0,$$

which describes asymptotic dynamics of a short capillary-gravity wave, where $v(t,x)$ denotes the fluid velocity on the surface (Faquir, Manna, Neveu, 2007).

3. Preliminaries

Consider the Cauchy problem

$$\begin{aligned} &m_t + k_1(2\mu(u)u - u_x^2)m)_x + k_2(2mu_x + um_x) = 0, \; t > 0, \; x \in R, \\ &u(0,x) = u_0(x), \;\; m = \mu(u) - u_{xx}, \;\; x \in R, \qquad (5) \\ &u(t, x+1) = u(t,x), \;\; t \geq 0, \;\; x \in R. \end{aligned}$$

In the following, all space of functions are defined over $S = R/Z$.

Definition 3.1 If $u \in \mathcal{C}([0,T]; H^s) \cap \mathcal{C}^1([0,T]; H^{s-1})$ with $s > \frac{5}{2}$ and some $T > 0$ satisfies (5), then u is called a strong solution on $[0,T]$. If u is a strong solution on $[0,T]$ for every $T > 0$, then it is called a global strong solution.

Applying the inverse operator of $A = \mu - \partial_x^2$ to equation (2) results in the equation

$$u_t + k_1 \left[(2u\mu(u) - \frac{1}{3}u_x^2)u_x + \partial_x A^{-1}(2\mu^2(u)u + \mu(u)u_x^2) + \frac{1}{3}\mu(u_x^3) \right]$$
$$+ k_2 \left[uu_x + A^{-1}\partial_x(2u\mu(u) + \frac{1}{2}u_x^2) \right] = 0.$$

The Green function of the operator A is (Lenells, Misiolek, Tiğlay, 2010)

$$g(x) = \frac{1}{2}(x - \frac{1}{2})^2 + \frac{23}{24}.$$

Its derivative can be assigned to zero at $x = 0$, so one has

$$g_x(x) = \begin{cases} 0, & x = 0, \\ x - \frac{1}{2}, & 0 < x < 1. \end{cases}$$

The above formulation allows us to define a weak solution as follows.

Definition 3.2 Given initial data $u_0 \in W^{1,3}$, the function $u \in L^\infty([0,T), W^{1,3})$ is said to be a weak solution to the initial-value problem (5) if it satisfies the following identity:

$$\int_0^T \int_{\mathbb{S}} \Big[u\,\varphi_t + k_1(\mu(u)u^2\varphi_x + \tfrac{1}{3}u_x^3\,\varphi - g_x * \big(2\mu^2(u)u + \mu(u)u_x^2\big)\,\varphi \\ -\tfrac{1}{3}\mu(u_x^3)\varphi) + k_2(\tfrac{1}{2}u^2\varphi_x - g_x * (2u\mu(u) + \tfrac{1}{2}u_x^2)\varphi)\Big]\,dx\,dt \\ + \int_{\mathbb{S}} u_0(x)\,\varphi(0,x)\,dx = 0,$$

for any smooth test function $\varphi(t,x) \in C_c^\infty([0,T) \times S)$. If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called a global weak solution.

A local well-posedess result is the following.

Theorem 3.1 (Qu, Fu, Liu, 2014) Suppose that $u_0 \in H^s(S)$ for $s > 5/2$. Then there exists $T > 0$, which depends only on $\|u_0\|_{H^s}$, such that problem (5) has a unique solution $u(t,x)$ in the space $C([0,T);H^s(S)) \cap C^1([0,T);H^{s-1}(S))$. Moreover, the solution u depends continuously on the initial data u_0 in the sense that the mapping of the initial data to the solution is continuous from the Sobolev space H^s to the space $C([0,T);H^s(S)) \cap C^1([0,T);H^{s-1}(S))$.

4. Stability of solitons of the g-KdV equation

Consider the generalized KdV equation

$$u_t + u_{xxx} + (u^p)_x = 0, \tag{6}$$

for $p > 1$ an integer, which, in the cases $p = 2$ and $p = 3$, are the KdV equation and mKdV equation, respectively. The case $p = 5$ is interesting due to the mass-critical property. For $p > 1$, it has the soliton

$$u(t, x) = c^{1/(p-1)} Q(c^{1/2}(x - x_0 - ct)) \tag{7}$$

where

$$Q(x) := \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}x)}\right)^{1/(p-1)}$$

is a positive, smooth, rapidly decreasing solution to the ODE

$$Q_{xx} + Q^p = Q.$$

Define the ground state curve

$$\Sigma = \{Q(\cdot - x_0) : x_0 \in R\} \subset H^1(R)$$

consisting of all translates of the ground state Q, then we see that $u(t)$ stays close to Σ all t.

