Analysis on steady subsonic solutions with both fixed and free boundaries

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Motivation

- The well-posedness theory for unsteady compressible Euler equations is widely open
- An important problem in the transonic flows

Three Dimensional Euler System and Divergent Nozzles

The three-dimensional steady full Euler system reads as

$$\begin{cases} \text{div } (\rho \mathbf{u}) = 0, \\ \text{div } (\rho \mathbf{u} \otimes \mathbf{u} + P I_n) = 0, \\ \text{div } (\rho(\frac{1}{2}|\mathbf{u}|^2 + e)\mathbf{u} + P\mathbf{u}) = 0, \end{cases} \tag{1}$$

where $\mathbf{u} = (u_1, u_2, u_3), \rho, P, e$ and S stand for the velocity, density, pressure, internal energy and specific entropy, respectively. The equation of state, the internal energy e, and the sound speed are given by

$$P = A\rho^{\gamma} e^{\frac{S}{c_v}}, \quad e = \frac{P}{(\gamma - 1)\rho}, \quad c(\rho, S) = \sqrt{\partial_\rho P(\rho, S)}.$$

The nozzle wall Γ^2 can be represented by

$$\sqrt{x_2^2 + x_3^2} = x_1 \tan(\theta_0 + \epsilon f(r)), x_1 > 0, r_1 < r < r_2 \tag{2}$$

and $\theta_0 \in (0, \frac{\pi}{2})$ and f is a smooth $C^{2,\alpha}$ function defined on $[r_1, r_2]$.

Background Transonic Shock Solutions

Given $U_b^-(r_1) > c(\rho_b(r_1), S_b^-) > 0$ and $P_b(r_1)$, S_b^-,

$$(\mathbf{u}^-, P_b^-, S_b^-)(\mathrm{x}) = (\mathrm{U}_\mathrm{b}^-(\mathrm{r}_1)\mathbf{e}_\mathrm{r}, \mathrm{P}_\mathrm{b}^-(\mathrm{r}_1), \mathrm{S}_\mathrm{b}^-) \quad \text{at } \mathrm{r} = \mathrm{r}_1,$$

there exists two positive constants P_1 and P_2 such that if the pressure $P_e \in (P_1, P_2)$ is posed at the exit $r = r_2$, there exists a unique spherical symmetric transonic shock solution

$$(\mathbf{u}_b^\pm, P_b^\pm, S_b^\pm)(\mathrm{x}) = (\mathrm{U}_\mathrm{b}^\pm(\mathrm{r})\mathbf{e}_\mathrm{r}, \mathrm{P}_\mathrm{b}^\pm(\mathrm{r}), \mathrm{S}_\mathrm{b}^\pm), \tag{3}$$

to (1) defined in

$$\Omega_{un}^- = \{x \in \mathbb{R}^3 : x_2^2 + x_3^2 \le x_1^2 \tan^2\theta_0, r \in (r_1, r_b)\}$$

and

$$\Omega_{un}^+ = \{x \in \mathbb{R}^3 : x_2^2 + x_3^2 \ge x_1^2 \tan^2\theta_0, r \in (r_b, r_2)\},$$

where $r = r_b \in (r_1, r_2)$ is a shock wave, and

$$[\rho U_b] = 0, \quad [\rho_b U_b^2 + P_b] = 0, \quad S_b^+ > S_b^-,$$

where $[f]$ denotes the jump of f at $r = r_b$.

The Axisymmetric Flows

Introduce the spherical coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta\cos\varphi, \quad x_3 = r\sin\theta\sin\varphi. \tag{4}$$

and decompose the velocity $\mathbf{u} = U_1\mathbf{e}_r + U_2\mathbf{e}_\theta + U_3\mathbf{e}_\varphi$. The axisymmpetric solutions do not depend on φ so that the Euler system reads

$$\begin{cases} \partial_r(r^2\rho U_1\sin\theta) + \partial_\theta(r\rho U_2\sin\theta) = 0, \\ \rho U_1\partial_r U_1 + \frac{1}{r}\rho U_2\partial_\theta U_1 + \partial_r P - \frac{\rho(U_2^2+U_3^2)}{r} = 0, \\ \rho U_1\partial_r U_2 + \frac{1}{r}\rho U_2\partial_\theta U_2 + \frac{1}{r}\partial_\theta P + \frac{\rho U_1 U_2}{r} - \frac{\rho U_3^2}{r}\cot\theta = 0, \\ \rho U_1\partial_r(rU_3\sin\theta) + \frac{1}{r}\rho U_2\partial_\theta(rU_3\sin\theta) = 0, \\ \rho U_1\partial_r S + \frac{1}{r}\rho U_2\partial_\theta S = 0. \end{cases} \tag{5}$$

