Global Existence of Weak Solutions to the Barotropic Compressible Navier-Stokes Flows with Degenerate Viscosities

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This talk is based on mainly the following paper.

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Abstract

This paper concerns the existence of global weak solutions to the barotropic compressible Navier-Stokes equations with degenerate viscosity coefficients. We construct suitable approximate system which has smooth solutions satisfying the energy inequality, the BD entropy one, and the Mellet-Vasseur type estimate. Then, after adapting the compactness results due to Mellet-Vasseur [Comm. Partial Differential Equations 32 (2007)], we obtain the global existence of weak solutions to the barotropic compressible Navier-Stokes equations with degenerate viscosity coefficients in two or three dimensional periodic domains or whole space for large initial data. This, in particular, solved an open problem in [P. L. Lions. Mathematical topics in fluid mechanics. Vol. 2. Compressible models. Oxford University Press, 1998].

Keywords. compressible Navier-Stokes equations; degenerate viscosities; global weak solutions; large initial data; vacuum.

AMS subject classifications. 35Q35, 35B65, 76N10

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Contents

Introduction

The barotropic compressible Navier-Stokes equations read:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}\mathbb{S} + \nabla P(\rho) = 0, \end{cases} \tag{1.1}$$

where $x \in \Omega \subset \mathbb{R}^N (N = 2, 3), t > 0,$

ρ : density,

$u = (u_1, \cdots, u_N)$: velocity,

$P(\rho) = \rho^\gamma (\gamma > 1)$: pressure,

Introduction

$\mathbb{S}$: viscous stress tensor with either

$$\mathbb{S} \equiv \mathbb{S}_1 \triangleq h\nabla u + g\text{div}u\mathbb{I}, \tag{1.2}$$

or

$$\mathbb{S} \equiv \mathbb{S}_2 \triangleq h\mathcal{D}u + g\text{div}u\mathbb{I}, \tag{1.3}$$

where $\mathcal{D}u = \frac{1}{2}(\nabla u + (\nabla u)^{\text{tr}})$: deformation tensor,

$\mathbb{I}$: the identical matrix,

h, g satisfy the physical restrictions

$$h > 0, \quad h + Ng \geq 0. \tag{1.4}$$

- **Global existence of classical solutions away from vacuum (h and g are both constants):**
  - **1D case:**

    Kanel (1968): isentropic case, large initial data;

    Kazhikhov & Shelukhin (1977): full NS case, large initial data.
  - **Multi-D case:**
    - Matsumura& Nishida (1980): initial data close to a non-vacuum equilibrium

Introduction

- **Global existence of weak solutions containing vacuum states (3D, h and g are both constants):**
  - Lions (1993,1998): large initial data, when $\gamma \geq 9/5$;
    Feireisl(2001): large initial data, when $\gamma > 3/2$;
    Jiang-Zhang(2001): $\gamma > 1$, for spherically symmetric solutions.

Theorem (Lions-Feireisl (1993,1998,2001))

If $\gamma > 3/2$ *and the initial energy*

$$C_0 \triangleq \frac{1}{2}\int \rho_0|u_0|^2 dx + \frac{1}{\gamma - 1}\int P(\rho_0)dx < \infty. \tag{1.5}$$

THEN $\exists$ *a global weak solution* (ρ, u).

Remark

In particular, for the whole space case, the density vanishes (in some weak sense).

Introduction

Variable viscosities degenerate at vacuum

- $h = h(\rho), g = g(\rho)$
  - Liu-Xin-Yang (1998)

    derived the compressible Navier-Stokes equations from the Boltzmann equation by the Chapman-Enskog expansions.
  - Gerbeau-Perthame (2001), Marche (2007), Bresch-Noble (2007) (2011)

    a friction shallow-water system used in Oceanography can be written in a two-dimensional space domain Ω with $h(\rho) = g(\rho) = \rho$.
  - Geophysical flow models etc.

Introduction

Open Problem (Lions (1998))

For $N = 2, 3$, the Cauchy problem is completely open with $\mathbb{S} = \rho \nabla u$ and $\gamma = 2$.

Known results

- 1D with $h = g = \rho^\alpha (\alpha > 0)$ (free boundaries) :
  Jiang, Kanel, Makino, Okada, Qin, Xin, Yang, Yao, Zhang, Zhu, et al.
- Weigant & Kazhikhov model$(h \equiv const., g = g(\rho))$:
  Weigant & Kazhikhov (1995), Huang-Li (2012),
  Jiu-Wang-Xin (2012): 2D, large initial data.

Introduction

Multi-dimensional case:

- BD entropy

  Bresch-Desjardins (2003) obtained a new a priori estimate on the spatial derivatives of the density (BD entropy) under the condition that

$$g(\rho) = h'(\rho)\rho - h(\rho)$$

  for the periodic boundary conditions and the Cauchy problem. They used the BD entropy to obtain the the global existence of weak solutions to (1.1) (1.2) and (1.1) (1.3) with some additional drag terms.

Introduction

- Log-type energy estimate

  Mellet-Vasseur (2007) obtained a new a priori Log-type energy estimate and study the stability of (1.1) (1.2) and (1.1) (1.3) without any additional drag term under the assumption of existence of smooth approximate approximation solutions satisfying energy estimate and BD entropy.

  .

Introduction

Remark

For the cases that h, g are both constants (non-degenerate viscosity), the construction of the approximation solutions can be achieved by using standard Galerkin methods (Lions (1998), Feireisl et al(2001)). However, the constructions of the smooth approximation solutions remain to be challenge, which does not seem standard in the case of appearance of vacuum due to the degeneracy of viscosities.