Theorem 4.1 (Benjamin, 1972; Bona, 1975; Weinstein, 1986) Let $1 < p < 5$. If $u_0 \in H^1(R)$ is such that $\mathrm{dist}_{H^1}(u_0, \Sigma)$ is sufficiently small (say less than σ for some small constant $\sigma > 0$), and u is the solution to (6) with initial data u_0, then we have

$$\mathrm{dist}_{H^1}(u(t), \Sigma) \lesssim \mathrm{dist}_{H^1}(u_0, \Sigma)$$

for all t. Here we use $X \lesssim Y$ or $X = O(Y)$ to denote the estimate $|X| \leq CY$ for some C that depends only on p, and $X \sim$ as shorthand for $X \lesssim Y \lesssim X$.

Proof. Find a functional $u \to L(u)$ on H^1 with the following properties:
1. If u is an H^1 solution to (6), then $L(u(t))$ is non-increasing in t.
2. Q is a local minimizer of L, thus $L(u) - L(Q) \geq 0$ for all u sufficiently close to Q in H^1.
3. Furthermore, the minimum is non-degenerate in the sense that $L(u) - L(Q) \geq \|u - Q\|_{H^1}^2$, for all u sufficiently close to Q in H^1.

5. Peaked solutions of (2)

Recall that the single peakon and multi-peakons for μ-CH equation and modified μ-CH equation. Their single peakons are given respectively by

$$u(t,x) = \frac{12}{13} c g(x - ct),$$

(Khesin, Lenells, Tiglay, 2010, CMP) and

$$u(t,x) = \frac{2\sqrt{3c}}{5} g(x - ct),$$

(Qu, Fu, Liu, 2014, JFA) where

$$g(x) = \frac{1}{2}(x - [x] - \frac{1}{2})^2 + \frac{23}{24}$$

with $[x]$ denoting the largest integer part of x. Their multi-peakons are given by

$$u(t, x) = \sum_{i=1}^{N} p_i(t) g(x - q^i(t)), \tag{8}$$

where $p_i(t)$ and $q^i(t)$ satisfy the following ODE system respectively for the μ-CH equation (Khesin, Lenells, Tiglay, 2010, CMP)

$$\dot{p}_i(t) = -\sum_{j=1}^{N} p_i p_j g_x(q^i - q^j),$$
$$\dot{q}^i(t) = \sum_{j=1}^{N} p_j g(q^i(t) - q^j(t)), \quad i = 1, 2, \cdots, N,$$

and for the modified μ-CH equation (Qu, Fu, Liu, 2014, JFA)

$$
\begin{aligned}
\dot{p}_i(t) &= 0,\\
\dot{q}^i(t) &= \frac{1}{12}[\sum_{j,k\neq i}^{N}(p_j+p_k)^2+25p_i^2]+p_i[\sum_{j\neq i}^{N}p_j((q^i-q^j+\frac{1}{2}\lambda_{ij})^2+\frac{49}{12})]\\
&\quad+\sum_{j<k,j,k\neq i}^{N}p_jp_k(q^j-q^k+\epsilon_{jk})^2,
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_{ij} &= \begin{cases} 1, & i<j\\ -1, & i>j,\end{cases}\\
\epsilon_{jk} &= \begin{cases} 1, & k-j\geq 2\\ 0, & k-j\leq 1,\end{cases}
\end{aligned} \tag{9}
$$

The existence of the single peakons of Eq.(2) is governed by the following result.

Theorem 5.1. For any $c \geq -\frac{169k_2^2}{1200k_1}$, equation (2) with $\gamma = 0$ admits the peaked periodic-one traveling wave solution $u_c = \phi_c(\xi)$, $\xi = x - ct$, where $\phi_c(\xi)$ is given by

$$\phi_c(\xi) = a_{1,2}[\frac{1}{2}(\xi - \frac{1}{2})^2 + \frac{23}{24}] \tag{10}$$

with

$$a_{1,2} = \frac{-13k_2 \pm \sqrt{169k_2^2 + 1200ck_1}}{50k_1}, \quad k_1 \neq 0, \tag{11}$$

for $\xi \in [-1/2, 1/2]$ and $\phi(\xi)$ is extended periodically to the real line.

