# Global small solutions for heat conductive compressible Navier-Stokes equations with vacuum

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# Outline

# Introduction

# Full compressible Navier-Stokes equations

Full compressible Navier-Stokes equations:

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho u) = 0, \\ \rho(\partial_t u + (u \cdot \nabla)u) - \mu\Delta u - (\mu + \lambda)\nabla \mathrm{div}\, u + \nabla p = 0, \\ c_v \rho(\partial_t \theta + u \cdot \nabla\theta) + p\mathrm{div}\, u - \kappa\Delta\theta = \mathcal{Q}(\nabla u), \end{cases}$$

where

$$\mathcal{Q}(\nabla u) = \frac{\mu}{2}|\nabla u + \nabla u^T|^2 + \lambda(\mathrm{div}\, u)^2.$$

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- Unknowns: $\rho$ density, $u$ velocity, $\theta$ temperature, $p$ pressure;
- Constants: $\mu, \lambda$ viscous coefficients, $\kappa$ heat conductive coefficient, satisfying

$$\mu > 0, \quad 2\mu + N\lambda \geq 0, \quad N \text{ space dimension}, \quad \kappa \geq 0.$$

# The entropy $s$

Constitutive equations:

$$p = R\rho\theta = Ae^{\frac{s}{c_v}}\rho^{\gamma}, \quad \gamma - 1 = \frac{R}{c_v}, R, c_v > 0.$$

$\Longrightarrow$

$$\boxed{s = c_v\left(\log\frac{R}{A} + \log\theta - (\gamma-1)\log\rho\right).}$$

Equation for $s$:

$$\rho(\partial_t s + u\cdot\nabla s) - \frac{\kappa}{c_v}\Delta s = \kappa(\gamma-1)\mathrm{div}\left(\frac{\nabla\rho}{\rho}\right) + \frac{\mathcal{Q}(\nabla u)}{\theta} + \kappa\left|\frac{\nabla\theta}{\theta}\right|^2,$$

where

$$\mathcal{Q}(\nabla u) := \frac{\mu}{2}|\nabla u + \nabla u^T|^2 + \lambda(\mathrm{div}\, u)^2.$$

# Non-vacuum: local and global

- **Local $\exists!$**: Serrin 1959, Nashi 1962, Itaya 1971, Vol'pert–Hudjaev 1972, Solonnikov 1976, Tani 1977, Valli 1982, $\cdots$
- **Global large in 1D**: Kazhikhov–Shelukhin 1977
- **Global small in 3D**: Matsumura–Nishida 1980s

$$\|(\rho_0 - 1, u_0, \theta_0 - 1)\|_{H^3} << 1 \qquad \text{(smallness condition)}$$

**Remark**

The solutions are obtained in Inhomogeneous spaces

$$\cdots\cdots \boxed{u \in L^\infty(0, T; L^2(\mathbb{R}^N))} \cdots\cdots$$

# With general vacuum (ignoring $s$): large solutions

- **Global weak solutions**:
  Lions 98, Feireisl–Novotn'y–Petzeltova 2001, Jiang–Zhang 2003, Feireisl 2004, Bresch–Jabin 2018, $\cdots\cdots$
- **Global strong solutions - 1D**:
  JL (SIMA 2019) — $\kappa > 0$; JL (arXiv:1908.00514) — $\kappa = 0$.
- **Local strong solutions - 3D**: Cho–Kim 2006

**Remark**

- The solutions are obtained in Homogeneous spaces

$$\cdots\cdots \boxed{\sqrt{\rho}u \in L^\infty(0, T; L^2(\mathbb{R}^N))} \cdots\cdots$$

- No information on $s$ can be provided in the vacuum region

# With general vacuum (ignoring $s$): small solutions

## Global small strong solutions

- **Isentropic:** Huang–Li–Xin 2012 3D, Li–Xin 2019 2D

$$\int \left(\frac{1}{2}\rho_0|u_0|^2 + \frac{P(\rho_0)}{\gamma - 1}\right) \le \varepsilon_0 << 1,$$

  where $\varepsilon_0$ depends on some higher order norms.