Introduction

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Introduction

- Part results for special cases are available :
  - 1D: Li-Li-Xin (2008)
  - Multi-D with the spherical initial data: Guo-Jiu-Xin (2008), Guo-Li-Xin (2012)
- The truly multidimensional case seems much more complicated. It has been an open problem mentioned by many researchers in the area.

Introduction

Open Problem

For general initial data, the problem proposed by Lions is essentially reduced to the constructions of the smooth approximation solutions satisfying the BD entropy inequality since the a priori energy estimate and Log-type one are relatively easy to be verified.

For the sake of simplicity, it is assumed that for $\alpha > 0$,

$$h(\rho) = \rho^{\alpha}, \quad g(\rho) = (\alpha - 1)\rho^{\alpha}. \tag{1.21}$$

For $\Omega = \mathbb{R}^N (N = 2, 3)$ or $\Omega = \mathbb{T}^N (N = 2, 3)$.

The initial conditions are imposed as

$$\rho(x, t = 0) = \rho_0, \quad \rho u(x, t = 0) = m_0. \tag{1.22}$$

Introduction

Definition (weak solutions)

(ρ, u) *is said to be a weak solution to* (1.1) (1.3) (1.21) (1.22) *if*

$$\begin{cases} 0 \le \rho \in L^\infty(0,T; L^1(\Omega) \cap L^\gamma(\Omega)), \\ \nabla \rho^{(\gamma+\alpha-1)/2} \in L^2(0,T; (L^2(\Omega))^N), \\ \nabla \rho^{\alpha-1/2}, \ \sqrt{\rho} u \in L^\infty(0,T; (L^2(\Omega))^N), \\ h(\rho)\nabla u, \ h(\rho)(\nabla u)^{\mathrm{tr}} \in L^2(0,T; (W^{-1,1}_{\mathrm{loc}}(\Omega))^{N\times N}), \\ g(\rho)\mathrm{div} u \in L^2(0,T; W^{-1,1}_{\mathrm{loc}}(\Omega)), \end{cases}$$

Introduction

Definition (weak solutions (Continued))

with $(\rho, \sqrt{\rho}u)$ *satisfying*

$$\begin{cases} \rho_t + \mathrm{div}(\sqrt{\rho}\sqrt{\rho}u) = 0, \\ \rho(x, t=0) = \rho_0(x), \end{cases} \quad \textit{in } \mathcal{D}',$$

and if the following equality holds for all smooth test function $\phi(x,t)$ *with compact support such that* $\phi(x,T) = 0$:

$$\int_\Omega m_0 \cdot \phi(x,0)dx + \int_0^\infty \left(\sqrt{\rho}(\sqrt{\rho}u)\phi_t + \sqrt{\rho}u \otimes \sqrt{\rho}u : \nabla\phi\right) dxdt$$
$$+ \int_0^\infty \rho^\gamma \mathrm{div}\phi dxdt - \frac{1}{2}\langle h(\rho)\nabla u, \nabla\phi\rangle - \frac{1}{2}\langle h(\rho)(\nabla u)^{\mathrm{tr}}, \nabla\phi\rangle$$
$$- \langle g(\rho)\mathrm{div}u, \mathrm{div}\phi\rangle = 0,$$

Introduction

Definition (weak solutions (Continued))

where

$$\langle h(\rho)\nabla u, \nabla\phi\rangle = -\int_0^\infty \rho^{\alpha-1/2}\sqrt{\rho}u\cdot\Delta\phi dxdt - \frac{2\alpha}{2\alpha-1}\int_0^\infty \sqrt{\rho}u_j\partial_i\rho^{\alpha-1/2}\partial_i\phi_j dxdt,$$

$$\langle h(\rho)(\nabla u)^{\mathrm{tr}}, \nabla\phi\rangle = -\int_0^\infty \rho^{\alpha-1/2}\sqrt{\rho}u\cdot\nabla\mathrm{div}\phi dxdt - \frac{2\alpha}{2\alpha-1}\int_0^\infty \sqrt{\rho}u_i\partial_j\rho^{\alpha-1/2}\partial_i\phi_j dxdt,$$

$$\langle g(\rho)\mathrm{div}u, \mathrm{div}\phi\rangle = -(\alpha-1)\int_0^\infty \rho^{\alpha-1/2}\sqrt{\rho}u\cdot\nabla\mathrm{div}\phi dxdt - \frac{2\alpha(\alpha-1)}{2\alpha-1}\int_0^\infty \sqrt{\rho}u\cdot\nabla\rho^{\alpha-1/2}\mathrm{div}\phi dxdt.$$

Condition (Conditions on the initial data)

For some $\eta_0 > 0$,

$$\begin{cases} 0 \le \rho_0 \in L^1(\Omega) \cap L^\gamma(\Omega),\ \nabla \rho_0^{\alpha - 1/2} \in L^2(\Omega), \\ m_0 \in L^{2\gamma/(\gamma+1)}(\Omega),\ \rho_0^{-1-\eta_0} |m_0|^{2+\eta_0} \in L^1(\Omega). \end{cases} \tag{2.1}$$

Main Theorems

Theorem 1

Let $\Omega = \mathbb{R}^2$ *or* $\mathbb{T}^2$. *Suppose that* α *and* γ *satisfy*

$$\alpha > 1/2, \quad \gamma > 1, \quad \gamma \geq 2\alpha - 1. \tag{2.1}$$

Moreover, assume that the initial data (ρ_0, m_0) *satisfies* (1.23). *Then there exists a global weak solution* (ρ, u) *to the problem* (1.1) (1.3) (1.21) (1.22).

Remark

Similar result holds for the problem (1.1) (1.2) (1.21) (1.22).