Remark 4.1 Note that the equation is invariant under $u \to -u$, $k_2 \to -k_2$. So it suffices to consider the peakon with amplitude a_1 .

Furthermore, Eq.(2) admits the multi-peakons of the form (8), where $p_i(t)$ and $q^i(t)$, $i = 1, 2, \ldots, N$, satisfy the following ODE system

$$
\begin{aligned}
&\dot{p}_i + k_2 \sum_{j=1}^{N} p_i p_j (q^i - q^j - \frac{1}{2}) = 0, \\
&\dot{q}_i - k_1 [\frac{1}{12}(23 \sum_{j,k \neq i} (p_j + p_k)^2 + 25 p_i^2) \\
&- p_i (\sum_{j \neq i} p_j (q^i - q^j)^2 + \frac{1}{2}\lambda_{ij})^2 + \frac{49}{12}) \\
&- \sum_{j<k, j,k \neq i} p_j p_k (q^j - q^k + \epsilon_{jk})^2] \\
&- k_2 \sum_{j=1}^{N} p_j (\frac{1}{2}(q^i - q^j)^2 - \frac{1}{2}|q^i - q^j| + \frac{13}{12}) = 0.
\end{aligned}
\tag{12}
$$

where λ_{ij} and ϵ_{jk} are given by (9).

In particular, when $N = 2$, system (12) can be solved explicitly, which yields

$$
\begin{aligned}
p_1 &= \frac{ae^{b(t-t_0)}}{1+e^{b(t-t_0)}}, \quad p_2 = \frac{a}{1+e^{b(t-t_0)}}, \\
q^1 &= -\frac{k_1 a^2}{b}\left(\frac{1}{12} + (\frac{1}{2} - a_1)^2\right)\frac{1}{1+e^{b(t-t_0)}} \\
&\quad +\frac{a}{12}(23k_1 a + 6a_1(a_1-1)k_2 + 13k_2)(t-t_0) \\
&\quad +\frac{a}{6b}(k_1 a - 3a_1(a_1-1)k_2)\ln(1+e^{b(t-t_0)}) + c_1, \\
q^2 &= q^1 + a,
\end{aligned}
$$

where a, a_1, $b > 0$ and t_0 are some constants.

6. Stability of peakons

◇ Case 6.1. $k_1 > 0$, $k_2 > 0$

Theorem 6.1 (Qu, Zhang, Liu, Liu, 2014, ARMA)
Let $c > 0$ and assume that $\gamma = 0$, $k_1 > 0, k_2 > 0$. For every $\epsilon > 0$, there is a $\delta > 0$ such that if $u \in C([0,T); H^1(S))$ is a solution to (2) with

$$\|u(\cdot,0) - \varphi_c\|_{H^1(S)} < \delta,$$

then

$$\|u(\cdot,t) - \varphi_c(\cdot - \xi(t))\|_{H^1(S)} < \epsilon \quad for\ t \in [0,T),$$

where $\xi(t) \in R$ is a point where $u(\cdot + \frac{1}{2}, t)$ attains its maximum.

Lemma 6.1 The peakon $\varphi_c(x)$ is continuous on S with peak at $x = \pm\frac{1}{2}$. The extrema of φ_c are

$$M_{\varphi_c} = \max_{x\in S}\{\varphi_c(x)\} = \varphi_c\left(\frac{1}{2}\right) = \frac{13}{12}H_0[\varphi_c],$$
$$m_{\varphi_c} = \min_{x\in S}\{\varphi_c(x)\} = \varphi_c(0) = \frac{23}{24}H_0[\varphi_c].$$

Moreover, we have

$$\lim_{x\uparrow\frac{1}{2}} \varphi_{c,x}(x) = \frac{1}{2}H_0[\varphi_c], \quad \lim_{x\uparrow-\frac{1}{2}} \varphi_{c,x}(x) = -\frac{1}{2}H_0[\varphi_c],$$

with

$$H_0[\varphi_c] = a_1, \quad H_1[\varphi_c] = \frac{13}{12}a_1^2, \quad H_2[\varphi_c] = \frac{1043}{960}k_1a_1^4 + \frac{47}{45}k_2a_1^3.$$

