Non-Existence in Finite Energy Space to the Cauchy Problem of Compressible Navier-Stokes Equations

Hailiang Li

Capital Normal University, Beijing

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Joint with

Y.-X Wang (LanZhou U), Z. Xin (CUHK)

Outline

- Models

- Motivations

- Main results

- Reformulation and Sketch of Proofs

Multi-D full compressible Navier-Stokes equations

Consider the full compressible Navier-Stokes equations (CNS) for the viscous compressible heat-conductive fluid in $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = \mu \Delta u + (\mu+\lambda)\nabla \operatorname{div} u \\ \partial_t(\rho e) + \operatorname{div}(\rho e u) + p \operatorname{div} u = \frac{\mu}{2}|\nabla u + (\nabla u)^*|^2 + \lambda(\operatorname{div} u)^2 + \frac{\kappa(\gamma-1)}{R}\Delta e \end{cases} \tag{1}$$

ρ — density; $u = (u_1, u_2, u_3)$ — velocity; e — internal energy.

μ, λ — coefficients of viscosity with $\mu > 0,\ 2\mu + n\lambda \geq 0$.

κ — coefficient of heat conduction with $\kappa \geq 0$.

p — pressure with $p = (\gamma - 1)\rho e$.

Compressible isentropic Navier-Stokes equations

Consider the isentropic compressible Navier-Stokes equations (CNS) for the viscous compressible heat-conductive fluid in $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mu \Delta u + (\mu + \lambda)\nabla \mathrm{div} u \end{cases} \tag{2}$$

ρ — density; $u = (u_1, u_2, u_3)$ — velocity, n=3.

μ, λ — coefficients of viscosity with $\mu > 0,\ 2\mu + n\lambda \geq 0$.

p — γ-law pressure with $p = a\rho_\gamma$, $a > 0$ constant, $\gamma > 1$ specific heat ratio.

Cauchy problem for compressible Navier-Stokes

Consider the Cauchy problem for the full CNS with vacuum

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mu \Delta u + (\mu + \lambda)\nabla \mathrm{div} u, \\ \partial_t(\rho e) + \mathrm{div}(\rho e u) + p \mathrm{div} u = \frac{\mu}{2}|\nabla u + (\nabla u)^*|^2 + \lambda(\mathrm{div} u)^2 + \frac{\kappa(\gamma-1)}{R}\Delta e. \end{cases} \tag{3}$$

The initial data are given by

$$(\rho, u, e)(x, 0) = (\rho_0, u_0, e_0)(x), \quad x \in \mathbb{R}^n, \tag{4}$$

and the initial density is compact supported on the open set $\Omega \subset \mathbb{R}^n$

$$\mathsf{supp}_x\, \rho_0 = \mathsf{supp}_x\, e_0 = \bar{\Omega}, \quad \rho_0(x) > 0,\ e_0(x) > 0,\ x \in \Omega. \tag{5}$$

- Vacuum — space-time region where density is zero, $\rho(x,t) = 0$.

Compressible isentropic Navier-Stokes equations

Consider the Cauchy problem for the isentropic CNS with vacuum

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \nabla p = \mu \Delta u + (\mu + \lambda)\nabla \mathrm{div} u, \end{cases} \tag{6}$$

The initial data are given by

$$(\rho, u)(x, 0) = (\rho_0, u_0)(x), \quad x \in \mathbb{R}^n. \tag{7}$$

with the initial density compact supported on the open set $\Omega \subset \mathbb{R}^n$

$$\mathrm{supp}_x\, \rho_0 = \bar{\Omega}, \quad \rho_0(x) > 0, \quad x \in \Omega. \tag{8}$$

Aim: Well-posedess/ill-posedness of classical solution in the inhomogeneous Sobolev space (finite energy space) to the Cauchy problem of Navier-Stokes equations in the presence of vacuum.

Motivations: Open problems

• Open problem

— To prove the global existence and uniqueness of classical (smooth) solution $(\rho, u, e)(x, t)$ to the CNS for general (arbitrarily large) smooth initial data $(\rho_0, u_0, e_0)(x)$

Motivations: Difficulties in mathematical analysis

- Strong nonlinearities

— Burger's type nonlinearity cause the formation of singularities

$u_t + uu_x = 0$, $u(x,0) = u_0(x)$ with $u_0'(x_0) < 0$ for some x_0.

