Some recent results on incompressible flows with helical symmetry

Dongjuan Niu

Capital Normal University, Beijing

Nonlinear PDEs and Related Topics, IMS, National University of Singapore

Dec. 26-30, 2019

The 3D incompressible Euler equations

$$\boxed{\begin{cases} \partial_t u + (u \cdot \nabla) u + \nabla p = 0 & \text{in } \mathbf{R}^3 \times (0,T), \\ \operatorname{div} \mathrm{u} = 0 & \text{in } \mathbf{R}^3 \times (0,T), \end{cases}}$$

with the initial data $u(t=0) = u_0$, where p is pressure, $u = (u_1, u_2, u_3)$ is velocity fields. $(u \cdot \nabla)u$ means that $\sum_{i=1}^{3} u_i \partial_{x_i} u_j,\ j = 1, \cdots, 3$.

Global existence and uniqueness of solutions to Euler equations is still open!

One physically relevant dynamic variable for incompressible flow is $\omega := curl\ u(t, x)$, the vorticity, which satisfies the evolution equation

$$\boxed{\partial_t \omega + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u.} \tag{0.1}$$

Note that ∇u is formally of the same order as ω. Thus the vortex stretching term $(\omega \cdot \nabla)u \approx \omega^2$.

♣ For 2D case, the vorticity $\omega = \partial_x u_2 - \partial_y u_1$ is a scalar function and the vortex stretching term $(\omega \cdot \nabla)u$ in (0.1) vanishes.

Known results

- 2D case (not comprehensive nor state-of-the-art)
  - Global existence and uniqueness of strong solutions when $\omega_0 \in L^\infty$ (Yudovich 1963);
  - Global existence of weak solutions when $\omega_0 \in L^p \cap L^1, p > 1$ (Diperna-Majda 1987);
  - Global existence of weak solutions when ω_0 is a Radon measure with one sign. (Delort 1991; Majda, Evans, Muler, Wu, Schochet, Xin, M. Lopes, L. Lopes, etc);
  - Global existence and uniquenss of solutions in Hölder spaces or Soblolev or Besov spaces (Wolibner, Bourgains, Temam, Kato, Vishik, Chae, Park, etc);
- 3D case (not comprehensive nor state-of-the-art)
  - Local results (Majda-Bertozzi 2002)
  - Global existence of the weak/admissible weak solution emanating from a L^2/vortex sheet initial data in periodic domain (Wiedemann 2011; Szekelyhidi 2011)

Axi-symmetric Flow

$$u(x,t) = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z, \tag{0.2}$$

$$p(x,t) = p(r,z,t). \tag{0.3}$$

The vorticity $\omega = \text{curl } u = -\partial_z u^\theta e_r + \omega^\theta e_\theta + \frac{1}{r}\partial_r(ru^\theta)e_z$, where $\omega^\theta = \partial_r u_z - \partial_z u_r$. Furthermore,

$$\partial_t(\frac{\omega^\theta}{r}) + u \cdot \tilde{\nabla}(\frac{\omega^\theta}{r}) = -\frac{\partial_z(ru^\theta)^2}{r^4}, \tag{0.4}$$

where

$$\tilde{\nabla} = v^r\partial_r + v^z\partial_z.$$

Known results

CASE I No Swirl ($u^\theta = 0$)(not comprehensive nor state-of-the-art)

- $\exists!$ Strong solutions to inviscid axisymmetrical flows with the assumption of $\frac{\omega_0}{r} \in L^\infty$ (Ukhovsky, Yudovich 1968; Danchin 2007 weaken with $\frac{\omega_0}{r} \in L^{3,1}$ and $\omega_0 \in L^\infty \cap L^{3,1}$)
- Global well-posedness of smooth solutions of Euler equations (Saint Raymond 1994)
- Global existence of weak solutions with initial vorticity in $L^{\frac{6}{5}} \cap L^p, p > 3$ (Chae-Kim 1997; Extended to $L^1 \cap L^p, p > 1$ by Jiu-Wu-Yang 2013)
- Global existence in Besov space with $u_0 \in B^{1+\frac{3}{p}}_{p,1}$ and $\frac{\omega_0}{r} \in L^{3,1}$. (Abidi-Hmidi-Keraani 2010)
- $\ldots$