Lemma 6.2 For every $u \in H^1(S)$ and $\xi \in R$,

$$H_1[u] - H_1[\varphi_c] = \|u - \varphi_c(\cdot - \xi)\|_\mu^2 + a_1(u(\xi + \frac{1}{2}) - M_{\varphi_c}).$$

Lemma 6.3 For any function $u \in H^1(S)$ with $\mu(u) > 0$, define the function

$$F_u : \{(M, m) \in R^2 : M \geq m\} \to R$$

by

$$\begin{aligned} & F_u(M, m) \\ = & \frac{4}{3}k_1(2M + m)H_0[u]H_1[u] - \frac{64}{45}k_1(M - m)(2H_0[u](M - m))^{\frac{3}{2}} \\ + & \frac{4}{3}k_1(2M + m)H_0^3[u] - \frac{4}{3}k_1 m(4M - m)H_0^2[u] + 2k_2(2m + M)H_0^2[u] \\ + & 2k_2 M H_1[u] - 4k_2 M m H_0[u] - \frac{32}{15}k_2(M - m)^{\frac{5}{2}}\sqrt{2H_0[u]} - 4H_2[u]. \end{aligned}$$

Then it satisfies

$$F_u(M_u, m_u) \geq 0,$$

where $M_u = \max_{x\in S}\{u(x)\}$ and $m_u = \min_{x\in S}\{u(x)\}$.

Proof. Introduce two functions

$$g(x) = \begin{cases} u_x + \sqrt{2\mu(u)(u-m)}, & \xi < x \leq \eta, \\ u_x - \sqrt{2\mu(u)(u-m)}, & \eta \leq x < \xi + 1. \end{cases} \tag{13}$$

and

$$h(x) = \begin{cases} k_1(\mu(u)u + \frac{1}{3}\sqrt{2\mu(u)(u-m)}\ u_x - \frac{1}{3}u_x^2) + k_2 u, & \xi < x \leq \eta, \\ k_1(\mu(u)u - \frac{1}{3}\sqrt{2\mu(u)(u-m)}\ u_x - \frac{1}{3}u_x^2) + k_2 u, & \eta \leq x < \xi + 1. \end{cases}$$

Lemma 6.4 For the peakon φ_c, it holds that

$$F_{\varphi_c}(M_{\varphi_c}, m_{\varphi_c}) = 0, \quad \frac{\partial F_{\varphi_c}}{\partial M}(M_{\varphi_c}, m_{\varphi_c}) = 0,$$
$$\frac{\partial F_{\varphi_c}}{\partial m}(M_{\varphi_c}, m_{\varphi_c}) = 0, \quad \frac{\partial^2 F_{\varphi_c}}{\partial M \partial m}(M_{\varphi_c}, m_{\varphi_c}) = 0,$$
$$\frac{\partial^2 F_{\varphi_c}}{\partial M^2}(M_{\varphi_c}, m_{\varphi_c}) = -\frac{16}{3}k_1 H_0^2[\varphi_c] - 4k_2 H_0[\varphi_c],$$
$$\frac{\partial^2 F_{\varphi_c}}{\partial m^2}(M_{\varphi_c}, m_{\varphi_c}) = -\frac{8}{3}k_1 H_0^2[\varphi_c] - 4k_2 H_0[\varphi_c].$$

Moreover, $(M_{\varphi_c}, m_{\varphi_c})$ is the unique maximum of F_{φ_c}.

Lemma 6.5 Let $u \in C([0,T); H^1(S))$ be a solution of (2). Given a small neighborhood U of $(M_{\varphi_c}, m_{\varphi_c})$ in R^2, there exists a $\delta > 0$ such that

$$(M_{u(t)}, m_{u(t)}) \in U \quad for \quad t \in [0,T) \tag{14}$$

if $\|u(\cdot,0) - \varphi_c\|_{H^1(S)} < \delta$.