- Coupled system of hyperbolic–parabolic type

— Solve the density via transport equation for given velocity $u \in C^1$

$\rho_t + (\rho u)_x = 0$, $\rho(x,0) = \rho_0(x)$

$\rho(x(t),t) = \rho_0(x_0)e^{-\int_0^t u_x(x(s),s)ds}, \quad x'(t) = u(x(t),t), x(0) = x_0$

- Strong degeneracy in the presence of vacuum

— The equation for velocity changes from parabolic type to elliptic one

Motivations: Progress

(A) Fluid density is away from vacuum: $\rho(x,t) > 0$

- **One-dimension**:

— **Local/global existence** for smooth data or discontinuous initial data

Kazhikhov–Shelukhin, Kanel, Hoff, ...

Key point: Non-formation of vacuum for solutions if no vacuum initially.

— Weak solution: Hoff–Smoller (CMP 2001) proved that weak solutions of the 1D compressible Navier-Stokes equations (1.1) do not exhibit vacuum states in a finite time provided that no vacuum is present initially,

$$\rho_0(x) > 0,\ a.e., \quad \Longrightarrow \quad \rho(x,t) > 0,\ a.e.$$

(A) Fluid density is away from vacuum: $\rho(x,t) > 0$

— Classical solution: Define (v,u) with $v = 1/\rho$ in one-dimension, and

$$\Phi(v) = \begin{cases} \ln\dfrac{\bar{v}}{v} + \dfrac{v}{\bar{v}} - 1, & \gamma = 1, \\ \\ \frac{1}{\gamma-1}v^{-\gamma+1} - \frac{1}{\gamma-1}\bar{v}^{-\gamma+1} + \bar{v}^{-\gamma}(v-\bar{v}), & \gamma > 1, \end{cases} \tag{9}$$

for a constant $\bar{v} > 0$. One can show (c.f. Kanel)

$$\int_{\mathbb{R}} \Big(|(\ln v)_x|^2 + \Phi(v) \Big) dx \le E(0) < +\infty, \tag{10}$$

which implies for some positive constants $0 < \rho_- < \rho_+ < +\infty$ that

$$\rho_- \le \rho(x,t) \le \rho_+, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+.$$

so long as it holds initially

$$0 < \rho_1 \le \rho(x,0) \le \rho_2 < +\infty, \quad x \in \mathbb{R}.$$

(A) Fluid density is away from vacuum: $\rho(x,t) > 0$

- **Multi-dimension**:

— **Local/global existence** for smooth initial data with small oscillations

Nash, Matsumura-Nishida, Hoff, Kawol, Jiang, Tani, Danchin, ...

— Global smooth solutions near to a non-vacuum equilibrium $\bar{\rho}, \bar{e} > 0$ is constructed in the energy space $C^1([0,T]; H^m(\mathbb{R}))$ for $T > 0$

$$\rho - \bar{\rho} \in C(0,T; H^3(\mathbb{R}^3)) \cap C^1(0,T; H^2(\mathbb{R}^3)),$$

$$u, e - \bar{e} \in C(0,T; H^3(\mathbb{R}^3)) \cap C^1(0,T; H^1(\mathbb{R}^3)).$$

Key point: Non-formation of vacuum for strong solution:

— The appearance of vacuum can be prevented for either smooth large initial data for short time, or smooth initial data with small perturbation for global time period.

(B) Fluid density contains vacuum: $\rho(x,t) \geq 0$

• Global weak solution in multi-D for general initial data with finite energy:

— Lions (Oxford 1998), Feireisl–Novotny–Petzeltová (JMFM, 2001), Jiang-Zhang (CMP, 2001), ...

— **Question**:

Does the homologous local/global existence to Nash, Matsumura-Nishida hold for CNS containing vacuum ?