Main Theorems

As for the three-dimensional case, it holds that

Theorem 2

Let $\Omega = \mathbb{R}^3$ *or* $\mathbb{T}^3$*. Suppose that* $\alpha \in [3/4, 2)$ *and* $\gamma \in (1, 3)$ *satisfy*

$$\gamma \in \begin{cases} (1, 6\alpha - 3), & \text{for } \alpha \in [3/4, 1], \\ [2\alpha - 1, 3\alpha - 1], & \text{for } \alpha \in (1, 2). \end{cases} \tag{2.3}$$

Assume that the initial data (ρ_0, m_0) *satisfies* (2.1)*. Then there exists a global weak solution* (ρ, u) *to the problem* (1.1) (1.2) (1.6) (1.7)*.*

Remark

In particular, $\gamma \in (1, 3)$ *for* $\alpha = 1$*.*

Main Theorems

Theorem 3

Let $\Omega = \mathbb{R}^3$ *or* $\mathbb{T}^3$. *Suppose that* $\alpha = 1$ *and* $\gamma \in (1, 3)$. *Assume that the initial data* (ρ_0, m_0) *satisfies* (1.23). *Then there exists a global weak solution* (ρ, u) *to the problem* (1.1) (1.3) (1.21) (1.22).

Main Theorems

Remark

If $\alpha = 1$ *and* $\gamma = 2$*, our results give a positive answer to the open problem proposed by Lions (1998).*

Remark

Recently, for a particular case: $\alpha = 1$ *and* $\Omega = \mathbb{T}^3$*, Vasseur-Yu [Invent. math. (2016) 206:935–974] tried to give another proof of our result (Theorem 3), that is, they used the weak solutions of*

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma - \operatorname{div}(\rho \mathcal{D}(u)) \\ = -r_1 u - r_2 \rho |u|^2 u - \kappa \rho \nabla (\frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}), \end{cases} \tag{2.3}$$

whose existence is shown in Vasseur-Yu [SIAM J. MATH. ANAL. 48 (2016) 1489–1511].

Main Theorems

Remark (Cont.)

However, as indicated by [Lacroix-Violet & Vasseur JMPA (2018)], those a priori estimates are not sufficient to define ∇u *as a function. In fact, in the proof of the key Lemma 4.2 of [Vasseur-Yu: Invent. math. (2016) 206:935-974], it seems to us that their need the assumption that* ∇u *is a function essentially.*

Invent. math. (2016) 206:935–974
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Existence of global weak solutions for 3D degenerate compressible Navier–Stokes equations

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Hence, $\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2)$ converges to $\rho^\alpha g(|\mathbf{u}|^2)$ almost everywhere. Since g is bounded and (4.1), $\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2)$ is uniformly bounded in $L^r((0,T)\times\Omega)$ for some $r>1$. Hence,

$$\rho_\kappa^\alpha g(|\mathbf{v}_\kappa|^2) \to \rho^\alpha g(|\mathbf{u}|^2) \quad \text{in } L^1((0,T)\times\Omega).$$

By the uniqueness of the limit, the convergence holds for the whole sequence.

Applying this result with $\alpha = 1$ and $g(|\mathbf{v}_\kappa|^2) = \varphi_n(\mathbf{v}_\kappa)$, we deduce (4.2).

Since $\gamma > 1$ in 2D, we can take $\alpha = 2\gamma - 1 < 2\gamma$; and take $\gamma < 3$ in 3D, we have $2\gamma - 1 < \frac{5\gamma}{3}$. Thus we use the above result with $\alpha = 2\gamma - 1$ and $g(|\mathbf{v}_\kappa|^2) = 1 + \tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)$ to obtain (4.3). □

With the lemma in hand, we are ready to recover the limits in (3.2) as $\kappa \to 0$ and $K \to \infty$. We have the following lemma.

Lemma 4.2 *Let $K = \kappa^{-\frac{3}{4}}$, and $\kappa \to 0$, for any $\psi \geq 0$ and $\psi' \leq 0$, we have*

$$\begin{aligned} &-\int_0^T \int_\Omega \psi'(t)\rho\varphi_n(\mathbf{u})\,dx\,dt \\ &\quad \leq 8\|\psi\|_{L^\infty}\left(\int_\Omega \left(\rho_0|\mathbf{u}_0|^2 + \frac{\rho_0^\gamma}{\gamma-1} + |\nabla\sqrt{\rho_0}|^2 - r_0 \log_- \rho_0\right) dx + 2E_0\right) \\ &\qquad + C(\|\psi\|_{L^\infty}) \int_0^T \int_\Omega (1 + \tilde{\varphi}_n'(|\mathbf{u}|^2))\rho^{2\gamma-1}\,dx\,dt + \psi(0)\int_\Omega \rho_0\varphi_n(\mathbf{u}_0)\,dx \end{aligned} \tag{4.5}$$

For the term $\mathbb{S}_\kappa = \phi_K(\rho_\kappa)\rho_\kappa(\mathbb{D}\mathbf{u}_\kappa + \kappa\frac{\Delta\sqrt{\rho_\kappa}}{\sqrt{\rho_\kappa}}\mathbb{I}) = \mathbb{S}_1 + \mathbb{S}_2$, we calculate as follows

$$\int_0^T \int_\Omega \psi(t)\mathbb{S}_1 : \nabla(\varphi_n'(\mathbf{v}_\kappa))\, dx\, dt$$
$$= \int_0^T \int_\Omega \psi(t)\phi_K(\rho_\kappa)\rho_\kappa\mathbb{D}\mathbf{u}_\kappa : \nabla(\varphi_n'(\mathbf{v}_\kappa))\, dx\, dt$$
$$= \int_0^T \int_\Omega \psi(t)[\nabla\mathbf{u}_\kappa\varphi_n''(\mathbf{v}_\kappa)\rho_\kappa] : \mathbb{D}\mathbf{u}_\kappa(\phi_K(\rho_\kappa))^2\, dx\, dt$$

$$+ \int_0^T \int_\Omega \psi(t)\rho_\kappa\phi_K(\rho_\kappa)\left(\mathbf{u}_\kappa^T\varphi_n''(\mathbf{v}_\kappa)\right)\mathbb{D}\mathbf{u}_\kappa\nabla(\phi_K(\rho_\kappa))\, dx\, dt$$
$$= A_1 + A_2. \tag{4.21}$$