Proof of Theorem 6.1: Let $u \in C([0,T); H^1(S))$ be a solution of (2) and suppose $\epsilon > 0$ be given. Pick a neighborhood U of $(M_{\varphi_c}, m_{\varphi_c})$ small enough such that $|M - M_{\varphi_c}| < \frac{25k_1\epsilon^2}{-78k_2+6\sqrt{169k_2^2+1200ck_1}}$ if $(M,m) \in U$. Choose a $\delta > 0$ as in Lemma 5.5 so that (14) holds. Taking a smaller δ if necessary, we may assume that $\mu(u) > 0$ and

$$|H_1[u] - H_1[\varphi_c]| < \frac{\epsilon^2}{6} \quad \text{if} \;\; \|u(\cdot,0) - \varphi_c\|_{H^1(S)} < \delta.$$

Then, by Lemma 6.2, we get

$$\begin{aligned}
&\|u(\cdot,t) - \varphi_c(\cdot - \xi(t))\|^2_{H^1(S)} \\
&\leq \; 3\|u(\cdot,t) - \varphi_c(\cdot - \xi(t))\|^2_{\mu} \\
&= \; 3(H_1[u] - H_1[\varphi_c]) + 3a_1(M_{\varphi_c} - M_{u(t)}) < \epsilon^2, \;\; t \in [0,T),
\end{aligned}$$

where $\xi(t) \in R$ is any point where $u(\xi(t) + \frac{1}{2}, t) = M_{u(t)}$. Thus Theorem 5.1 is then proved. □

◇ Case 6.2. $k_1 > 0$, $k_2 < 0$

Theorem 6.2 Let $k_1 > 0$, $k_2 < 0$, and assume that $c > 23k_2^2/(64k_1)$. For every $\epsilon > 0$, there is a $\delta > 0$ such that if $u \in C([0,T); H^1(S))$ is a solution to (2) with

$$\|u(\cdot,0) - \varphi_c\|_{H^1(S)} < \delta,$$

then

$$\|u(\cdot,t) - \varphi_c(\cdot - \xi(t))\|_{H^1(S)} < \epsilon \quad for\ t \in [0,T),$$

where $\xi(t) \in R$ is a point where $u(\cdot + \frac{1}{2}, t)$ attains its maximum.

Lemma 6.6. For any function $u \in H^1(S)$ with $\mu(u) > 0$, $k_1 > 0$, $k_2 \le 0$, and $4k_1\mu(u) + 3k_2 > 0$, define the function

$$F_u : \{(M, m) \in R^2 : M \ge m\} \to R$$

by

$$\begin{aligned} F_u(M,m) \;=\; & \frac{1}{3}k_1(2M+m)H_0[u]H_1[u] - \frac{16}{45}k_1(M-m)^{\frac{5}{2}}(2H_0[u])^{\frac{3}{2}} \\ & +\frac{1}{3}k_1(2M+m)H_0^3[u] - \frac{1}{3}k_1 m(4M-m)H_0^2[u] \qquad (15) \\ & +\frac{1}{2}k_2(2m+M)H_0^2[u] + \frac{1}{2}k_2 M H_1[u] - k_2 M m H_0[u] \\ & -\frac{8}{15}k_2(M-m)^{{}_5 2}\sqrt{2H_0[u]} - H_2[u]. \end{aligned}$$

Then it satisfies

$$F_u(M_u, m_u) \ge 0,$$

where $M_u = \max_{x\in S}\{u(x)\}$ and $m_u = \min_{x\in S}\{u(x)\}$.

Proof. Let $u \in H^1(S) \subset C(S)$ with $\mu(u) > 0$. Denote $M = M_u = \max_{x\in S}\{u(x)\}$, $m = m_u = \min_{x\in S}\{u(x)\}$. Let ξ and η be such that $u(\xi) = M$ and $u(\eta) = m$. Define

$$\begin{aligned}\tilde{H}_2[u] &= k_1 \int_S \left(\mu^2(u)(u-m)^2 + \mu(u)(u-m)u_x^2 - \frac{1}{12}u_x^4\right) dx \\ &\quad + k_2 \int_S \left(\mu(u)(u-m)^2 + \frac{1}{2}(u-m)u_x^2\right) dx \\ &\equiv k_1\tilde{J}_1[u] + k_2\tilde{J}_2[u],\end{aligned}$$

with

$$\begin{aligned}\tilde{J}_1[u] &= \int_S \left(\mu^2(u)(u-m)^2 + \mu(u)(u-m)u_x^2 - \frac{1}{12}u_x^4\right) dx, \\ \tilde{J}_2[u] &= \int_S \left(\mu(u)(u-m)^2 + \frac{1}{2}(u-m)u_x^2\right) dx.\end{aligned}$$

