Optimal global regularity for elliptic equations which is degenerate or singular on the boundary

Huai-Yu Jian

Tsinghua University

Presented at NUS

The Original Problem

$$(MA) \qquad \begin{aligned} \det D^2 u = F(x,u) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega \end{aligned}$$

where Ω is a bounded convex domain in R^n, and F satisfies the assumption (H):

$F(x,t) \in C(\Omega \times (-\infty, 0))$ *is non-decreasing in* t *for any* $x \in \Omega$

and

$$0 < F(x,t) \leq A d_x^{\beta-n-1} |t|^{-\alpha}, \ \forall (x,t) \in \Omega \times (-\infty, 0)$$

for some constants $A > 0$, $\beta \geq n+1$ *and* $\alpha \geq 0$, *where* $d_x = dist(x, \partial\Omega)$.

Motivations (1)

When $F(x,t) = |t|^{-n-2}$ and u is a solution to problem (MA), then

• $(-u)^{-1}u_{x_ix_j}dx_idx_j$ is a Hilbert metric in Ω

—[Loewner-Nirenberg: 1974]

• The Legendre transform

$$y = Du(x), \quad u^*(y) = x \cdot y - u(x).$$

The graph of u^* defines an Affine hyperbolic spheres

— [Calabi: Symp Math, 1972] and [Cheng-Yau: CPAM, 1977]

• Affine hyperbolic sphere is a well-known important model in Affine geometry as well as a fundamental model in Affine Sphere Relativity

— [Minguzzi: CMP, 2017]

Motivations (2)

• When $F = f(x)|t|^{-p}$, Problem (MA) is the Projection of the equation

$$\det D^2 u = f(x) u^{-p} \text{ in } S^n \subset R^{n+1}$$

on the plane $\{x_{n+1} = -1\}$ from the unit sphere S^n, which is L_p-Minkowski problem in the affine geometry

—[Lutwak: JDG, 1993; Chou-Wang: Adv Math, 2006; Jian-Lu-Zhu: Calculus PDE, 2016; Jian-Lu-Wang: JFA, 2018]

—[Lutwak etc: JAMS, 2013; Huang etc: Acta Math, 2016; Jian-Lu-Wang: Adv Math, 2015; Jian-Lu: Adv Math, 2019]

• Also, for general F, Problem (MA) may be obtained from the constructing non-homogeneous complete Einstein-K$\ddot{a}$hler metrics on a tubular domain

— [Cheng-Yau: CPAM, 1986].

Cheng-Yau's Results

Cheng and Yau in [Cheng-Yau: CPAM, 1977] proved that *if* Ω *is a strictly convex* C^2*-domain and* $F \in C^k$ *(*$k \geq 3$*) satisfies* (H)*, then the problem* (MA) *admits an unique convex solution* $u \in C^{k+1,\varepsilon}(\Omega) \bigcap C(\bar{\Omega})$ *for any* $\varepsilon \in (0, 1)$

Questions

• Q 1: For the Affine Hyperbolic Sphere

$$(AHS) \quad \det D^2 u = |u|^{-n-2} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

what is the (optimal) boundary regularity of the Affine Hyperbolic Sphere? —-(mentioned a few times by S. T. Yau)

• Q 2: What is the existence, uniqueness and the optimal boundary regularity for the problem (MA) *even if F is not in C^3, or $\partial\Omega$ is not in C^2, or Ω is not strictly convex*?

Answer to Q1 (Yau's Question)

Theorem 1 [J-Wang: JDG, 2013] *Suppose Ω is a bounded, uniformly convex domain in R^n with $C^{k,\alpha}$ boundary, where $3 \le k \le n+2$ and $\alpha \in (0,1)$. Then the graph M_v is $C^{k,\alpha}$ up to its boundary.*

Theorem 2 [J-Wang-Zhao: JDE, 2017] *If n is even, then the graph M_v is C^∞ up to its boundary if $\partial\Omega \in C^\infty$. But if n is odd, the result in Theorem 1 is optimal.*

Regularity results for uniformly elliptic case

$$\det D^2 u = f(x) \text{ in } \Omega, \;\; u = \phi \text{ on } \partial\Omega,$$

where Ω is bounded and strictly convex.