(B) Fluid density contains vacuum: $\rho(x,t) \geq 0$

• Ill-posedness (Blowup) of classical solution containing vacuum in finite energy space for large time

— Xin (CPAM, 1998) showed that there is no global smooth solution in energy space $C^1([0,T];H^m(\mathbb{R}))$ to Cauchy problem for (3) and (6) with a nontrivial compactly supported initial density, which gives results for finite time blow-up in the presence of vacuum.

-- Classical solutions of viscous compressible fluids without heat conduction will blow up in finite time, as long as the initial data has an isolated mass group.

(B) Fluid density contains vacuum: $\rho(x,t) \geq 0$

• Local existence of classical solution in homogeneous Sobolev space:

— Cho-Choe-Kim (JMPA, 2004; JDE, 2003; MM ,2006) proved the local well-posedness of classical solutions to the Cauchy problem of isentropic CNS and full CNS for the initial density containing vacuum for some $T > 0$ in the homogeneous Sobolev space

$$\rho \in C(0,T;H^3(\Omega)) \cap C^1(0,T;H^2(\Omega)),$$

$$u \in C(0,T;D^3(\Omega)) \cap L^2(0,T;D^2(\Omega)), \quad D^3(\Omega) =: \{f;\ \nabla f \in H^2(\Omega)\};$$

• Global existence of classical solution in homogeneous Sobolev space for smooth initial data with small initial energy:

Huang–Li–Xin (CPAM,2012), Ding-Wen-Yao-Zhu (SIAM JMA 2012), ...

Main results

Question: Well-posedess/ill-posedness of classical solution in energy s-pace $C^1([0,T]; H^m(\mathbb{R}))$ to the Cauchy problem of Navier-Stokes equations in the presence of vacuum.

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Question: Well-posedess/ill-posedness of classical solution in energy space $C^1([0,T];H^m(\mathbb{R}))$ to the Cauchy problem of Navier-Stokes equations in the presence of vacuum.

Thm: There exist initial data so that the isentropic compressible Navier-Stokes (3) or the full compressible Navier-Stokes (6) **has no classical solution** (ρ, u) **in the finite energy space** $C^1([0,T];H^m(\mathbb{R})), m > 2$ **for any positive time** $T > 0$.

Main results

Consider the one-dimensional isentropic CNS (6)

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p)_x = \nu u_{xx}, \\ (\rho, u)(x, 0) = (\rho_0, u_0), \quad x \in \mathbb{R}, \end{cases} \tag{11}$$

where the initial density satisfies

$$\mathrm{supp}_x\, \rho_0 = [0, 1], \quad \rho_0(x) > 0, \quad x \in (0, 1), \tag{12}$$

and for some positive numbers $\lambda_i, i = 1, 2, 3, 4$ with $0 < \lambda_3, \lambda_4 < 1$ that

$$\frac{(\rho_0)_x}{\rho_0} \geq \lambda_1 > 0, \quad u_0(\lambda_3) < 0, \quad u_0 \leq 0, \quad in\ (0, \lambda_3), \tag{13}$$

or

$$\frac{(\rho_0)_x}{\rho_0} \leq -\lambda_2 < 0, \quad u_0(\lambda_4) > 0, \quad u_0 \geq 0, \quad in\ (\lambda_4, 1). \tag{14}$$

Thm 1. (L.-Wang-Xin, ARMA 2018) Let (13)–(14) hold for $n = 1$. The Cauchy problem (11)–(12) for 1D isentropic Navier-Stokes equations with initial density satisfying (12) has no classical solution $(\rho, u) \in C^1([0,T]; H^m(\mathbb{R})), m > 2$ (inhomogeneous Sobolev space) for any positive time $T > 0$, if the initial data (ρ_0, u_0) satisfy either (13) or (14).