Perturbed Domain and Boundary Conditions

The perturbed nozzle is
$\Omega = \{(r,\theta,\varphi) : r_1 < r < r_2, 0 \le \theta \le \theta_0 + \epsilon f(r), \varphi \in [0, 2\pi]\}$,
where $f \in C^{2,\alpha}([r_1, r_2])$ satisfying

$$f(r_1) = f'(r_1) = 0. \tag{6}$$

Suppose the supersonic incoming flow at the inlet $r = r_1$ is given by

$$\mathbf{\Phi}_{en}^- = (U_1^-, U_2^-, U_3^-, P^-, S^-) = \mathbf{\Phi}_b^- + \epsilon \mathbf{\Psi}(\theta), \tag{7}$$

where $\mathbf{\Phi}_b^- = (U_b^-(r), 0, 0, P_b^-(r), S_b^-)$ and $\mathbf{\Psi}(\theta) \in (C^{2,\alpha}([0,\theta_0]))^5$
At the exit of the nozzle, the end pressure is prescribed by

$$P^+(x) = P_e + \epsilon P_0(\theta) \text{ on } \ r = r_2, \tag{8}$$

here $\epsilon > 0$ is sufficiently small, and $P_0 \in C^{1,\alpha}([0, 2\theta_0])$.

Denote the transonic shock surface by $\mathcal{S}$ and the upstream and downstream flows by $x_1 = \eta(x_2, x_3)$ and $(\mathbf{u}^{\pm}, P^{\pm}, S^{\pm})(x)$, respectively. Then the Rankine-Hugoniot conditions on $\mathcal{S}$ become

$$\begin{cases} [(1, -\nabla_{x'}\eta(x')) \cdot \rho\mathbf{u}] = 0, \\ [((1, -\nabla_{x'}\eta(x')) \cdot \rho\mathbf{u})\mathbf{u}] + (1, -\nabla_{x'}\eta(x'))^t [P] = 0, \\ [(1, -\nabla_{x'}\eta(x')) \cdot (\rho(e + \frac{1}{2}|\mathbf{u}|^2) + P)\mathbf{u}] = 0, \end{cases} \tag{9}$$

where $\nabla_{x'} = (\partial_{x_2}, \partial_{x_3})$. Moreover, the physical entropy condition is also satisfied

$$S^+(x) > S^-(x), \quad \text{on} \quad x_1 = \eta(x_2, x_3). \tag{10}$$

Stability of Transonic Shocks

Theorem 1 (Weng, Xie, Xin) Given the supersonic incoming flow $\mathbf{\Phi}_{en}^{-}$ satisfying the certain compatibility conditions, the transonic shock problem has a unique solution $\mathbf{\Phi}^{+} = (U_1^{+}, U_2^{+}, U_3^{+}, P^{+}, S^{+})(r, \theta)$ and $\xi(\theta)$ satisfying

(i) $\xi(\theta) \in C_{3,\alpha;(0,\theta_*)}^{(-1-\alpha;\{\theta_*\})}$ and

$$\|\xi(\theta) - r_b\|_{3,\alpha;(0,\theta_*)}^{(-1-\alpha;\{\theta_*\})} \leq C_0\epsilon, \tag{11}$$

where (r_*, θ_*) stands for the intersection circle of the shock surface with the nozzle wall and C_0 is a positive constant depending only on the supersonic incoming flow.