For A_1, by part a. of Lemma 2.2, we have

$$A_1 = \int_0^T \int_\Omega \psi(t)[\nabla\mathbf{u}_\kappa\varphi_n''(\mathbf{v}_\kappa)\rho_\kappa] : \mathbb{D}\mathbf{u}_\kappa(\phi_K(\rho_\kappa))^2\, dx\, dt$$
$$= 2\int_0^T \int_\Omega \psi(t)\tilde{\varphi}_n'(|\mathbf{v}_\kappa|^2)(\phi_K(\rho_\kappa))^2\rho_\kappa\mathbb{D}\mathbf{u}_\kappa : \nabla\mathbf{u}_\kappa\, dx\, dt$$
$$+ 4\int_0^T \int_\Omega \psi(t)\rho_\kappa(\phi_K(\rho_\kappa))^2\tilde{\varphi}_n''(|\mathbf{v}_\kappa|^2)(\nabla\mathbf{u}_\kappa\mathbf{v}_\kappa \otimes \mathbf{v}_\kappa) : \mathbb{D}\mathbf{u}_\kappa\, dx\, dt$$
$$= A_{11} + A_{12}. \tag{4.22}$$

Notice that

$$\mathbb{D}\mathbf{u}_\kappa : \nabla\mathbf{u}_\kappa = |\mathbb{D}\mathbf{u}_\kappa|^2,$$

GLOBAL WEAK SOLUTIONS TO THE COMPRESSIBLE QUANTUM NAVIER-STOKES EQUATION AND ITS SEMI-CLASSICAL LIMIT

INGRID LACROIX-VIOLET AND ALEXIS F. VASSEUR

Abstract. This paper is dedicated to the construction of global weak solutions to the quantum Navier-Stokes equation, for any initial value with bounded energy and entropy. The construction is uniform with respect to the Planck constant. This allows to perform the semi-classical limit to the associated compressible Navier-Stokes equation. One of the difficulty of the problem is to deal with the degenerate viscosity, together with the lack of integrability on the velocity. Our method is based on the construction of weak solutions that are renormalized in the velocity variable. The existence, and stability of these solutions do not need the Mellet-Vasseur inequality.

1. Introduction

Quantum models can be used to describe superfluids [12], quantum semiconductors [6], weakly interacting Bose gases [8] and quantum trajectories of Bohmian mechanics [16]. They have attracted considerable attention in the last decades due, for example, to the development of nanotechnology applications.

In this paper, we consider the barotropic compressible quantum Navier-Stokes equations, which has been derived in [5], under some assumptions, using a Chapman-Enskog expansion in Wigner equation. In particular, we are interested in the existence of global weak solutions together with the associated semi-classical limit. The quantum Navier-Stokes equation that we are considering read as:

$$\begin{aligned} &\rho_t + \operatorname{div}(\rho u) = 0, \\ &(\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla \rho^\gamma - 2\operatorname{div}(\sqrt{\nu\rho}\mathbb{S}_\nu + \sqrt{\kappa\rho}\mathbb{S}_\kappa) = \sqrt{\rho} f + \sqrt{\kappa}\operatorname{div}(\sqrt{\rho}\mathbb{M}), \end{aligned} \tag{1.1}$$

where

$$\sqrt{\nu\rho}\mathbb{S}_\nu = \rho \mathbb{D} u, \qquad \operatorname{div}(\sqrt{\kappa\rho}\mathbb{S}_\kappa) = \kappa\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right), \tag{1.2}$$

and with initial data

$$\rho(0,x) = \rho_0(x), \quad (\rho u)(0,x) = (\rho_0 u_0)(x) \quad \text{ in } \Omega, \tag{1.3}$$

2. Preliminary results and main ideas

We are first working on the System (1.9) with drag forces. The definitions will be valid for all the range of parameter, $r_0 \geq 0, r_1 \geq 0, \kappa \geq 0, \nu > 0$. The energy and the BD entropy on solutions to (1.9) provide controls on

$$\mathcal{E}_r(\sqrt{\rho}, \sqrt{\rho}u) = \int_\Omega \left(\rho\frac{|u|^2}{2} + (2\kappa + 4\nu^2)|\nabla\sqrt{\rho}|^2 + \rho^\gamma + r_0(\rho - \ln\rho)\right) dx,$$

$$\mathcal{D}_{\mathcal{E}}^r(\sqrt{\rho}, \sqrt{\rho}u) = \int_\Omega \left(\nu|\nabla\rho^{\gamma/2}|^2 + \nu\kappa\left(|\nabla\rho^{1/4}|^4 + |\nabla^2\sqrt{\rho}|^2\right) + |\mathbb{T}_\nu|^2 + r_0|u|^2 + r_1\rho|u|^4\right) dx.$$

From these quantities, we can obtain the following a priori estimates. For the sake of completeness we show how to obtain them in the appendix.