Known results

CASE II Swirl ($u^\theta \neq 0$) (not comprehensive nor state-of-the-art)

- Well-posedness still Open
- Lower bound of lifespan (Danchin 2013)
- Blow up criteria (Chae et al; Jiu-Xin 2001; Kubica-Porkorny-Zajaczkowski 2012; Wang-Zhang 2012)
- Liouville theorem (Z. Xin, Q. Jiu, 2015)
- Global regularity under some assumptions (Chen-Strain-Tsai-Yau 2008, 2009; Koch-Nadirashvili-Seregin-Sverak 2009; Z. Lei, Q.S. Zhang 2011)
- Singular formation in special domain (Elgindi, Jeong 2018)
- . . .

Helically Symmetrical Flows:

- Invariant under rotation and simultaneous translation along axis of rotation.
- This invariance (helical symmetry) preserved by fluid flow.

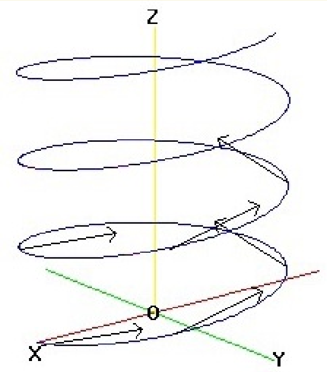

Figure: Helically symmetric vector fields u

Helical flows

Definition

- Vector field u is helical symmetry if $u(S_\theta(x)) = R_\theta u(x)$,
- A scalar function f is helical symmetry if $f(S_\theta(x)) = f(x)$.

$$S_\theta(X) := R_\theta X + \begin{pmatrix} 0 \\ 0 \\ \frac{\sigma}{2\pi}\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ \frac{\sigma}{2\pi}\theta \end{pmatrix} \tag{0.5}$$

$\sigma > 0$ is called step, which is defined as the translation displacement along the symmetry axis after one full clockwise turn around the axis.

Figure: Picture of tornada & simulation in Purdue vortex chamber in 1988

Observations

- Helical flow is periodic in term of x_3 variable with σ period

Observations

- Helical flow is periodic in term of x_3 variable with σ period
- $\sigma = 0 \iff$ axisymmetric flow ?

Observations

- Helical flow is periodic in term of x_3 variable with σ period
- $\sigma = 0 \iff$ axisymmetric flow ? Radial flows (Lopes-N.-Lopes-Titi, Preprint, 2010)

Observations

- Helical flow is periodic in term of x_3 variable with σ period
- $\sigma = 0 \iff$ axisymmetric flow ? Radial flows (Lopes-N.-Lopes-Titi, Preprint, 2010)
- $\sigma = \infty \iff$ the helix becomes straight line ?

Observations

- Helical flow is periodic in term of x_3 variable with σ period
- $\sigma = 0 \iff$ axisymmetric flow ? Radial flows (Lopes-N.-Lopes-Titi, Preprint, 2010)
- $\sigma = \infty \iff$ the helix becomes straight line ? 2.5D flows (Lopes-Mazzucato-N.-Lopes-Titi, 2014)

Beltrami flow (Helical flow)

Definition (Gromeka (1881); Beltrami (1889); Dritschel (1991))

A steady 3D fluid flow is called a Beltrami flow if the vorticity $\omega = curl v$ satisfies the Beltrami condition

$$\boxed{\omega(x) = \lambda(x) v(x) \text{ for some } \lambda(x) \neq 0} \tag{0.6}$$

for all x.

Results: Any steady, divergence-free velocity field $v(x)$ in $\mathbb{R}^3$ that satisfies Beltrami condition (0.6), is a solution to the 3D Euler equation.

Experiments show that flows in which the vorticity ω is locally roughly parallel to the velocity generate interesting 3D instabilities. This can result from the self-induced velocity of vortex lines deforming the vortex lines into very unstable horseshoes.