(ii) $\mathbf{\Phi}^{+}(r, \theta) \in C_{2,\alpha;R_+}^{(-\alpha;\Gamma_{w,s})}$, and

$$\|\mathbf{\Phi}^{+} - \mathbf{\Phi}_b^{+}\|_{2,\alpha;R_+}^{(-\alpha;\Gamma_{w,s})} \leq C_0\epsilon, \tag{12}$$

where

$$\Gamma_{w,s} = \{(r, \theta) : \xi(\theta) \leq r \leq r_2, \theta = \theta_0 + \epsilon f(r)\}.$$

Know Results and Remarks

Known Results

- Potential flows: G.-Q. Chen and Feldman (Dirichlet condition for velocity potential at the exit), Xin and Yin (the problem is in general ill-posedenss given the exit pressure), Bae and Feldman (Non-isentropic potential flows)
- flat nozzle for the Euler system: G. Q. Chen et al for velocity boundary conditions at the exit, S. X. Chen etc for the particular pressure at the exit
- Divergent nozzle for the Euler system: Li-Xin-Yin for 2D and 3D axisymmetric without swirl, S. X. Chen for 2D case

Remark

- The nozzle wall Γ^2 can depend on both r and θ.
- There is another result on the stability of transonic shock for 3D axisymmetric case with swirl via a different approach by Park after we uploaded the paper

Main Difficulty and Key Observation

- There is a singular factor $\sin\theta$ in the density equation of (5), the standard Lagragian coordinate used by Li-Xin-Yin is not invertible near the axis $\theta = 0$.
- Observation: $\sin\theta$ is of order $O(\theta)$ near $\theta = 0$. Define $(\tilde{y}_1, \tilde{y}_2) = (r, \tilde{y}_2(r,\theta))$ such that

$$\frac{\partial \tilde{y}_2}{\partial r} = -r\rho^{\pm}U_2^{\pm}\sin\theta, \quad \frac{\partial \tilde{y}_2}{\partial \theta} = r^2\rho^{\pm}U_1^{\pm}\sin\theta, \quad \text{if } (r,\theta) \in \overline{R_{\pm}},$$
$$\tilde{y}_2(r_1, 0) = 0, \quad \tilde{y}_2(r_2, 0) = 0.$$

It is clear that $\tilde{y}_2 \geq 0$ in $\overline{R_-} \cup \overline{R_+}$. Setting

$$y_1 = \tilde{y}_1 = r, \quad y_2 = \tilde{y}_2^{\frac{1}{2}}(r,\theta).$$

The transformation $\mathcal{L} : (r,\theta) \in \bar{R} \mapsto (y_1, y_2) \in \bar{D}$ satisfies

$$\det\begin{pmatrix} \frac{\partial y_1}{\partial r} & \frac{\partial y_1}{\partial \theta} \\ \frac{\partial y_2}{\partial r} & \frac{\partial y_2}{\partial \theta} \end{pmatrix} = \frac{r^2\rho U_1 \sin\theta}{2y_2} \geq C_3 > 0. \tag{13}$$

Reformulated System and the Domain

The reformulated system can be written as

$$
\begin{cases}
\partial_{y_1}\left(\dfrac{2y_2}{y_1^2\rho U_1\sin\theta}\right)-\partial_{y_2}\left(\dfrac{U_2}{y_1U_1}\right)=0,\\
\partial_{y_1}(U_1+\frac{P}{\rho U_1})-\frac{y_1\sin\theta}{2y_2}\partial_{y_2}(\frac{PU_2}{U_1})-\frac{2P}{y_1\rho U_1}-\frac{PU_2\cos\theta}{y_1\rho U_1^2\sin\theta}-\frac{(U_2^2+U_3^2)}{y_1U_1}=0,\\
\partial_{y_1}(y_1U_2)+\frac{y_1^2\sin\theta}{2y_2}\partial_{y_2}P-\frac{U_3^2}{U_1}\cot\theta=0,\\
\partial_{y_1}(y_1U_3\sin\theta)=0,\\
\partial_{y_1}B=0.
\end{cases}
$$

The nozzle wall $\Gamma_{w,s}$ is straighten to be $\Gamma_{w,y}=(\phi(M),r_2)\times\{M\}$.