• Assume $0 < c_1 \leq f(x) \leq c_2$ and $f \in C^k$ ($k \geq 2$), and $\partial\Omega, \phi$ are sufficiently smooth. The C^{k+1}-regularity for the solution was obtained by

— Clabi (1958)

— Pogorelov (1971) —Cheng-Yau (1976, 1977)

—Caffarelli-Nirenberg-Spruck (1983)

—Tian (1983), Trudinger-Urbas (1983), $\cdots$, etc

Assume $0 < c_1 \leq f(x) \leq c_2$ and $f \in C^\alpha$

• The interior $C^{2,\alpha}$-regularity for the solution was obtained by

— [Caffarelli: Ann Math, 1990] for $\alpha \in (0,1)$, which was re-proved by [J-Wang: Siam J Math Anal, 2007] for $\alpha \in [0,1]$

• The boundary $C^{2,\alpha}$-regularity was obtained by

—[Trudinger-Wang: Ann Math, 2008] when and $\partial\Omega, \phi \in C^3$

— [Savin: JAMS, 2012] when $\partial\Omega, \phi \in C^{2,\alpha}$.

Regularity results for degenerate elliptic case

• If $f \geq 0$ and $f^{1/(n-1)} \in C^{1,1}$, then $u \in C^{1,1}$.
— [Guan P.F., Trudinger N.S., Wang X.J.: Acta Math, 1999]

• If $f(x) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, the regularity of the solution and its asymptotic expansion near the origin was studied by
—[Rios C., Sawyer E.T., Wheeden R.L: Adv Math, 2006]
—[Savin: CPAM, 2010]

• Assume $0 < c_1 \leq f(x) \leq c_2$ and $f \in C^\alpha$ ($\alpha \in (0, 1]$). The global $C^{2,\alpha}$-regularity for the problem

$$\det D^2 u = f(x)(d_x)^\alpha \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega,$$

was obtained by
— [Savin: Invent Math, 2017].

• The $C^\infty(\bar{\Omega})$ solution to the Eigenvalue Problem

$$\det D^2 u = (-\lambda u)^n) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega,$$

was obtained by
— [Hong-Huang-Wang: CPDE, 2011] when $n = 2$
— [Savin: Invent Math, 2017] when $n \geq 2$.

• Suppose that $\mu(u) > 0$ is nondecreasing in u, $p > n + 1$, $\alpha \in [0, 2(p - n - 1))$, and $\partial\Omega \in C^{1,1}$. The $C^2(\Omega) \bigcap C^\delta(\bar{\Omega})$ (for some $\delta \in (0, 1)$) solution for the problem

$$\det D^2 u = \mu(u)(d_x)^\alpha (1 + |Du|^2)^p \text{ in } \Omega, \quad u = \phi \text{ on } \partial\Omega$$

was obtained by
—[Chen: Lecture Notes Math, No 1306, 1986]
— [Urbas: Invent Math, 1986]

Question 2

Q2: How are about the existence, uniqueness and the optimal boundary regularity of the solution to the

$$(MA)\qquad \begin{aligned} \det D^2 u &= F(x,u) \ \text{ in } \Omega, \\ u &= 0 \ \text{ on } \partial\Omega \end{aligned}$$

when F is not in C^3, or $\partial\Omega$ is not in C^2, or Ω is not strictly convex? Here, F satisfies the assumption (H):

$F(x,t) \in C(\Omega \times (-\infty, 0))$ *is non-decreasing in t for any $x \in \Omega$*

and

$$0 < F(x,t) \le A d_x^{\beta-n-1} |t|^{-\alpha}, \ \forall (x,t) \in \Omega \times (-\infty, 0)$$

for some constants $A > 0$, $\beta \ge n+1$ and $\alpha \ge 0$, where $d_x = dist(x, \partial\Omega)$.