Recall

$$\frac{(\rho_0)_x}{\rho_0} \geq \lambda_1 > 0, \quad u_0(\lambda_3) < 0, \quad u_0 \leq 0, \quad in\ (0, \lambda_3), \tag{13}$$

or

$$\frac{(\rho_0)_x}{\rho_0} \leq -\lambda_2 < 0, \quad u_0(\lambda_4) > 0, \quad u_0 \geq 0, \quad in\ (\lambda_4, 1). \tag{14}$$

Rem 2. (i) There exists a class of initial data (ρ_0, u_0) satisfying the conditions (13) or (14). Let k, l be positive numbers. Taking

$$\rho_0(x) = \begin{cases} x^k(1-x)^k, \ for \ x \in \ [0,1], \\ 0, \ for \ x \in \mathbb{R} \setminus [0,1] \end{cases} \tag{15}$$

and

$$u_0(x) = \begin{cases} -x^l, \ for \ x \in \ [0, \frac{1}{4}], \\ \ smooth \ connection, \ for \ x \in \ (\frac{1}{4}, \frac{3}{4}), \\ (1-x)^l, \ for \ x \in \ in \ [\frac{3}{4}, 1], \\ 0, \ for \ x \in \mathbb{R} \setminus [0,1], \end{cases} \tag{16}$$

then (ρ_0, u_0) meet all the needs of (13) or (14).

Rem 3. (i) It is well known that the CNS (11)-(12) is well-posed in homogeneous Sobolev space in classical sense if ρ_0 and u_0 satisfy the following compatibility condition (Cho-Kim, MM (2006), JDE (2006))

$$\begin{cases} -\mu\Delta u_0 - (\mu+\lambda)\nabla \mathrm{div} u_0 + \nabla p_0 = \rho_0 g, \\ g \in D^1, \quad \sqrt{\rho_0} g \in L^2. \end{cases} \tag{17}$$

In one-dimensional case, for (ρ_0, u_0) given by (15) and (16), we have

$$g = \begin{cases} O(x^{l-k-2}) + O(x^{l-k-1}) + O(x^{k(\gamma-1)-1}),\ for\ x \in\ [0, \frac{1}{4}], \\ \ smooth\ connection,\ for\ x \in\ (\frac{1}{4}, \frac{3}{4}), \\ O((1-x)^{l-k-2}) + O((1-x)^{l-k-1}) + O((1-x)^{k(\gamma-1)-1}),\ for\ x \in\ in\ [\frac{3}{4}, 1], \\ 0,\ for\ x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

The direct calculations show that the initial data (ρ_0, u_0) satisfy (17) so long as

$$k > \frac{3}{2(\gamma-1)}, \qquad l > k + \frac{5}{2}. \tag{18}$$

(ii) In particular, if we want to search solutions for the CNS (6)-(7) in the inhomogenous energy space $C^1([0,T];H^m(\mathbb{R})), m > 2$, then the compatibility condition should be

$$\begin{cases} -\mu\Delta u_0 - (\mu+\lambda)\nabla \mathrm{div} u_0 + \nabla p_0 = \rho_0 g, \\ g \in D^1, \quad \sqrt{\rho_0} g \in L^2, \quad u_0 \in L^2. \end{cases}$$

For the initial data (ρ_0, u_0) satisfying (15)–(18), the CNS (6)-(7) is well-posedness in the homogeneous Sobolev space but ill-posedness in the inhomogeneous Sobolev space $C^1([0,T];H^m(\mathbb{R}))$, $m > 2$ for any positive time T. Therefore, the solution constructed by Cho-Kim (MM (2006), JDE (2006)) has no finite energy in $C^1([0,T];H^m(\mathbb{R})), m > 2$ for any positive time T even if the initial data has finite energy in $H^m(\mathbb{R})$.

Precisely, it may imply for the strong solution (ρ, u) to 1D CNS that

$$\int_{\mathbb{R}} u(x,t)^2 dx = +\infty, \quad \forall\, t > 0, \tag{19}$$

even if it holds initially

$$\int_{\mathbb{R}} u_0(x)^2 dx < \infty.$$

Other Cases: Non-existence of finite energy solution

Thm 4. (L.-Wang-Xin, ARMA 2018) There exists a class of initial data with the initial density being compactly supported, such that there is no classical solution (ρ, u, e) in inhomogenous Sobolev space $C^1([0,T]; H^m(\mathbb{R}))$, $m > 2$ for any positive time $T > 0$ to the following Cauchy problem for

- The one-dimensional full compressible Navier-Stokes with zero heat conductivity ($\kappa = 0$),

- The multi-dimensional full compressible Navier-Stokes with positive heat conductivity ($\kappa > 0$).