$$\begin{aligned}
&\sqrt{\rho} \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \qquad \nabla\sqrt{\rho} \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \qquad \nabla\rho^{\gamma/2} \in L^2(\mathbb{R}^+; L^2(\Omega))\\
&\sqrt{\rho}u \in L^\infty(\mathbb{R}^+; L^2(\Omega)), \qquad \mathbb{T}_\nu \in L^2(\mathbb{R}^+; L^2(\Omega)), \qquad \sqrt{\kappa}\nabla^2\sqrt{\rho} \in L^2(\mathbb{R}^+; L^2(\Omega)),\\
&\kappa^{1/4}\nabla\rho^{1/4} \in L^4(\mathbb{R}^+; L^4(\Omega)), \qquad r_1^{1/4}\rho^{1/4}u \in L^4(\mathbb{R}^+; L^4(\Omega)),\\
&r_0^{1/2}u \in L^2(\mathbb{R}^+; L^2(\Omega)), \qquad r_0\ln\rho \in L^\infty(\mathbb{R}^+; L^1(\Omega)).
\end{aligned} \tag{2.1}$$

Note that those a priori estimates are not sufficient to define ∇u as a function. The statement that $\sqrt{\rho}\nabla u$ is bounded in L^2 means that there exists a function $\mathbb{T}_\nu \in L^2(\mathbb{R}^+; L^2(\Omega))$ such that:

$$\sqrt{\nu}\sqrt{\rho}\mathbb{T}_\nu = \operatorname{div}(\rho u) - \sqrt{\rho}u \cdot \nabla\sqrt{\rho},$$

Global existence of weak solutions to the compressible quantum Navier-Stokes equations with degenerate viscosity

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Remark 1.1 It should be noted that the arguments in the work of Vasseur-Yu[34,35] rely crucially on the assumption that the gradient of velocity field ∇u is a well-defined function, which indeed does not make sense in the presence of vacuum. In particular, in the proof of Ref. 35, Lemma 4.2, which is crucial to deduce the key Mellet-Vasseur type estimate in Ref. 35, it requires essentially that ∇u is a well-defined function.

Very recently, Lacroix-Violet and Vasseur[25] also studied the QNS equations and considered a new function $\mathbb{T}_\nu \in L^2(\Omega \times (0,T))$ satisfying

$$\sqrt{\nu\rho}\,\mathbb{T}_\nu = \nu\nabla(\rho u) - 2\nu\sqrt{\rho}u \otimes \nabla\sqrt{\rho}. \tag{1.16}$$

More precisely, they[25] used the function $\mathbb{T}_\nu$ to give a new understanding of $\sqrt{\rho}\nabla u$. However, as mentioned in Ref. 25, it still does not allow one to define the gradient of velocity ∇u as a function.

[25] I. Lacroix-Violet and A. Vasseur, "Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit," J. Math. Pures Appl. **114**, 191–210 (2018).

[34] A. Vasseur and C. Yu, "Global weak solutions to the compressible quantum Navier-Stokes equations with damping," SIAM J. Math. Anal. **48**, 1489–1511 (2016).
[35] A. Vasseur and C. Yu, "Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations," Invent. Math. **206**, 935–974 (2016).

Main Theorems

Remark

For 3D case, our Theorems 2 & 3 are valid for $\alpha = 1$ *and all* $\gamma \in (1, 3)$*, which are in sharp contrast to the case that* h *and* g *are both constants, where the condition* $\gamma > 3/2$ *is essential in the analysis of Lions (1998) and Feireisl et al (2001). In fact, for* h *and* g *being both constants and* $\gamma \in (1, 3/2]$*, it remains completely open to obtain the global existence of weak solutions to* (1.1) (1.2) *for general initial data except for the spherically symmetric case (Jiang-Zhang (2001)).*

Sketch of Proof

GOAL: To construct an approximate system which has smooth solutions satisfying

1) Standard and Log-type energy estimates

2) BD entropy

- Observation: Standard and Log-type energy estimates hold for parabolized system.
- Key issue is to derive BD entropy independent of perturbation parameters
- Main technical difficulty is to obtain the positive lower bound and upper one for the density (which may depend on the parameters)

Sketch of Proof

Our constructions: Consider the following approximate system

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = G, \\ \rho u_t + \rho u \cdot \nabla u - \mathrm{div}(h_\varepsilon(\rho)\mathcal{D}u) - \nabla(g_\varepsilon(\rho)\mathrm{div} u) + \nabla P \\ \qquad = H, \end{cases} \tag{3.1}$$

where

$$\begin{cases} h_\varepsilon(\rho) = \rho^\alpha + \varepsilon^{1/3}(\rho^{7/8} + \rho^{\tilde{\gamma}}), \quad g_\varepsilon(\rho) = \rho h'_\varepsilon(\rho) - h_\varepsilon(\rho), \\ 0 < \varepsilon \le \varepsilon_0 \triangleq (2\alpha - 1)(16(\alpha + \gamma))^{-10}, \quad \tilde{\gamma} \triangleq \gamma + 1/6. \end{cases}$$

Sketch of Proof

- $G \triangleq \varepsilon\rho^{1/2}\mathrm{div}(\rho^{-1/2}h'_\varepsilon(\rho)\nabla\rho)$ parabolization

  KEY for BD entropy and lower and upper bounds of the density

- $H \triangleq H_1 + H_2$

  $H_1 \triangleq \sqrt{\varepsilon}[\mathrm{div}(h_\varepsilon(\rho)\nabla u) + \nabla(g_\varepsilon(\rho)\mathrm{div}u)]$

  $\Rightarrow \int \rho|u|^{2+\varepsilon}dx$

  $H_2 \triangleq -e^{-\varepsilon^{-3}}(\rho^{\varepsilon^{-2}} + \rho^{-\varepsilon^{-2}})u$

  $\Rightarrow \int (\rho^p + \rho^{-p})dx, \ \forall p > 1.$

- $\int \rho|u|^{2+\varepsilon}dx + \int (\rho^p + \rho^{-p})dx$ +De Giorgi type method