Answer to Question 2

Theorem 3 [J-Li-Tu: Preprint, 2018] *Supposed that* Ω *is a bounded convex domain in* R^n *and* $F(x,t)$ *satisfies* (H). *Let*

$$\gamma_1 := \begin{cases} \frac{\beta-n+1}{n+\alpha}, & \text{if} \beta < \alpha+2n-1, \\ \textit{any number in}(0,1), & \text{if} \beta \geq \alpha+2n-1. \end{cases}$$

Then problem (MA) *admits an unique convex (Alexandrov) generalized solution* $u \in C^{\gamma_1}(\overline{\Omega})$. *Furthermore,* $u \in C^{2,\gamma_1}(\Omega)$ *if* $F(x,t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

Improved the Regularity for (a, η) type domain

Denote

$$x = (x_1, x_2, \ldots, x_n) = (x', x_n), \quad x' = (x_1, ..., x_{n-1})$$

Definition . *Supposed that* Ω *is a bounded convex domain in* R^n*, and* $x_0 \in \partial\Omega$*. We say* x_0 *is* (a, η) *type if there are numbers* $a \in [1, +\infty)$ *and* $\eta > 0$*, after translation and rotation transforms, we have*

$$x_0 = 0 \ \text{ and } \ \Omega \subseteq \{x \in R^n | x_n \geq \eta |x'|^a\}.$$

Ω *is called* (a, η) *type domain if every point of* $\partial\Omega$ *is* (a, η) *type.*

Remarks on (a, η) type domain

Remark 1. The convexity requires that the number a should be no less than 1. The less is a, the more convex is the domain. There is no (a, η) type domain for $a \in [1, 2)$, although part of $\partial\Omega$ may be (a, η) type for $a \in [1, 2)$.

Remark 2. $(2, \eta)$ type domain is equivalent to the domain satisfies exterior sphere condition.

Hölder exponent can be described by the convexity for Ω.

Theorem 4 [J-Li-Tu: Preprint, 2018] *Supposed that* Ω *is* (a, η) *type domain in* R^n *with* $a \in [2, +\infty)$, *and* F *satisfies* (H). *Let*

$$\gamma_2 := \begin{cases} \frac{\beta-n+1}{n+\alpha} + \frac{2n-2}{a(n+\alpha)}, & \text{if} \beta < \alpha + 2n - 1 - \frac{2n-2}{a}, \\ \text{any number in } (0,1), & \text{if} \beta \geq \alpha + 2n - 1 - \frac{2n-2}{a}. \end{cases}$$

Then the convex generalized solution obtained in Theorem 3 satisfies

$$u \in C^{\gamma_2}(\overline{\Omega}).$$

Furthermore $u \in C^{2,\gamma_2}(\Omega)$ *if* $F(x,t) \in C^{0,1}(\Omega \times (-\infty, 0))$.

Remark This result was obtained by J-Li in JDE, 2018 for $F \equiv t^{-n-2}$

The boundary regularity of Theorems 3 and 4 is optimal

Consider the equation for affine hyperbolic sphere

$$\det D^2 u = \frac{1}{|u|^{n+2}} \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

The $\gamma_1 = \frac{1}{n+1}$ for general convex domain, and $\gamma_2 = \frac{1}{2}$ for $(2, \eta)$ type domain, any of which can not be improved. In fact,

(1) If $\Omega = B_1^{n-1}(0) \times R^+$, then the solution is

$$u(x) = -\frac{(n+1)^{\frac{1}{2}}}{n^{\frac{n}{2(n+1)}}} x_n^{\frac{1}{n+1}} [1 - (x_1 + \cdots + x_{n-1})^2]^{\frac{n}{2(n+1)}};$$

(2) If $\Omega = B_1(0)$, then $u(x) = -\sqrt{1 - |x|^2}$.

The sketch of the proof of Theorem 3-(1)

- Lemma: *Let Ω be a bounded convex domain and u be a convex function with $u|_{\partial\Omega} = 0$. If there is a $\gamma \in (0, 1]$ and a $M > 0$ such that*

$$|u(y)| \leq M{d_y}^{\gamma}, \ \ \forall y \in \Omega$$

where $d_y = dist(y, \partial\Omega)$, then

$$|u|_{C^{\gamma}(\overline{\Omega})} \leq M[1 + (\frac{diam(\Omega)}{2})^{\gamma}].$$

- For any point $y \in \Omega$, letting $z \in \partial\Omega$ be the nearest boundary point to y. Since problem (MA) is invariant under translation and rotation transforms, we assume $z = 0$, $\Omega \subseteq R^n_+$ and the line yz is the $x_n - axis$.

The sketch of the proof of Theorem 3-(2)

- Let

$$W(x) = -Mx_n^\gamma \cdot \sqrt{N^2 l^2 - r^2}$$

where $l = diam(\Omega)$ and $r = \sqrt{x_1^2 + ... + x_{n-1}^2}$. Choosing positive constants γ, M, N and after tedious calculation we find that W is an sub-solution to problem (MA).