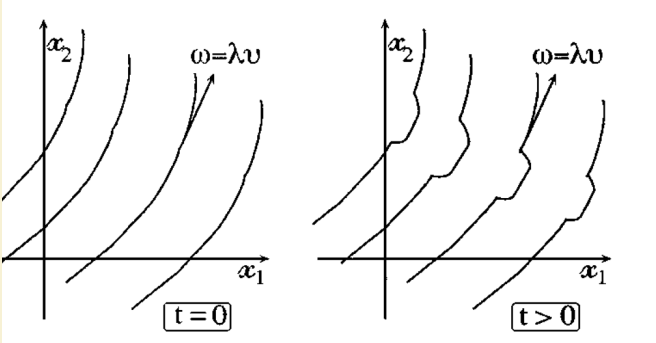

Figure: The figure on the left shows vortex lines at time $t = 0$. The figure on the right shows what can happen to these lines if the vorticity is roughly aligned with the velocity field as in the case of a Beltrami flow

Known results

1) No helical swirl (i.e., $u_0 \cdot \xi = 0$):

- Well-posedness for smooth data (A. Dutrifoy 1999).
- $\exists!$ strong solution if $(curl u_0)_3 \in L^\infty_c$ (B. Ettinger and E. S. Titi 2008).
- $\exists$ weak solution with $(curl u_0)_3 \in L^p_c, p > \frac{4}{3}$. (A. Bronzi, M. Lopes, H. Lopes 2015).
- $\exists!$ strong solutions with initial data lying in critical spaces, e.g., $\omega_0 \in \mathring{\mathcal{B}}^0_{\infty,1}$ with additional assumptions. (H. Abidi,S. Sakrani 2016)

- The limits of the 3D viscous and inviscid incompressible flows respectively approach the flow, the dynamic of which are two-dimensional when the step σ goes to ∞. (Mazzucato, M. Lopes, N., H. Lopes, Titi 2014)
- Vanishing viscosity limit problem of 3D Navier-Stokes equations (Jiu, M. Lopes, N., H. Lopes 2018)

2) With helical swirl (i.e., $u_0 \cdot \xi \neq 0$) : Open!

Properties of Helical Flow

- The helical symmetry reduces the periodicity of u with σ with respect to the x_3 variable. Thus, the effective domain is $\mathbf{R}^2 \times (0, \sigma)$ instead of $\mathbf{R}^3$.
- Define $\eta := \mathbf{u} \cdot \boldsymbol{\xi}$, called helical swirl, which satisfies

$$\partial_t \eta + \mathbf{u} \cdot \nabla \eta = 0,$$

  where $\boldsymbol{\xi} = (-y, x, 1)^t$ is the tangential direction of the flows along the helical line.
- If $\mathbf{u}$ is helical, then the vorticity

$$\omega = curlu = \omega_3 \frac{\vec{\xi}}{\sigma} + \frac{1}{\sigma}(\frac{\partial \eta}{\partial y}, -\frac{\partial \eta}{\partial x}, 0),$$

- The vorticity equation

$$\frac{D\omega}{Dt} + \frac{1}{\sigma}\omega_3(u_2, -u_1, 0)^t + \frac{1}{\sigma}(\partial_x \eta \partial_y u - \partial_y \eta \partial_x u) = 0.$$

- when $\eta = 0$, $\omega = \omega_3 \frac{\vec{\xi}}{\sigma}$ and

$$\frac{\partial \omega_3}{\partial t} + (u \cdot \nabla)\omega_3 = 0.$$

- Two-dimensional property of helical flows. The following proposition shows that sufficiently smooth functions and vector fields with helical symmetry are essentially two-dimensional.