Elliptic Modes

Put $\varpi = \frac{U_2}{U_1}$, then one has

$$
\begin{cases}
\partial_{y_1}\varpi - \frac{y_1\rho U_1 \varpi \sin\theta}{2y_2}\partial_{y_2}\varpi - \frac{\varpi}{y_1} - \frac{\varpi^2}{y_1}\cot\theta + \frac{y_1\sin\theta}{2y_2 U_1}\partial_{y_2}P \\
\quad - \frac{\varpi}{\rho c^2(\rho,S)}\partial_{y_1}P - \frac{U_3^2}{y_1 U_1^2}\cot\theta = 0, \\
\partial_{y_1}P - \frac{\rho c^2(\rho,S)U_1^2}{y_1(c^2(\rho,S)-U_1^2)}\left(\frac{y_1^2\rho U_1\sin\theta}{2y_2}\partial_{y_2}\varpi + \varpi\cot\theta\right) \\
\quad - \frac{y_1\rho c^2(\rho,S)U_1\varpi\sin\theta}{2y_2(c^2(\rho,S)-U_1^2)}\partial_{y_2}P - \frac{\rho c^2(\rho,S)U_1^2}{y_1(c^2(\rho,S)-U_1^2)}(\varpi^2+2) \\
\quad - \frac{\rho c^2(\rho,S)U_3^2}{y_1(c^2(\rho,S)-U_1^2)} = 0.
\end{cases}
$$

The corresponding boundary conditions become

$$
\begin{cases}
\varpi(y_1,0) = 0, \quad \forall y_1 \in [r_1, r_2], \\
\varpi(y_1,M) = \epsilon y_1 f'(y_1), \quad \forall y_1 \in [r_1, r_2], \\
P(r_2,y_2) = P_e + \epsilon P_0(\theta(r_2,y_2)), \quad \forall y_2 \in [0, M].
\end{cases}
$$

Fix the Domain

Introduce the coordinate transformation

$$z_1 = \frac{y_1 - \psi(y_2)}{r_2 - \psi(y_2)} N, \quad z_2 = y_2, \quad N = r_2 - r_b$$

so that the free boundary becomes a fixed boundary. Setting

$$\begin{aligned} &W_1(z) = \tilde{U}_1(z) - \tilde{U}_0^+(z_1),\ W_2(z) = \tilde{\varpi}(z), \\ &W_3(z) = \tilde{U}_3(z),\ W_4(z) = \tilde{P}(z) - \tilde{P}_b^+(z_1), && (14) \\ &W_5(z) = \tilde{S}(z) - S_b^+,\ W_6(z_2) = \psi(z_2) - r_b. && (15) \end{aligned}$$

After this coordinate transformation, the equation for the shock becomes

$$\psi'(z_2) = \frac{2z_2}{\sin\theta} \frac{(\tilde{U}_b^+(0) + W_1)W_2 - U_2^-(r_b + W_6(z_2), z_2)}{(r_b + W_6(z_2))((\tilde{P}_b^+(0) + W_4) - P^-(r_b + W_6(z_2), z_2))},$$

where the functions are evaluated at $(0, z_2)$.

Iteration

Define the solution class

$$\Xi_\delta = \Big\{ \mathbf{W} : \|\mathbf{W}\|_{\Xi_\delta} : \sum_{i=1}^{5} \|W_i\|_{2,\alpha;E_+}^{(-\alpha;\Gamma_{w,z})} + \|W_6\|_{3,\alpha;(0,M)}^{(-1-\alpha;\{M\})} \leq \delta;$$

$$\partial_{z_2} W_j(z_1,0) = 0,\ j = 1,3,4,5;\ W_6'(0) = W_6^{(3)}(0) = 0;$$

$$W_2(z_1,0) = \partial_{z_2}^2 W_2(z_1,0) = W_5(z_1,0) = 0 \Big\}.$$

Given any $\hat{\mathbf{W}} \in \Xi_\delta$, we will develop an iteration to produce a new $\mathbf{W} \in \Xi_\delta$ so we get a mapping $\mathcal{T}$ from Ξ_δ to itself by choosing suitable small δ. To design a good iteration, we first need to find the explicit form of the leading linear order term, and all the $\mathbf{W}$ in the remaining nonlinear error terms will be replaced by $\hat{\mathbf{W}}$ and finally the error terms should be bounded by $C(\|\hat{\mathbf{W}}\|_{\Xi_\delta}^2 + \epsilon)$.