By the Cauchy inequality, we have the estimate

$$\tilde{J}_1[u] \leq \frac{4}{3}\mu(u)\tilde{J}_2[u]. \tag{16}$$

The equality holds if and only if u is the peakon of Eq. (2). On the other hand, a straightforward computation leads to

$$\begin{aligned} \tilde{J}_1[u] &= J_1[u] - mH_0^3[u] + m^2H_0[u]^2 - mH_0[u]H_1[u], \quad \text{and} \\ \tilde{H}_2[u] &= H_2[u] - k_1m(H_0^3[u] - mH_0^2[u] + H_0[u]H_1[u]) \\ &\quad -k_2m\left(\frac{3}{2}H_0^2[u] - mH_0[u] + \frac{1}{2}H_1[u]\right), \end{aligned}$$

where

$$J_1[u] = \int_S \left(\mu^2(u)u^2 + \mu(u)uu_x^2 - \frac{1}{12}u_x^4\right) dx.$$

By virtue of the result in (Liu, Qu, Zhang, Phys. D, 2013), we have

$$\begin{aligned}\int \tilde{h}(x)g^2(x)dx &= 4J_1[u] - 2mH_0^3[u] - 2mH_0[u]H_1[u] \\ &\quad -\frac{8}{15}(m+4M)(2H_0[u](M-m))^{\frac{3}{2}},\end{aligned}$$

where

$$\tilde{h}(x) = \begin{cases} 2\mu(u)u + \frac{2}{3}\sqrt{2\mu(u)(u-m)}\, u_x - \frac{1}{3}u_x^2, & \xi < x \leq \eta, \\ 2\mu(u)u - \frac{2}{3}\sqrt{2\mu(u)(u-m)}\, u_x - \frac{1}{3}u_x^2, & \eta \leq x < \xi + 1, \end{cases}$$

and $g(x)$ is given by (13). Notice that

$$\tilde{h}(x) \leq 2MH_0[u] + \frac{2}{3}(M-m)H_0[u] = \frac{2}{3}(4M-m)H_0[u].$$

It then follows that

$$
\begin{aligned}
4J_1[u] \; & -2mH_0^3[u] - 2mH_0[u]H_1[u] - \frac{8}{15}(m+4M)(2H_0[u](M-m))^{3}2 \\
& \leq \frac{2}{3}(4M-m)H_0[u](H_1[u] + H_0^2[u] - 2mH_0[u] \\
& -\frac{4}{3H_0[u]}(2H_0[u](M-m))^{\frac{3}{2}}).
\end{aligned}
$$

Using this inequality and combining expressions, we can get (15). This completes the proof of the lemma.

7. Classification of the 2-μ-CH system

In this section, we classify the system (4), specifically, we consider the following system:

$$\begin{aligned} m_t = &\ m(u_x + a_1 v_x) + m_x(b_1 u + c_1 v) \\ &+ n(d_1 u_x + f_1 v_x) + n_x(g_1 u + h_1 v), \\ n_t = &\ n(v_x + a_2 u_x) + n_x(b_2 v + c_2 u) \\ &+ m(d_2 v_x + f_2 u_x) + m_x(g_2 v + h_2 u), \end{aligned} \tag{17}$$

where $m = \mu(u) - u_{xx}$, $n = \mu(v) - v_{xx}$, $a_i, b_i, c_i, d_i, f_i, g_i$, and $h_i, i = 1, 2$ are some constants.

- **Step 1.** Assume that system (4) possesses the weak solution

$$\begin{aligned}\int_S(\mu(u_t)\phi - u_t\phi_{xx})dx = \int_S(F_1\phi + F_2\phi_x + F_3\phi_{xx})dx,\\ \int_S(\mu(v_t)\phi - v_t\phi_{xx})dx = \int_S(G_1\phi + G_2\phi_x + G_3\phi_{xx})dx,\end{aligned}$$

for some functions $F_i(u, v, u_x, v_x)$, $G_i(u, v, u_x, v_x)$, $i = 1, 2, 3$ and $\phi(t, x) \in C_0^\infty([0, +\infty) \times S)$.

Then we find the constants satisfy

$$a_1 - c_1 = d_1 - g_1, \quad a_2 - c_2 = d_2 - g_2.$$

In this case, $\mu(u)$ and $\mu(v)$ are conserved.