Mazzucato-Lopes-N.-Lopes-Titi,2014

Let $\mathbf{u} = \mathbf{u}(x)$ be a smooth helical vector field and let $p = p(x)$ be a smooth helical function. Then there exist unique $\mathbf{w} = (w^1, w^2, w^3)(y_1, y_2)$ and $q = q(y_1, y_2)$ such that

$$\mathbf{u}(x) = R_{2\pi x_3/\sigma}\mathbf{w}(y(x)), \quad p = p(x) = q(y(x)), \tag{0.7}$$

Conversely, if $\mathbf{u}$ and p are defined through (0.7) for some $\mathbf{w} = \mathbf{w}(y_1, y_2)$, $q = q(y_1, y_2)$, then $\mathbf{u}$ is a helical vector field and p is a helical scalar function,

where

$$y(x) = \begin{bmatrix} y_1 \\ \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos(2\pi x_3/\sigma) & -\sin(2\pi x_3/\sigma) \\ \\ \sin(2\pi x_3/\sigma) & \cos(2\pi x_3/\sigma) \end{bmatrix} \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix}.$$

Questions

- Can we prove the global existence of solutions to 3D incompressible Euler equations with helical symmetry WITHOUT/WITH helical swirl?
- Can we prove the global existence of weak solutions in the Critical space without helical swirl compared with Bronzi, M. Lopes, H. Lopes's result, i.e., $(curl\, u_0)_3 \in L^p, p > \frac{4}{3}$ with compact support?
- Can we prove the global existence of solution with helical swirl in any regular space?

Motivation

- Helical flow seems more close to two-dimensional flow instead of three-dimensional flow (without helical swirl case).
- There exists Sobolev embedding $W^{1,p} \hookrightarrow\hookrightarrow L^2, p > 1$ without helical swirl in some sense.
- The case with helical swirl is very Different!

Case I: Without helical swirl

We look at the global existence of weak solutions to 3D Euler equations with helical symmetry WITHOUT helical swirl.

Jiu-Li-N., 2017

Given a scalar helical function $\omega_0 \in L^1_{per} \cap L^p_{per}(\mathbb{R}^2 \times [-\pi, \pi])$ with some $p > 1$, for any $T > 0$, there exist weak solutions $\mathbf{u} = \mathbf{u}(t, x) \in L^\infty([0, T]; W^{1,p}_{loc}(\mathbb{R}^2 \times [-\pi, \pi]))$ to the three-dimensional Euler equations with helical symmetry with the initial vorticity $\boldsymbol{\omega_0} = \omega_0 \boldsymbol{\xi}$ in the sense of distribution. Moreover, $\mathbf{u} \cdot \boldsymbol{\xi} = 0$.

Difficulties

I. We should prove that the following inequality by virtue of Biot-Savart law

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \le c(p)\|\omega\|_{L^p(\mathbb{R}^N)}, \ \ 1 < p < \infty.$$

Compared with the classical case:

$$\mathbf{u}(t,x) = \int_{\mathbb{R}^N} K_N(x-y)\omega(t,y)dy, \ x \in \mathbb{R}^N.$$

Singular Integral Operator theory tells us that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \le c(p)\|\omega\|_{L^p(\mathbb{R}^N)}, \ 1 < p < \infty.$$

In fact, Biot-Savart Law with helical symmetry has the additional requirement

$$\begin{cases} \mathrm{curl}\mathbf{u} = \boldsymbol{\omega} = \omega_3 \boldsymbol{\xi}, & \text{in} \quad \mathbb{R}^2 \times [-\pi, \pi], \\ \mathrm{div}\mathbf{u} = 0, & \text{in} \quad \mathbb{R}^2 \times [-\pi, \pi], \\ \mathbf{u} \cdot \boldsymbol{\xi} = 0, & \text{in} \quad \mathbb{R}^2 \times [-\pi, \pi], \\ \mathbf{u}(x) = O(|x'|), & \text{as} \ |x'| \to \infty, \\ \mathbf{u} \text{ periodic in } x_3, & \text{in } \mathbb{R}^2. \end{cases}$$

- We express the form of $\mathbf{u} = \nabla G * \omega$, where G is the Green function with the complicated form as

$$G(x) = -\frac{1}{2\pi} \ln|x'| + \frac{1}{\pi} \sum_{n=1}^{\infty} K_0(n|x'|) \cos(nx_3).$$

Bounded domain case

Using the condition of vanishing helical swirl, the divergence free condition can be rewritten as

$$
\begin{aligned}
&\partial_x u_1 + \partial_y u_2 + \partial_z (y u_1 - x u_2) = 0, \\
&\partial_\xi u_1 = (-y\partial_x + x\partial_y + \partial_z) u_1 = -u_2, \\
&\partial_\xi u_2 = (-y\partial_x + x\partial_y + \partial_z) u_2 = u_1.
\end{aligned}
$$