  $\Rightarrow C^{-1}(\varepsilon) \leq \rho \leq C(\varepsilon).$

Sketch of Proof (2D Case)

Step 2: BD entropy inequality

Lemma

$\exists C$ independent of ε and T s.t.

$$\sup_{0\le t\le T}\int\left(\rho^{-1}(h_\varepsilon'(\rho))^2|\nabla\rho|^2+\varepsilon^{13/3}e^{-\varepsilon^{-3}}(\rho^{\varepsilon^{-2}+\tilde{\gamma}-1}+\rho^{-\varepsilon^{-2}-1/8})\right)dx$$
$$+\int_0^T\int h_\varepsilon(\rho)|\nabla u|^2dxdt+\int_0^T\int\rho^{\gamma-3}h_\varepsilon(\rho)|\nabla\rho|^2dxdt\le C. \tag{4.2}$$

Sketch of Proof (2D Case)

Proof. Set

$$G \triangleq \varepsilon\rho^{1/2}\mathrm{div}(\rho^{-1/2}h_\varepsilon'(\rho)\nabla\rho), \quad \varphi_\varepsilon'(\rho) \triangleq \rho^{-1}h_\varepsilon'(\rho) \geq 0.$$

Direct computations give

$$\begin{aligned}
&\frac{1}{2}\Big(\int \rho|\nabla\varphi_\varepsilon(\rho)|^2dx\Big)_t + \int \nabla h_\varepsilon(\rho)\cdot\nabla u\cdot\nabla\varphi_\varepsilon(\rho)dx \\
&+ \int \nabla h_\varepsilon(\rho)\cdot\nabla(\rho\varphi_\varepsilon'(\rho)\mathrm{div}u)dx \\
&+ \int \varphi_\varepsilon'(\rho)G\left(\Delta h_\varepsilon(\rho) - \frac{1}{2}\varphi_\varepsilon'(\rho)|\nabla\rho|^2\right)dx = 0.
\end{aligned} \tag{4.3}$$

Sketch of Proof (2D Case)

Multiplying $(3.1)_2$ by $\nabla\varphi_\varepsilon(\rho)$ leads to

$$\begin{aligned}
&\frac{1}{1+\sqrt{\varepsilon}}\Big(\int u\cdot\nabla h_\varepsilon(\rho)dx\Big)_t - \int h_\varepsilon(\rho)\nabla \mathrm{div} u\cdot\nabla\varphi_\varepsilon(\rho)dx\\
&-\int \nabla h_\varepsilon(\rho)\cdot\nabla u\cdot\nabla\varphi_\varepsilon(\rho)dx + \int g_\varepsilon(\rho)\mathrm{div} u\Delta\varphi_\varepsilon(\rho)dx\\
&+\frac{1}{2(1+\sqrt{\varepsilon})}\int h_\varepsilon(\rho)|\nabla u|^2dx + \frac{1}{1+\sqrt{\varepsilon}}\int P'(\rho)\varphi_\varepsilon'(\rho)|\nabla\rho|^2dx\\
&+\frac{e^{-\varepsilon^{-3}}}{1+\sqrt{\varepsilon}}\int(\rho^{\varepsilon^{-2}}+\rho^{-\varepsilon^{-2}})u\cdot\nabla\varphi_\varepsilon(\rho)dx\\
&+\frac{1}{1+\sqrt{\varepsilon}}\int \mathrm{div} u h_\varepsilon'(\rho)Gdx\\
&\le C\int h_\varepsilon(\rho)|\mathcal{D}u|^2dx,
\end{aligned} \tag{4.4}$$

Sketch of Proof (2D Case)

(4.3)+(4.4)

$\varphi_\varepsilon'(\rho) = \rho^{-1} h_\varepsilon'(\rho), \quad g_\varepsilon(\rho) = \rho h_\varepsilon'(\rho) - h_\varepsilon(\rho) \Rightarrow$

$$\int \nabla h_\varepsilon(\rho) \cdot \nabla(\rho \varphi_\varepsilon'(\rho) \mathrm{div} u) dx - \int h_\varepsilon(\rho) \nabla \mathrm{div} u \cdot \nabla \varphi_\varepsilon(\rho) dx$$

$$+ \int g_\varepsilon(\rho) \mathrm{div} u \Delta \varphi_\varepsilon(\rho) dx$$

$$= \int \left(\nabla h_\varepsilon(\rho) \cdot \nabla(\rho \varphi_\varepsilon'(\rho)) - \nabla g_\varepsilon(\rho) \cdot \nabla \varphi_\varepsilon(\rho) \right) \mathrm{div} u dx$$

$$+ \int \left(\rho \varphi_\varepsilon'(\rho) \nabla h_\varepsilon(\rho) - h_\varepsilon(\rho) \nabla \varphi_\varepsilon(\rho) - g_\varepsilon(\rho) \nabla \varphi_\varepsilon(\rho) \right) \cdot \nabla \mathrm{div} u dx = 0$$

THEN (4.3)+(4.4)$\Rightarrow$

$$
\begin{aligned}
&\frac{1}{2}\Big(\int \rho|\nabla\varphi_\varepsilon(\rho)|^2dx\Big)_t + \frac{1}{1+\sqrt{\varepsilon}}\Big(\int \rho u\cdot\nabla\varphi_\varepsilon(\rho)dx\Big)_t \\
&\quad + \frac{1}{2(1+\sqrt{\varepsilon})}\int h_\varepsilon(\rho)|\nabla u|^2dx + \frac{1}{1+\sqrt{\varepsilon}}\int P'(\rho)\varphi_\varepsilon'(\rho)|\nabla\rho|^2dx \\
&\quad + \frac{e^{-\varepsilon^{-3}}}{1+\sqrt{\varepsilon}}\int (\rho^{\varepsilon^{-2}}+\rho^{-\varepsilon^{-2}})u\cdot\nabla\varphi_\varepsilon(\rho)dx \\
&\quad + \int \varphi_\varepsilon'(\rho)G\left(\Delta h_\varepsilon(\rho) - \frac{1}{2}\varphi_\varepsilon'(\rho)|\nabla\rho|^2 + \frac{1}{1+\sqrt{\varepsilon}}\rho\,\mathrm{div}u\right)dx \\
&\le C\int h_\varepsilon(\rho)|\mathcal{D}u|^2dx,
\end{aligned}
\tag{4.5}
$$