- By comparison principle for generalized solutions, we have

$$|u(y)| \le |W(y)| \le MNl y_n^{\frac{\beta-n+1}{n+\alpha}} = MNl d_y^{\frac{\beta-n+1}{n+\alpha}},$$

which, together with the Lemma, implies the following

The sketch of the proof of Theorem 3-(3)

• A Priori Estimate: *Under the assumptions of Theorem 3, if $u \in C(\overline{\Omega})$ is a convex generalized solution to problem (MA), then $u \in C^{\gamma_1}(\overline{\Omega})$ and*

$$|u|_{C^{\gamma_1}(\overline{\Omega})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n).$$

• The above method can be used to prove Theorem 4, but constructing the sub-solution to problem (MA) is much more complicated. Its form is

$$W(x_1, ..., x_n) = -[(\frac{x_n}{\varepsilon})^{\frac{2}{a}} - x_1^2 - ... - x_{n-1}^2]^{\frac{1}{b}}$$

where b and ε need to be chosen according three cases $a = 2$; $a \geq \frac{2\alpha+2}{\beta-n+1}$; $2 < a < \frac{2\alpha+2}{\beta-n+1}$ if $\frac{2\alpha+2}{\beta-n+1} > 2$.

The sketch of the proof of Theorem 3-(4)

• Suppose that Ω is bounded convex but $F(x,t) \in C^k(\Omega \times (-\infty, 0))$ $(k \geq 3)$ satisfies (H).

Choose a sequence of bounded and strictly convex domains $\{\Omega_i\}$ such that

$$\Omega_i \in C^2 \text{ and } \Omega_i \subseteq \Omega_{i+1}, i = 1, 2, \cdots, \bigcup_{i=1}^{\infty} \Omega_i = \Omega.$$

Then by Cheng-Yau's result, there exists a convex generalized solution u_i to problem (MA) in the domain Ω_i for each i. Set $u_i \equiv 0$ in $R^n \setminus \Omega_i$ and extend u in R^n. By the a priori estimate, we have the uniform estimations

$$|u_i|_{C^{\gamma_1}(\overline{\Omega})} = |u_i|_{C^{\gamma_1}(\overline{\Omega_i})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n),$$

The sketch of the proof of Theorem 3-(5)

which implies that there is a subsequence, still denoted by itself, convergent to a u in the space $C(\overline{\Omega})$ and

$$\textit{Hölder estimate} \quad |u|_{C^{\gamma_1}(\overline{\Omega})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n).$$

By the well-known convergence result for convex generalized solutions, we see that u is a convex generalized solution to problem (MA).

• Drop the restriction on the smoothness for F.

Suppose $F_j \in C^k(\Omega \times (-\infty, 0))$ $(k \geq 3)$ satisfy satisfies (H) as above, and F_j locally uniform convergence to F in as $j \to \infty$. Then by the above result, for each j, there exists a convex generalized solution $u_j \in C^{\gamma_1}(\overline{\Omega})$ to problem (MA) with F replaced by F_j.

The sketch of the proof of Theorem 3-(6)

Moreover, by the Hölder estimate we have

$$|u_j|_{C^{\gamma_1}(\overline{\Omega})} \leq C(\alpha,\ \beta,\ A,\ diam(\Omega),\ n)$$

for all j. Using this estimate, we obtain a generalized solution u to problem (MA), which is the limit of a subsequence of u_j in the space space $C(\overline{\Omega})$. Furthermore, the solution u still satisfies the Hölder estimate. The uniqueness for (MA) is directly from the comparison principle.

• It remains to prove $u \in C^{2,\,\gamma_1}(\Omega)$ if $F(x,t) \in C^{0,1}(\Omega \times (-\infty, 0))$. The Hölder estimate implies $F(x, u(x)) \in C^{\gamma_1}(\overline{\Omega})$. Hence we can use the Caffarelli's local $C^{2,\alpha}$ regularity to obtain $u \in C^{2,\,\gamma_1}(\Omega)$. [Caffarelli: Ann Math, 1990; Jian-Wang: Siam J Math Anna, 2007]

Application to proper affine hyperspheres-1

Finding proper affine hyperspheres with mean curvature H which is asymptotic to a cone in R^{n+k+1} is reduced to solve

$$(PAS) \qquad \det D^2 u = \frac{[x \cdot \nabla u(x) - u(x)]^{-k}}{[Hu(x)]^{n+k+2}} \text{ in } \Omega \subset R^n, \qquad u = 0 \text{ on } \partial\Omega$$

where Ω is a bounded convex domain containing the origin, $H < 0$ and $k \geq 0$ are constants.