• Step 2. Assume that system (4) admits two-peaked solutions of the form

$$\begin{aligned} u &= p_1(t)g(x - q_1(t)) + p_2(t)g(x - q_2(t)), \\ v &= r_1(t)g(x - q_1(t)) + r_2(t)g(x - q_2(t)), \end{aligned} \tag{18}$$

where $g(x) = \frac{1}{2}(x - [x] - \frac{1}{2})^2 + \frac{23}{24}$, and $[x]$ denotes the largest integer part of x, are the usual weak solutions in the sense of distribution, then the constants must satisfy

$$g_1 = h_1 = g_2 = h_2 = 0, \quad b_1 = c_2, \quad b_2 = c_1,$$
$$a_1 - c_1 = d_1, \quad a_2 - c_2 = d_2.$$

• Step 3. Assume that system (4) enjoys the H^1-conservation law

$$h_1[u] = \int_S (u_x^2 + v_x^2)dx.$$

then the constants satisfy

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 = f_1 = f_2 = \frac{1}{2},$$
$$d_1 = d_2 = g_1 = g_2 = h_1 = h_2 = 0.$$

Hence system (4) reduces to (3).

We have shown that system (3) has the two-peaked solutions (18) with satisfying

$$\begin{aligned}
p_1' &= ((1-b_1)p_1p_2 + (a_1-c_1)(p_1r_2+p_2r_1) \\
&\qquad +f_1r_1r_2)sgn(q_2-q_1)(-|q_1-q_2|+\tfrac{1}{2}), \\
p_2' &= ((b_1-1)p_1p_2 + (c_1-a_1)(p_1r_2+p_2r_1) \\
&\qquad -f_1r_1r_2)sgn(q_2-q_1)(-|q_1-q_2|+\tfrac{1}{2}), \\
r_1' &= ((1-b_2)r_1r_2 + (a_2-c_2)(p_1r_2+p_2r_1) \\
&\qquad +f_2p_1p_2)sgn(q_2-q_1)(-|q_1-q_2|+\tfrac{1}{2}), \\
r_2' &= ((b_2-1)r_1r_2 + (c_2-a_2)(p_1r_2+p_2r_1) \\
&\qquad -f_2p_1p_2)sgn(q_2-q_1)(-|q_1-q_2|+\tfrac{1}{2}),
\end{aligned}$$

$$
\begin{aligned}
q_1' = & -\frac{1}{2}(b_1p_2 + c_1r_2)(-|q_1 - q_2| + \frac{1}{2})^2 \\
& -\frac{13}{12}(b_1p_1 + c_1r_1) - \frac{23}{24}(b_1p_2 + c_1r_2), \\
q_2' = & -\frac{1}{2}(b_1p_1 + c_1r_1)(-|q_1 - q_2| + \frac{1}{2})^2 \\
& -\frac{13}{12}(b_1p_2 + c_1r_2) - \frac{23}{24}(b_1p_1 + c_1r_1).
\end{aligned}
$$

8. Stability of peakons for the cubic systems

Consider the following integrable two-component Novikov system

$$\begin{aligned} m_t + uvm_x + (2vu_x + uv_x)m = 0, \quad m = u - u_{xx}, \\ n_t + uvn_x + (2uv_x + vu_x)n = 0, \quad n = v - v_{xx}. \end{aligned} \tag{19}$$

It has the peaked solitions

$$\begin{aligned} u(t,x) = \varphi_c(x - ct) = ae^{-|x-ct|}, \\ v(t,x) = \psi_c(x - ct) = be^{-|x-ct|}, \end{aligned} \tag{20}$$

where $c = ab \neq 0$, and the following conserved densities

$$E_0[u,v] = \int_R (mn)^{\frac{1}{3}}\, dx,$$

$$E_u[u] = \int_R \left(u^2 + u_x^2\right)\, dx, \quad E_v[v] = \int_R \left(v^2 + v_x^2\right)\, dx,$$
$$H[u,v] = \int_R \left(uv + u_x v_x\right) dx$$

and

$$F[u,v] = \int_R \left(u^2v^2 + \frac{1}{3}u^2v_x^2 + \frac{1}{3}v^2u_x^2 + \frac{4}{3}uvu_xv_x - \frac{1}{3}u_x^2v_x^2\right)\, dx,$$

while the corresponding three conserved quantities of Novikov equation are

$$H_0[u] = \int_R m^{\frac{2}{3}}\, dx, \quad E[u] = \int_R \left(u^2 + u_x^2\right) dx,$$
$$F[u] = \int_R \left(u^4 + 2u^2u_x^2 - \tfrac{1}{3}u_x^4\right) dx.$$