Hyperbolic Mode

It is easy to derive that

$$\partial_{z_1} W_5 = 0, \quad \partial_{z_1} \tilde{B} = 0, \ \forall z \in [0, N] \times [0, M). \tag{16}$$

Furthermore, one has

$$\begin{cases} \partial_{z_1}[(r_b + z_1 + \frac{N - z_1}{N} W_6(z_2)) W_3 \sin\theta(z_1, z_2)] = 0, \\ W_3(0, z_2) = U_3^-(r_0 + W_6(z_2), z_2). \end{cases} \tag{17}$$

The equation for the shock can be written as

$$W_6'(z_2) = \frac{2z_2}{\sin\theta} \frac{(\tilde{U}_b(0) + W_1) W_2 - U_2^-(r_b + W_6(z_2), z_2)}{(r_b + W_6(z_2))\{\tilde{P}_b^+(0) - P_b^-(r_b) + W_4 - (P^- - P_b^-(r_b))\}},$$

where W_i are evaluated at $(0, z_2)$ and P^- is evaluated at the corresponding point on the shock.

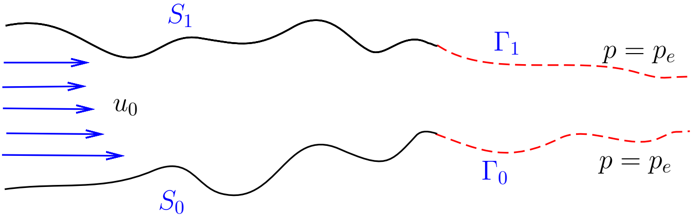
Second Order Elliptic Equation

The elliptic modes can be governed by a problem for second order equation

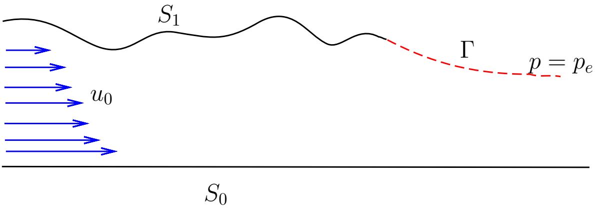
$$
\begin{cases}
\partial_{z_1}\left(\dfrac{\lambda_4(z_1)}{\lambda_2(z_1)}\partial_{z_1}\phi\right) - \left\{a\lambda_6(z_1) + \dfrac{d}{dz_1}\left(\dfrac{\lambda_4(z_1)\lambda_3(z_1)}{\lambda_2(z_1)}\right)\right\}\left(\phi(0,z_2) - \dfrac{W_6(M)}{a}\right) \\
\quad + \dfrac{\lambda_5(z_1)}{\lambda_1(z_1)}\left(\dfrac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}\left(\dfrac{\sin\theta_b(z_2)}{2z_2}\partial_{z_2}\phi\right) + \dfrac{\kappa_b\cos\theta_b(z_2)}{2z_2}\partial_{z_2}\phi\right) = \mathcal{F}, \\
\partial_{z_1}\phi(0,z_2) + \beta\left(\phi(0,z_2) - \dfrac{W_6(M)}{a}\right) = \mathcal{G}, \\
\partial_{z_1}\phi(N,z_2) = \epsilon\lambda_2(N)P_0(\hat{\theta}(N,z_2)) - \displaystyle\int_{z_2}^{M} G_1(N,s)ds, \\
\partial_{z_2}\phi(z_1,0) = 0, \\
\partial_{z_2}\phi(z_1,M) = -\dfrac{2M}{\sin\theta_b(M)}\lambda_1(z_1)\epsilon(r_0 + z_1 + \dfrac{N-z_1}{N}\hat{W}_6(M))f'.
\end{cases}
$$

The solvability condition for this problem determines the location of the shock.

Jet Problems



General Jet Problems for Two Dimensional Flows



A Simpler Case for Two Dimensional Flows

Steady Euler System

2D steady Euler System:

$$\begin{cases} div(\rho \mathbf{u}) = 0, \\ div(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = 0, \end{cases} \tag{19}$$

where $p = p(\rho)$. If we denote $p'(\rho) = c^2(\rho)$, and

$$A = \begin{pmatrix} \frac{uc^2(\rho)}{\rho} & c^2(\rho) & 0 \\ c^2(\rho) & \rho u & 0 \\ 0 & 0 & \rho u \end{pmatrix}, \; B = \begin{pmatrix} \frac{vc^2(\rho)}{\rho} & 0 & c^2(\rho) \\ 0 & \rho v & 0 \\ c^2(\rho) & 0 & \rho v \end{pmatrix}, \; U = \begin{pmatrix} \rho \\ u \\ v \end{pmatrix}$$

then, 2-D system can be written as

$$AU_{x_1} + BU_{x_2} = 0.$$

$$\det(\lambda A - B) = 0 \Longrightarrow \lambda_1 = \frac{v}{u}, \; \lambda_\pm = \frac{uv \pm c(\rho)\sqrt{u^2 + v^2 - c^2(\rho)}}{u^2 - c^2}.$$

Boundary Conditions

- The nozzle walls are assumed to be impermeable

$$(u, v) \cdot \vec{n} = 0, \qquad \text{on} \quad \partial\Omega, \tag{20}$$

where $\vec{n}$ is the unit outer normal of the nozzle walls.