Then we introduce the stream function ψ similar to 2D case, i.e.,

$$
\begin{cases}
-\dfrac{\partial \psi}{\partial y} = (1+y^2) u_1 - xy u_2, \\
\dfrac{\partial \psi}{\partial x} = (1+x^2) u_2 - xy u_1.
\end{cases}
$$

The stream function ψ satisfies the second-order elliptic equations with variable coefficients, i.e.,

$$\boxed{\begin{cases} L_H\psi =: \nabla \cdot (K\nabla\psi) = \omega, \\ \psi \mid_{\partial\Omega} = 0, \end{cases}}$$

where $K(x, y) = \frac{1}{1+x^2+y^2} \begin{pmatrix} 1+y^2 & -xy \\ -xy & 1+y^2 \end{pmatrix}$.

Then the regularity theory of second-order elliptic equations shows that

$$\|\psi\|_{w^{2,p}} \leq c\|\omega\|_{L^p},$$

which implies that

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} \leq c(p)\|\omega\|_{L^p(\mathbb{R}^N)}, \ 1 < p < \infty.$$

Difficulties

II. We lose the L^2 integrability of velocity field, i.e., we DONOT have $\int_{\mathbb{R}^2\times[-\pi,\pi]} |\mathbf{u}|^2 dx < \infty$.

- For two-dimensional flow, the velocity must satisfy the strong restriction that $\int_{\mathbb{R}^2} \omega dx = 0$ because the decay behavior of the two-dimensional kernel at infinity is like $\frac{1}{r}$, which is not square integrable.
- The standard strategy to decompose the velocity fields into an explicit and steady solution, and the other part, which is recovered from the vorticity with the zero average by classical Biot-Savart law.
- L^2_{loc} estimate instead (which helps us to remove the assumption of compact support).

Case II: With helical swirl

We look at the global existence of 3D Euler equations WITH helical swirl.

N.& Swierczewska-Gwiazda, 2018

Let $\mathbf{u}_0$ be a helical divergence free vector fields with $\mathbf{u}_0 \in L^2 \cap L^\infty(\mathbb{R}^2 \times [-\pi, \pi])$ and its initial vorticity $x\omega_0 \in \mathring{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2 \times [-\pi, \pi])$. In addition, we assume that $(\nabla_h \eta_0, \mathbf{u}_0) \in \mathring{\mathcal{B}}^0_{\infty,1}(\mathbb{R}^2 \times [-\pi, \pi])$. Then the Euler system with helical symmetry has a local existence of strong solutions. Morever, the lifespan $T^\star$ satisfies that

$$T^\star \geq \frac{\log\left(1 + \frac{1}{2}\log\left(1 + \frac{\|\omega_0\|_{\mathring{\mathcal{B}}^0_{\infty,1} \cap L^\infty} + \|x_h\omega_0\|_{\mathring{\mathcal{B}}^0_{\infty,1} \cap L^\infty}}{\|\nabla_h \eta_0\|_{\mathring{\mathcal{B}}^0_{\infty,1}} + \|\mathbf{u}_0\|_{L^\infty}}\right)\right)}{\|\omega_0\|_{\mathring{\mathcal{B}}^0_{\infty,1} \cap L^\infty} + \|x_h\omega_0\|_{\mathring{\mathcal{B}}^0_{\infty,1} \cap L^\infty}}.$$

Besov space

Definition

Let u be a mean free function in $\mathcal{S}'(\mathbb{R}^2 \times (-\pi, \pi))$, 2π-periodic with respect the third variable $(p, r) \in [1, +\infty]^2$ and $s \in \mathbb{R}$ be given real numbers. Then u belongs to the Besov space $\mathring{\mathcal{B}}^s_{p,r}$ if and only if

$$\|\mathbf{u}\|_{\mathring{\mathcal{B}}^s_{p,r}} := \sum_{n \in \mathbb{Z}} \|\mathbf{u}_n\|_{\mathring{B}^s_{p,r}} < \infty, \tag{0.8}$$

where u_n is the Fourier coefficient, is computed as follows

$$u_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\cdot, \cdot, z) e^{-inz} dz,$$

and we recall the defintition of standard homogenous Besov type spaces with

$$\|\mathbf{u}\|_{\mathring{B}^s_{p,r}} := (2^{js} \|\mathring{\Delta}_j \mathbf{u}\|_{L^p})_{l^r}.$$

The sketch of the proof

The main DIFFICULTY lies in the a priori estimates involved with η and ω.