Theorem 8.1. (He, Liu, Qu, 2019) Let φ_c and ψ_c be the peaked solitons traveling with speed $c = ab > 0$. Then φ_c and ψ_c are orbitally stable in the following sense. Assume that $u_0, v_0 \in H^s(R)$ for some $s \geq 3$, $0 \not\equiv (1-\partial_x^2)u_0(x)$ and $0 \not\equiv (1-\partial_x^2)v_0(x)$ are nonnegative, and there is a $\delta > 0$ such that

$$\|(u_0, v_0) - (\varphi_c, \psi_c)\|_{H^1(R)\times H^1(R)} < \delta.$$

Then the corresponding solution $(u(t,x), v(t,x))$ of the Cauchy problem for the two-component Novikov equations (19) with the initial data $u(0,x) = u_0(x)$ and $v(0,x) = v_0(x)$ satisfies

$$\sup_{t\in[0,T)} \|(u(t,\cdot), v(t,\cdot)) - (\varphi_c(\cdot - \xi(t)), \psi_c(\cdot - \xi(t)))\|_{H^1(R)\times H^1(R)} < A\delta^{\frac{1}{4}},$$

where $T > 0$ is the maximal existence time, $\xi(t) \in R$ is the maximum point of the function $u(t,x)v(t,x)$, the constant A depends only on a, b as well as the norms $\|u_0\|_{H^s(R)}$ and $\|v_0\|_{H^s(R)}$.
On stability of the train of peakons, we have the following result.

Theorem 8.2. (He, Liu, Qu, 2019) Let be given N velocities $c_1, c_2, \cdots, c_N$ such that $0 < a_1 < a_2 < \ldots < a_N$, $0 < b_1 < b_2 < \ldots < b_N$ and $c_i = a_i b_i$ for any $i \in \{1, \ldots, N\}$. There exist $A > 0$, $L_0 > 0$ and $\epsilon_0 > 0$ such that if the initial data $(u_0, v_0) \in H^s(R) \times H^s(R)$ for some $s \geq 3$ with $0 \not\equiv (1-\partial_x^2)u_0(x)$ and $0 \not\equiv (1-\partial_x^2)v_0(x)$ being nonnegative, satisfy

$$\left\|u_0 - \sum_{i=1}^N \varphi_c(\cdot - z_i^0)\right\|_{H^1} + \left\|v_0 - \sum_{i=1}^N \psi_c(\cdot - z_i^0)\right\|_{H^1} \leq \epsilon$$

for some $0 < \epsilon < \epsilon_0$ and $z_i^0 - z_i^0 \geq L$ with $L > L_0$, then there exist $x_1(t), ..., x_N(t)$ such that the corresponding strong solution $(u(t,x), v(t,x))$ satisfies

$$\left\| u(t,\cdot) - \Sigma_{i=1}^N \varphi_c(\cdot - x_i(t)) \right\|_{H^1} + \left\| v(t,\cdot) - \Sigma_{i=1}^N \psi_c(\cdot - x_i(t)) \right\|_{H^1} \leq A\left(\epsilon^{\frac{1}{4}} + L^{-\frac{1}{8}}\right),$$

for all $t \in [0, T)$, where $x_j(t) - x_{j-1}(t) > L/2$.

Conclusions and remarks

• Orbitally stable of peaked solutions to (3) in the energy space H^1?

• The classifications of the nonlocal equations with cubic nonlinear terms?

$$m_{l,t} + \sum_{i,j,k=1}^{2} a^l_{i,j,k} u_i u_j m_{k,x} + \sum_{i,j,k=1}^{2} b^l_{i,j,k} u_i u_{j,x} m_k = 0,$$

$l = 1, 2$, $m_l = \mu(u_l) - u_{l,xx}$ or $m_l = u_l - u_{l,xx}$. (Zhao, Qu, 2019)

• Geometric formulations to the cubic-type equations?

• Inverse scattering method for the μ-type equations?

• Nonlocal equations for the classical integrable systems?

Thank you!!!