Reformulation and proof

One-dimensional isentropic CNS

Assume $\Omega \triangleq I = (0,1)$ and set $A = 1$ for simplicity. Let $(\rho, u) \in C^1([0,T]; H^m(\mathbb{R})), m > 2$ be one solution to the system (6)-(7) with the initial density satisfying (12). Let $a(t)$ and $b(t)$ be the particle paths stating from $a(0) = 0$ and $b(0) = 1$ respectively,

$$a'(t) = u(a(t), t), \quad a(0) = 0, \quad b'(t) = u(b(t), t), \quad b(0) = 1. \tag{20}$$

Following from $(6)_1$ and $(6)_2$ respectively, it holds

$$\text{supp}_x\, \rho = [a(t), b(t)], \quad t \in (0, T], \tag{21}$$

$$u_{xx}(x,t) = 0, \quad \forall\, x \in \mathbb{R} \backslash [a(t), b(t)] \times (0, T]. \tag{22}$$

By (22), it leads to

$$u(x,t)=\begin{cases} u(b(t),t)+(x-b(t))u_x(b(t),t), \ if \ x>b(t) \\ u(a(t),t)+(x-a(t))u_x(a(t),t), \ if \ x<a(t). \end{cases}$$

which together with $u(\cdot,t)\in H^m(\mathbb{R}), m>2$, implies

$$u(x,t)=u_x(x,t)=0, \forall\ x\in\mathbb{R}\backslash[a(t),b(t)]\times(0,T], \tag{23}$$

and then

$$[a(t),b(t)]=[0,1], \quad i.e., \quad \mathsf{supp}_x\,\rho(x,t)=[0,1]. \tag{24}$$

• To prove the well-posedness of the one-dimensional IVP (11)-(12) for CNS with the initial density satisfying (12) is equivalent to show the well-posedness of the following over-determined IBVP problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \ \ in\ I \times (0, T], \\ (\rho u)_t + (\rho u^2 + p)_x = \nu u_{xx}, \ \ in\ I \times (0, T], \\ (\rho, u, u_x)(0, t) = (\rho, u, u_x)(1, t) = 0, \quad t \in (0, T], \\ (\rho, u)(x, 0) = (\rho_0, u_0), \ \ on\ I, \end{cases} \tag{25}$$

where $I = (0, 1)$ and $\nu = 2\mu + \lambda$. As for (13) and (14), we have either

$$\begin{cases} \frac{(\rho_0)_x}{\rho_0} \geq \lambda_1, \quad in\ (0, \lambda_3), \\ u_0(\lambda_3) < 0, \quad u_0 \leq 0, \quad in\ (0, \lambda_3), \end{cases} \tag{26}$$

or

$$\begin{cases} \frac{(\rho_0)_x}{\rho_0} \leq -\lambda_2, \quad in\ (\lambda_4, 1), \\ u_0(\lambda_4) > 0, \quad u_0 \geq 0, \quad in\ (\lambda_4, 1). \end{cases} \tag{27}$$

Thm 5. The over-determined IBVP (25) has no classical solution $(\rho, u)(x, t)$ in the space $C^{2,1}(I \times (0, T]) \cap C(\bar{I} \times [0, T])$ for any positive time $T > 0$, if the initial data (ρ_0, u_0) satisfy the conditions (26) or (27).

• Key point: Let $(\rho, u)(x, t)$ be the classical solution to the over-determined IBVP problem (25) for one-dimensional compressible Navier-Stokes equations. Then, under the assumptions (26) or (27) of initial data, it holds

$$\frac{\partial u(x, t)}{\partial \nu}\Big|_{x=0,1} > 0, \quad t \in (0, T],$$

with ν being the outward unit normal vector. But, it holds $u(x, t) = u(x, t) = 0$ for $x = 0, 1$ by assumptions. Contradiction!

THANK YOU FOR YOUR ATTENTION!