Sketch of Proof (2D Case)

By the definition of G

$$G \triangleq \varepsilon\rho^{1/2}\mathrm{div}(\rho^{-1/2}h_\varepsilon'(\rho)\nabla\rho) = \varepsilon(\Delta h_\varepsilon(\rho) - \frac{1}{2}\varphi_\varepsilon'(\rho)|\nabla\rho|^2).$$

We have the following KEY observation (Since $\varphi_\varepsilon'(\rho)\geq 0$):

$$\begin{aligned}
&\Rightarrow \int \varphi_\varepsilon'(\rho)G\left(\Delta h_\varepsilon(\rho) - \frac{1}{2}\varphi_\varepsilon'(\rho)|\nabla\rho|^2 + \frac{1}{1+\sqrt{\varepsilon}}\rho\mathrm{div}u\right)dx \\
&\geq \frac{1}{2\varepsilon}\int \varphi_\varepsilon'(\rho)G^2dx - \frac{\varepsilon}{2}\int \rho^2\varphi_\varepsilon'(\rho)\left(\mathrm{div}u\right)^2 dx \\
&\geq \frac{1}{2\varepsilon}\int \varphi_\varepsilon'(\rho)G^2dx - C\varepsilon\int h_\varepsilon(\rho)|\mathcal{D}u|^2dx.
\end{aligned} \tag{4.6}$$

Sketch of Proof (2D Case)

Step 3: Log-type energy estimate

Lemma

Assume that $\gamma > 1$ *satisfies* $\gamma \geq (1+\alpha)/2$ *in addition. Then there exists some generic constant* C *but independent of* ε *such that*

$$\sup_{0 \leq t \leq T} \int \rho(e+|u|^2)\ln(e+|u|^2)dx \leq C. \tag{4.7}$$

Sketch of Proof (2D Case)

Step 4: Lower and upper bounds of the density

Lemma

There exists some positive constant C depending on ε and T such that for all $(x,t) \in \Omega \times (0,T)$

$$C^{-1} \leq \rho(x,t) \leq C. \tag{4.8}$$

Sketch of Proof (2D Case)

Proof. 1) It is easy to show that

$$\sup_{0\le t\le T}\int \rho|u|^{2+\varepsilon}dx+\sqrt{\varepsilon}\int_0^T\int h_\varepsilon(\rho)|u|^\varepsilon|\nabla u|^2dxdt\le C. \quad (4.9)$$

Sketch of Proof (2D Case)

2) $v \triangleq \sqrt{\rho}$ satisfies

$$2v_t - 2\varepsilon \mathrm{div}(h'_\varepsilon(v^2)\nabla v) + \mathrm{div}(uv) + u \cdot \nabla v = 0. \qquad (4.10)$$

For $k \geq \|v(\cdot, 0)\|_{L^\infty(\Omega)} = \|\rho_0\|_{L^\infty(\Omega)}^{1/2}$, (4.10) $\times (v-k)_+ \Rightarrow$

$$\begin{aligned} &\frac{d}{dt}\int (v-k)_+^2 dx + 2\alpha\varepsilon \int v^{2\alpha-2}|\nabla(v-k)_+|^2 dx \\ &\leq C\int_{A_k(t)} v^{4-2\alpha}|u|^2 dx + \alpha\varepsilon \int v^{2\alpha-2}|\nabla(v-k)_+|^2 dx, \end{aligned} \qquad (4.11)$$

where $A_k(t) \triangleq \{x \in \Omega | v(x,t) > k\}$.

Sketch of Proof (2D Case)

$$
\begin{aligned}
&\int_{A_k(t)} v^{4-2\alpha}|u|^2 dx \\
&\le C\left(\int_{A_k(t)} v^2|u|^{2+\varepsilon}dx\right)^{2/(2+\varepsilon)}\left(\int_{A_k(t)} v^{(4+4\varepsilon-2(2+\varepsilon)\alpha)/\varepsilon}dx\right)^{\varepsilon/(2+\varepsilon)} \\
&\le C\left(\int_{A_k(t)} (\rho^{4(\alpha+1)\varepsilon^{-1}} + \rho^{-4(\alpha+1)\varepsilon^{-1}})dx\right)^{\varepsilon/(2+\varepsilon)} \\
&\le C\left(\int_{A_k(t)} (\rho^{\varepsilon^{-2}} + \rho^{-\varepsilon^{-2}})dx\right)^{\varepsilon(4-\varepsilon)/(6(2+\varepsilon))} |A_k(t)|^{\varepsilon/6} \\
&\le C|A_k(t)|^{\varepsilon/6},
\end{aligned} \tag{4.12}
$$

Sketch of Proof (2D Case)

(4.12)+(4.11) $\Rightarrow$

$$I_k'(t) + \alpha\varepsilon \int \rho^{\alpha-1} |\nabla (v-k)_+|^2 dx \le C \nu_k^{\varepsilon/6}, \tag{4.13}$$

where

$$I_k(t) \triangleq \int (v-k)_+^2 (x,t) dx, \quad \nu_k \triangleq \sup_{0\le t\le T} |A_k(t)|.$$

Sketch of Proof (2D Case)