- the mass flux crossing any section transversal to the x_1-axis remains a positive constant m_0,

$$\int_S (\rho u, \rho v) \cdot \vec{l} dS = m, \tag{21}$$

where $\vec{l}$ is the unit normal of S in the positive x_1-direction.

- prescribe horizontal velocity of the flow in the upstream,

$$u(x_1, x_2) \to u_0(x_2) \quad \text{as} \quad x_1 \to -\infty. \tag{22}$$

Remark One can also prescribe the Bernoulli function in the upstream.

Jet Problem

Problem Given the incoming horizontal velocity u_0 and the total flux m, find (ρ, u, v), the free boundary Γ, and the outer pressure p_e such that Γ connects with S_1, (ρ, u_1, u_2) satisfies the Euler system (19) in Ω, and

$$p(\rho) = p_e \quad \text{and} \quad (u_1, u_2) \cdot \mathbf{n} = 0 \text{ on } \Gamma,$$

where Ω is the region bounded by S_0, S_1, and Γ.

Major Progress:

- Early works: Gilbarg, Serrin, ...
- Alt, Caffarelli, Friedman (JDE, 1985): Existence of an irrotational solution via variational formulation (some recent reformulation by Lili Du, etc);
- Wang and Xin: Existence of a subsonic and sonic jet for potential flows via hodograph transformation

Main Results on Subsonic Flows with Jet

Theorem 2 (Shi, Tang, Xie) Suppose that
$S_1 = \{(x_1, x_2) | x_1 = \xi(x_2), x_2 \in [1/2, 1]\}$ and
$S_0 = \{(x_1, 0) : x_1 \in \mathbb{R}\}$. Without loss of generality, we assume
$\lim_{x_2 \to 1} \xi(x_2) = -\infty$. There exists an $\epsilon_0 > 0$ such that

$$u_0'(1) = 0, \quad |u_0'| + |u_0''| \leq \epsilon_0. \tag{23}$$

There exists an m_{cr} such that as long as $m > m_{cr}$, the jet problem has a unique solution. Furthermore, at far field, the free boundary has a representation $x_2 = k(x_1)$ satisfying

$$\lim_{x_1 \to \infty} k(x_1) = \bar{a}$$

where $\bar{a}$ is unique determined by m and $\bar{u}_1$.

Remarks:

- Jets and cavities for 2D full Euler and 3D axisymmetric Euler system

Proposition 1

$$AU_{x_1} + BU_{x_2} = 0 \Leftrightarrow \begin{cases} (\rho u)_{x_1} + (\rho v)_{x_2} = 0, \\ (u, v) \cdot \nabla \left(\dfrac{u^2 + v^2}{2} + h(\rho) \right) = 0, \\ (u, v) \cdot \nabla \left(\dfrac{\omega}{\rho} \right) = 0, \end{cases} \tag{24}$$

where $\omega = v_{x_1} - u_{x_2}$, if the given flows satisfy

$$u > 0 \text{ in } \Omega, \tag{25}$$

and the following asymptotic behavior

$u, \ \rho$ and v_{x_2} are bounded, while $v, \ v_{x_1}$ and $\rho_{x_2} \to 0, \ $ as $x_1 \to -\infty$.