<u>STEP 1:</u> Utilizing the helical property that $\partial_\xi \mathbf{u} = \mathbf{u}^\perp$, we rewrite the equations of η as

- $$\partial_t \eta + (\tilde{\mathbf{u}}_h \cdot \nabla_h)\eta = 0,$$

  where $\tilde{\mathbf{u}}_h = (u_1 + yu_3, u_2 - xu_3)^t$ satisfying $div_h \tilde{\mathbf{u}}_h = 0$.
- From the propogation of Besov norm for transport equations, we know that

$$\|\eta(t)\|_{\mathring{\mathcal{B}}^1_{\infty,1}} \lesssim \|\eta_0\|_{\mathring{\mathcal{B}}^1_{\infty,1}} e^{\int_0^t \|\nabla_h \tilde{\mathbf{u}}_h(s)\|_{\mathring{\mathcal{B}}^0_{\infty,1}} ds} \tag{0.9}$$

STEP 2: The equation of ω satisfies that

- $\partial_t \omega + (\tilde{\mathbf{u}}_h \cdot \nabla_h)\omega = \partial_y \eta \partial_x \mathbf{u} - \partial_x \eta \partial_y \mathbf{u} + (\omega_3 \mathbf{u}^{\perp} - u_3 \boldsymbol{\omega}^{\perp}) =: G_1(t,x)$,
- with the estimate of $G_1(t,x)$ as

$$\|G_1(t,x)\|_{\mathring{\mathcal{B}}^0_{\infty,1}} \lesssim \|\nabla_h \eta\|_{\mathring{\mathcal{B}}^0_{\infty,1}} \|\nabla_h \mathbf{u}\|_{\mathring{\mathcal{B}}^0_{\infty,1}} + \|\nabla_h \mathbf{u}\|_{\mathring{\mathcal{B}}^0_{\infty,1}} \|\mathbf{u}\|_{L^\infty}.$$

- Then we similarly prove that

$$\|\omega(t)\|_{\mathring{\mathcal{B}}^0_{\infty,1}} \leq (\|\omega_0\|_{\mathring{\mathcal{B}}^0_{\infty,1}} + \int_0^t \|G_1(t,x)\|_{\mathring{\mathcal{B}}^0_{\infty,1}})(1 + \int_0^t \|\nabla_h \tilde{\mathbf{u}}_h\|_{L^\infty} ds$$

STEP 3: To close the above estimates, we need to look for the estimate of $||\nabla_h \tilde{\mathbf{u}}_h||_{L^\infty}$. Indeed, we have the following lemma

lemma

Let $\mathbf{v} = (v_1, v_2, v_3)^t$ be a divergence free helical vector field, 2π-periodic with respect the third variable, then

$$||\nabla \mathbf{v}||_{\mathring{\mathcal{B}}^0_{\infty,1}} \lesssim ||curl \mathbf{v}||_{\mathring{\mathcal{B}}^0_{\infty,1}}.$$

Moreover,

$$||\nabla_h(x_h \mathbf{v})||_{\mathring{\mathcal{B}}^0_{\infty,1}} \lesssim ||curl \mathbf{v}||_{\mathring{\mathcal{B}}^0_{\infty,1}} + ||x_h curl \mathbf{v}||_{\mathring{\mathcal{B}}^0_{\infty,1}} + ||\mathbf{v}||_{L^2}.$$

It implies that

$$||\nabla_h \tilde{\mathbf{u}}_h||_{L^\infty} \leq C(||\omega||_{\mathring{\mathcal{B}}^0_{\infty,1}} + ||x_h \omega||_{\mathring{\mathcal{B}}^0_{\infty,1}} + ||\mathbf{u}||_{L^2}).$$

STEP 4: Then we need to prove the estimate of $\|x_h\omega\|_{\mathring{\mathcal{B}}^0_{\infty,1}}$.