- Haodi Chen and Genggeng Huang in [JDE: 267 (2019)] proved that (PAS) admits a unique convex solution $u \in C(\bar{\Omega}) \bigcap C^{\infty}(\Omega)$.

Application to proper affine hyperspheres-2

Since Ω contains the origin, u is convex and $u = 0$ on $\partial\Omega$, then

$$x \cdot \nabla u(x) - u(x) \geq -u(0) > 0.$$

Therefore $f(x,u) := \frac{[x\cdot\nabla u(x) - u(x)]^{-k}}{[Hu(x)]^{n+k+2}}$ satisfies

$$0 < f(x,u) \leq \frac{(-u(0))^{-k}}{(-H)^{n+k+2}} |u(x)|^{-n-k-2}.$$

By Theorem 4, we have

Theorem 5 *Supposed that* Ω *is (a,* η*) type domain in* R^n *with* $a \in [2, +\infty)$. *Let* $\gamma_3 = \frac{a+n-1}{a(n+1+k/2)}$. *Then the convex solution to* (AFS) *satisfies* $u \in C^{\gamma_3}(\overline{\Omega})$.

Our method can be applied to Chaplygin gas and minimal graph

$$(CM) \quad \begin{aligned} \Delta u - \frac{u_i u_j}{1+|\nabla u|^2} u_{ij} &= -\frac{n}{u} \text{ in } \Omega \\ u &= 0 \text{ on} \partial\Omega \\ u &> 0 \text{ in } \Omega, \end{aligned}$$

Where Ω is a bounded domain in R^n.

- $n = 2$: two dimensional Riemann problem with four-shock data (the vortex and the saddle) for Chaplygin gas
—[D. Serre: Arch Rational Mech Anal, 191(2009)]

- $n \geq 2$: the graph of u defines a minimal graph in hyperbolic space
—[Anderson, Invent Math 1982; Hardt and Lin, Invent Math 1987; Lin, Invent Math 1989]

The Existence and Uniqueness

• There is a unique solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ if $\Omega \in C^2$ and the mean curvature $H|_{\partial\Omega} \geq 0$.
—[Lin, Invent Math 1989]

• There is a unique solution $u \in C^\infty(\Omega) \cap C^0(\bar{\Omega})$ if $n = 2$ and Ω is piecewise C^2-convex domain and the curvature $K|_{\partial\Omega} > 0$.
—[D. Serre: Arch Rational Mech Anal, 191(2009)]

Questions

• Q 3: Is the solution u $Hölder$ continuous up to the boundary?

• Answered by [Lin: Invent Math 1989] if $\Omega \in C^2$ and curvature $H|_{\partial\Omega} > 0$;

• Answered by [Han-Shen and Yue Wang: Car Var PDE, 2016] if Ω is piecewise C^2 and curvature $H|_{\partial\Omega} > 0$;

Concave Solutions

Assume that Ω is a bounded convex domain. Then the problem admits a unique solution $u \in C(\overline{\Omega}) \bigcap C^{\infty}(\Omega)$, and u is concave. Moreover,

$$u \in C^{\frac{1}{n+1}}(\overline{\Omega})$$

.

This result was proved by Qing Han, Weiming Shen and Yue Wang in Car Var PDE, 2016

The regularity depends on the convexity

Applying our method and constructing the super-solution in the form

$$W(x_1, ..., x_n) = ((\frac{x_n}{\varepsilon})^{\frac{2}{a}} - x_1^2 - ... - x_{n-1}^2)^{\frac{1}{b}}$$

where $b \geq 2$ and $\varepsilon > 0$ are to be determined, we obtain

Theorem 6 [J-You Li: Preprint, 2018)] *Let* Ω *be* (a, η) *type domain with* $a \in [2, +\infty)$. *Then*

$$u \in C^{\frac{1}{\bar{a}}}(\overline{\Omega})$$

where $\bar{a} = max\{\frac{1}{a}, \frac{1}{n+1}\}$.

Thank You!