Stream Function

Stream function ψ:

$$\psi_{x_1} = -\rho v, \ \psi_{x_2} = \rho u. \Longrightarrow \nabla^{\perp}\psi \cdot \nabla \left(\frac{u^2+v^2}{2} + h(\rho) \right) = 0,$$

where $\nabla^{\perp} = (-\partial_{x_2}, \partial_{x_1})$.

$$h(\rho) + \frac{|\nabla\psi|^2}{2\rho^2} = h(\rho) + \frac{1}{2}(u^2+v^2) = \mathcal{B}(\psi). \tag{26}$$

In the upstream,

$$\psi = \int_0^{X_2} \rho_0 u_0(s) ds \Longrightarrow X_2 = \kappa(\psi). \tag{27}$$

Set

$$f(\psi) = u_0'(\kappa(\psi)), \text{ and } F(\psi) = u_0(\kappa(\psi)). \tag{28}$$

Then f and F are well-defined on $[0, m]$. Furthermore,

$$f(\psi) = \rho_0 F(\psi) F'(\psi). \tag{29}$$

Representation of Density and Vorticity

$$
\begin{aligned}
(h(\rho)+\frac{|\nabla\psi|^2}{2\rho^2})(x_1,x_2) &= (h(\rho)+\frac{u^2+v^2}{2})(-\infty,\kappa(\psi)) \\
&= h(\rho_0)+\frac{F^2(\psi(x_1,x_2))}{2}.
\end{aligned}
$$

$$
\rho = H(|\nabla\psi|^2,\psi) = J\left(|\nabla\psi|^2, h(\rho_0)+\frac{F^2(\psi)}{2}\right), \tag{30}
$$

$$
\nabla\psi\cdot\nabla(\frac{\omega}{\rho})=0 \Rightarrow \frac{\omega}{\rho}(x_1,x_2) = -\frac{f(\psi(x_1,x_2))}{\rho_0} = -F(\psi)F'(\psi). \tag{31}
$$

The density ρ can be represented by

$$
\rho = H(|\nabla\psi|^2,\psi).
$$

One has the following boundary conditions

$$
\psi = 0 \text{ on } S_1, \text{ and } \psi = m \text{ on } S_2. \tag{32}
$$

Stream Function Formulation for the Jet Problem

Using the stream function formulation, the jet problem can be formulated into the following boundary value problem

$$
\begin{aligned}
&\nabla \cdot \left(g(|\nabla\psi|^2, \psi)\nabla\psi\right) - \frac{F(\psi)F'(\psi)}{g(|\nabla\psi|^2, \psi)} = 0 \text{ in } \{\psi < m\},\\
&\psi = 0 \text{ on } \mathbb{R} \times \{0\},\\
&\psi = m \text{ on } S_1 \cup \partial\{\psi < m\},\\
&|\nabla\psi| = \Lambda \text{ on } \partial\{\psi < m\}
\end{aligned}
\tag{33}
$$

and we also ask ψ satisfies

$$|\nabla\psi|^2 < \Sigma^2(\psi) \text{ on } \{\psi < m\},$$

where $g = 1/H$ and $p(H(\Lambda^2, m)) = p_e$.

Lemma 1 Let ψ be a minimizer of the problem

$$\min_{\psi \in \mathcal{K}_{\mu,R}} J^{\epsilon}_{\mu,R}(\psi), \tag{34}$$

with

$$\mathcal{K}_{\mu,R} := \{\psi \in H^1(\Omega_{\mu,R}) : \psi = \phi_{\mu,R} \text{ on } \partial\Omega_{\mu,R}\}.$$

$$J^{\epsilon}_{\mu,R}(\psi) := \int_{\Omega_{\mu,R}} G_\epsilon(|\nabla\psi|^2, \psi) + \lambda_\epsilon^2 \chi_{\{\psi<m\}}\, dx, \tag{35}$$

where

$$G_\epsilon(t,z) := \frac{1}{2}\int_0^t g_\epsilon(\tau,z)d\tau + \frac{1}{\gamma}\left(g_\epsilon(0,z)^{-\gamma} - g_\epsilon(0,m)^{-\gamma}\right)$$

and

$$\lambda_\epsilon^2 := 2\partial_t G_\epsilon(\Lambda^2, m)\Lambda^2 - G_\epsilon(\Lambda^2, m).$$

Then ψ is a weak solution to the equation in (33) and satisfies the boundary conditions in (33) in the weak sense.

L^∞ Estimate and Hölder Estimate

Let ψ be a minimizer for (34).

- ψ is a supersolution, i.e.

$$\int_\Omega \partial_p \mathcal{G}(\nabla\psi,\psi)\cdot\nabla\zeta + \partial_z\mathcal{G}(\nabla\psi,\psi)\zeta \geq 0, \text{ for all } \zeta \geq 0,\ \zeta \in C_0^\infty(\Omega).$$

- If $0 \leq \psi_0 \leq m$ on $\partial\Omega$, then

$$0 \leq \psi \leq m.$$

- $\psi \in C^{0,\alpha}_{loc}(\Omega)$ for any $\alpha \in (0,1)$. Moreover, $\|\psi\|_{C^{0,\alpha}(K)} \leq C(m, K, \epsilon_0, \lambda, \alpha, n)$ for any $K \Subset \Omega$.