- $$\begin{aligned} &\partial_t(x_h\omega)+(\tilde{\mathbf{u}}_h\cdot\nabla_h)(x_h\omega)\\ &=\partial_y\eta\partial_x(x_h\mathbf{u})-\partial_x\eta\partial_y(x_h\mathbf{u})+(x_h\omega_3\mathbf{u}^\perp-u_3x_h\omega^\perp)\\ &\quad+\tilde{u}_{1,h}\omega-\partial_y\eta\mathbf{u}\\ &=:G_2(t,x)\end{aligned}$$

- with
$$\begin{aligned} &\|G_2(t,x)\|_{\mathring{\mathcal{B}}^0_{\infty,1}\cap L^\infty}\\ &\lesssim\|\nabla_h\eta\|_{\mathring{\mathcal{B}}^0_{\infty,1}}(\|\nabla_h(x_h\mathbf{u})\|_{\mathring{\mathcal{B}}^0_{\infty,1}\cap L^\infty}+\|\nabla_h\mathbf{u}\|_{\mathring{\mathcal{B}}^0_{\infty,1}\cap L^\infty})\\ &\quad+(\|\nabla_h\mathbf{u}\|_{\mathring{\mathcal{B}}^0_{\infty,1}}+\|\mathbf{u}\|_{\mathring{\mathcal{B}}^0_{\infty,1}\cap L^\infty})\|x_h\omega\|_{\mathring{\mathcal{B}}^0_{\infty,1}\cap L^\infty}.\end{aligned}$$

- $$\|x_h\omega(t)\|_{\mathring{\mathcal{B}}^0_{\infty,1}}\leq(\|\omega_0\|_{\mathring{\mathcal{B}}^0_{\infty,1}}+\int_0^t\|G_2(t,x)\|_{\mathring{\mathcal{B}}^0_{\infty,1}})(1+\int_0^t\|\nabla_h\tilde{\mathbf{u}}_h\|_{L^\infty}$$

STEP 5:

- Define $A(t) := \|\omega(t)\|_{\mathring{\mathcal{B}}^0_{\infty,1}\cap L^\infty}$, $Z(t) := \|x_h\omega(t)\|_{\mathring{\mathcal{B}}^0_{\infty,1}\cap L^\infty}$,
- $B(t) := \|\eta(t)\|_{\mathring{\mathcal{B}}^1_{\infty,1}} + \|\mathbf{u}\|_{L^\infty}$
- Then

$$B(t) \leq B(0)e^{C\int_0^t \|\nabla_h \tilde{\mathbf{u}}_h(s)\|_{L^\infty} ds},$$

$$A(t) \leq (A(0) + \int_0^t B(s)A(s)ds)(1 + \int_0^t (A(s) + Z(s))ds),$$

$$Z(t) \leq (Z(0) + \int_0^t B(s)Z(s)ds)(1 + \int_0^t (A(s) + Z(s))ds).$$

- Define $M(t) := A(t) + Z(t)$, then

$$M(t) \lesssim (M(0) + B(0)\int_0^t M(s)ds\, exp\{\int_0^t M(s)ds\})(1 + \int_0^t M(s)d$$

- We assume that for some $T' > 0$

$$B(0)\int_0^{T'} M(s)ds\, exp\{\int_0^{T'} M(s)ds\} \leq M(0). \tag{0.10}$$

- Gronwall inequality implies that

$$M(t) \leq 2M(0)e^{2tM(0)} \quad \text{for all } t \in [0, T']. \tag{0.11}$$

Therefore, for (0.10) is valid, it suffices that

$$exp\{2(e^{2T'M(0)} - 1)\} - 1 \leq M(0).$$

- The lifespan T^* satisfies

$$T^* \geq \frac{1}{2M(0)} \log(1 + \frac{1}{2}\log(1 + \frac{M(0)}{B(0)})), \tag{0.12}$$

which is the desired inequality.

Thank you