Let

$$I_k(\sigma) = \sup_{0 \le t \le T} I_k(t).$$

$$\text{(4.13)} \Rightarrow I_k(\sigma) + \int \rho^{\alpha-1} |\nabla (v-k)_+|^2 (x,\sigma) dx \le C \nu_k^{\varepsilon/6},$$

$$\Rightarrow I_k(\sigma) + \|\nabla (v-k)_+(\cdot,\sigma)\|^2_{L^{24/(12+\varepsilon)}(\Omega)}$$
$$\le I_k(\sigma) + \int \rho^{\alpha-1} |\nabla (v-k)_+|^2 (x,\sigma) dx \left(\int \rho^{12(1-\alpha)/\varepsilon} (x,\sigma) dx \right)^{\varepsilon/12}$$
$$\le C \nu_k^{\varepsilon/6}.$$

Sketch of Proof (2D Case)

Then, for any $h > k \geq \|v(\cdot, 0)\|_{L^\infty(\Omega)}$,

$$
\begin{aligned}
&|A_h(t)|(h-k)^2 \\
&\leq \|(v-k)_+(\cdot, t)\|^2_{L^2(\Omega)} \\
&\leq \|(v-k)_+(\cdot, \sigma)\|^2_{L^2(\Omega)} \\
&\leq C\|(v-k)_+(\cdot, \sigma)\|^2_{L^{24/\varepsilon}(\Omega)} |A_k(\sigma)|^{1-\varepsilon/12} \\
&\leq C \left(\|(v-k)_+(\cdot, \sigma)\|^2_{L^2(\Omega)} + \|\nabla(v-k)_+(\cdot, \sigma)\|^2_{L^{24/(12+\varepsilon)}(\Omega)} \right) \nu_k^{1-\varepsilon/12} \\
&\leq C\nu_k^{1+\varepsilon/12},
\end{aligned}
$$

Sketch of Proof (2D Case)

$$\Rightarrow \nu_h \leq C(h-k)^{-2}\nu_k^{1+\varepsilon/12},$$

De Giorgi-type lemma $\Rightarrow \sup_{0\leq t\leq T} \|\rho\|_{L^\infty(\Omega)} \leq \tilde{C}.$ (4.14)

3) Applying similar arguments to the equation of ρ^{-1} shows

$$\sup_{(x,t)\in\Omega\times(0,T)} \rho^{-1}(x,t) \leq C,$$

for some positive constant $C \geq \tilde{C}$.

Sketch of Proof (2D Case)

Step 5: Higher order estimates

Lemma

For any $p > 2$, there exists some constant C depending on ε, p, and T such that

$$\int_0^T \left(\|(\rho, u)_t\|_{L^p(\Omega)}^p + \|(\rho, u)\|_{W^{2,p}(\Omega)}^p \right) dt \le C. \tag{4.15}$$

Sketch of Proof (3D Case)

For constants p_0 and ε satisfying

$$p_0 = 50, \quad 0 < \varepsilon \le \varepsilon_1 \triangleq \min\{10^{-10}, \eta_0\},$$

with η_0 as in (1.23), we consider the following approximate system

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = \varepsilon v \Delta v + \varepsilon v \operatorname{div}(|\nabla v|^2 \nabla v) + \varepsilon \rho^{-p_0}, \\ \rho u_t + \rho u \cdot \nabla u - \operatorname{div}(\rho \mathcal{D} u) + \nabla P \\ = \sqrt{\varepsilon} \operatorname{div}(\rho \nabla u) + \varepsilon v |\nabla v|^2 \nabla v \cdot \nabla u - \varepsilon \rho^{-p_0} u - \varepsilon \rho |u|^3 u, \end{cases} \tag{5.1}$$

where $v \triangleq \rho^{1/2}$.

Sketch of Proof (3D Case)

Lemma

There exists some generic constant C independent of ε such that

$$\sup_{0\le t\le T}\int(\rho|u|^2+\rho+\rho^\gamma+\varepsilon\rho^{-p_0})dx+\int_0^T\int\rho|\mathcal{D}u|^2dxdt$$
$$+\varepsilon\int_0^T\int\left(|\nabla v|^4+|\nabla v|^2|u|^2+|\nabla v|^4|u|^2+\rho^{-p_0}|u|^2+\rho|u|^5\right)dxdt$$
$$+\varepsilon^2\int_0^T\int\rho^{-2p_0-1}dxdt\le C. \tag{5.2}$$

Sketch of Proof (3D Case)

Lemma (BD entropy)

$\exists$ C independent of ε such that

$$\sup_{0\le t\le T}\int \left(|\nabla v|^2+\varepsilon|\nabla v|^4\right)dx+\int_0^T\int \left(\rho|\nabla u|^2+\rho^{\gamma-2}|\nabla\rho|^2\right)dxdt$$
$$+\,\varepsilon\int_0^T\int((\Delta v)^2+|\nabla v|^2|\nabla^2 v|^2)dxdt+\varepsilon^2\int_0^T\int|\nabla v|^4|\nabla^2 v|^2dxdt$$
$$\le C. \tag{5.3}$$

Sketch of Proof (3D Case)

Observation:

$$\begin{aligned}
&\int \operatorname{div}(|\nabla v|^2 \nabla v)\Delta v dx \\
&= -\int |\nabla v|^2 \nabla v \cdot \nabla \Delta v dx \\
&= \int |\nabla v|^2 |\nabla^2 v|^2 dx + \frac{1}{2}\int |\nabla |\nabla v|^2|^2 dx.
\end{aligned} \tag{5.4}$$

Open Problems:

- Shallow water models;
- Full compressible Navier-Stokes system with viscosity and heat conduction depending on temperature;
- $\cdots$

Thank You!