- Let ψ be a supersolution in the sense of (27). Let ϕ be a solution

$$\int_{\Omega} \partial_p \mathcal{G}(\nabla\phi, \phi)\cdot\nabla\zeta + \partial_z \mathcal{G}(\nabla\phi, \phi)\zeta = 0, \text{ for all } \zeta \in C_0^{\infty}(\Omega), \tag{36}$$

  and $\phi \le \psi$ on $\partial\Omega$. Then if ϵ_0 is sufficiently small, we have $\phi \le \psi$ in Ω.

- Let $x_0 \in \{\psi < m\}$ such that $\mathrm{dist}(x_0, \Gamma_\psi) \le \min\{1, \frac{1}{4}\,\mathrm{dist}(x_0, \partial\Omega)\}$. Then if ϵ_0 is sufficiently small, there exists $C > 0$ such that

$$\psi(x_0) \ge m - C\lambda\,\mathrm{dist}(x_0, \Gamma_\psi).$$

Lipschitz Regularity, Non-Degeneracy, and Fine Properties

Let ψ be a minimizer for (34). Then

- $\psi \in C^{0,1}_{loc}(\Omega)$.
- For any $p > 1$ and any $0 < r < 1$, there exists a constant $c_r > 0$ such that for any $B_R \subset \Omega$ with $R \leq 1$, if
$$\frac{1}{R}\left(\frac{1}{|B_R|}\int_{B_R}|m-\psi|^p\right)^{1/p} \leq c_r\lambda,$$
then $\psi = m$ in B_{rR}.
- Assume that u_0 satisfies (23). Then
$$\psi_0(-\mu, x_2) < \psi(x_1, x_2) < \psi_0(R, x_2), \text{ for all } (x_1, x_2) \in \Omega_{\mu,R}.$$
- ψ is the unique minimizer and furthermore, $\partial_{x_1}\psi \geq 0$.

Unique Continuation

Inspired by the unique continuation results by Koch and Tataru, we have the following proposition.

Proposition 2 Let $\psi, \psi_0 \in W^{1,2}_{loc}(\mathbb{R} \times [0, \bar{\xi}])$, $\bar{\xi} > 0$, be two solutions to the Cauchy problem

$$\nabla \cdot \partial_p \mathcal{G}(\nabla\psi, \psi) + \mathcal{H}(\nabla\psi, \psi) = 0 \text{ in } \mathbb{R} \times (0, \bar{\xi}),$$
$$\psi = m, \ \partial_{x_2}\psi = \Lambda \text{ on } \mathbb{R} \times \{\bar{\xi}\},$$

where m, Λ are constants. Assume that $\mathbb{R}^2 \times \mathbb{R} \ni (p, z) \mapsto \mathcal{G}(p, z)$ are C^2 and $(p, z) \mapsto \mathcal{H}(p, z)$ are C^1. Then $\psi_0 = \psi$.

We combine the comparison principle and unique continuation type results.

- If $\Lambda_n \to \Lambda$, then $\psi_{\Lambda_n} \to \psi_\Lambda$ uniformly in $\Omega_{\mu,R}$ and $k_{\Lambda_n}(x_2) \to k_\Lambda(x_2)$ for each $\bar{a} < x_2 \le 1/2$.
- If $\Lambda > 0$ is large, then the free boundary $\Gamma_{\mu,R,\Lambda}$ is nonempty and it satisfies $k_\Lambda(1/2) < 0$; if Λ is small, then $k_\Lambda(1/2) > 0$.
- $N \cup \Gamma$ is C^1 in a $\{\psi < m\}$-neighborhood of A (the connecting point).

Summary and Ongoing Projcts

Summary

- Stability of transonic shocks for 3D axisymmetric solutions
- Subsonic flow with jet

Ongoing Projects

- Stability of transonic shocks under 3D perturbations for the exit pressure
- Well-posedness for 3D jet for potential flows
- 2D problem with both transonic shock and jet
- ...

Thanks!