- **Full CNS:** Huang–Li 2018 $\rho_\infty > 0$, Wen–Zhu 2017 $\rho_\infty = 0$.

**Remarks**

- The solutions are obtained in Homogeneous spaces

$$\cdots\cdots \boxed{\sqrt{\rho}u \in L^\infty(0, T; L^2(\mathbb{R}^N))} \cdots\cdots$$

- **Entropy $s$ is unbounded** in Huang–Li 2018 and Wen–Zhu 2017

# With "strong" vacuum: $\nexists$ entropy-bounded solutions

$$\emptyset \neq supp\, \rho_0 \subset\subset \mathbb{R}^N$$

$\Longrightarrow$

- $\nexists$ **in inhomogeneous** spaces (Li-Wang-Xin 2019):

$$(\rho, u) \in C^1([0, T]; H^m(\mathbb{R}^N)), \quad m > \left[\frac{N}{2}\right] + 2$$

- $\nexists$ **global** solution **with finite** $s$:
  - Xin 1998: No global solution such that

$$(\rho, u, s) \in C^1([0, \infty); H^m(\mathbb{R}^N)), \quad m > \left[\frac{N}{2}\right] + 2$$

  - Xin–Yan 2013: No global solution such that

$$\begin{cases} \rho \in C^1_{x,t}, (u, \theta) \in C^{2,1}_{x,t}, & \text{if } \kappa = 0, \\ \rho \in C^1_{x,t}, (u, \theta) \in C^{2,1}_{x,t}, s \text{ is finitely valued}, & \text{if } \kappa > 0 \end{cases}$$

# With "mild" vacuum: $\exists$ entropy-bounded solutions

**Theorems (global in 1D): JL–Xin (Adv. Math. 2019, & 2019 preprint)**

$$\cdots\cdots \rho_0 > 0 \text{ on } \mathbb{R} \quad \text{and} \quad \boxed{\rho_0(x) \geq \frac{K}{|x|^2}} \text{ for } x >> 1 \cdots\cdots$$

$\Longrightarrow$

**exist** { **global-in-time entroy-bounded solutions in inhomogenous spaces** }

**Remark (local in 3D): JL–Xin in preparation**

For 3D, local-in-time existence of entropy-bounded solutions can be also achieved, under similar slow decay assumptions on $\rho_0$.

# A summary: vacuum VS non-vacuum

| The Compressible Navier-Stokes Equations | | | | |
|---|---|---|---|---|
| | **Non-vacuum** | **Strong vacuum** ($\rho_0 \subset\subset R^d$) | **Mild vacuum** ($\rho_0$ decays slowly) | |
| **Solution spaces** | Existence in inhomogeneous spaces | Non-existence in inhomogeneous spaces | Existence in inhomogeneous spaces | |
| **Entropy** | Finite | Can be infinite | Finite | |
| **Entropy-bounded solutions** | Can exist globally | Do not exist globally | Exist globally | 1D √ <br> 3D ? |

# Motivation I: Global entropy-bounded solutions in 3D?

Available results:

- 1D Global entropy-bounded solutions, if $\rho_0(x) \geq \frac{K}{1+|x|^2}$
- 3D Local entropy-bounded solutions, if $\rho_0(x) \geq \frac{K}{1+|x|^2}$
- 3D Global small solutions but with unbounded entropy, for general vacuum, and requires $\rho_0 \in L^1(\mathbb{R}^3)$

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$$\boxed{\left\{ \rho_0 \,\middle|\, \rho_0(x) \geq \frac{K}{1+|x|^2} \right\}} \cap \boxed{L^1(\mathbb{R}^3)} = \emptyset$$

# Motivation II: scaling invariant property of CNS

If $(\rho, u, \theta)$ is a solution to the full CNS with initial data $(\rho_0, u_0, \theta_0)$, then, $(\rho_\lambda, u_\lambda, \theta_\lambda)$ is also a solution with initial data $(\rho_{0\lambda}, u_{0\lambda}, \theta_{0\lambda})$ for any $\lambda \neq 0$, where

$$\begin{cases} \rho_\lambda(x,t) := \rho(\lambda x, \lambda^2 t) \\ u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t) \\ \theta_\lambda(x,t) := \lambda^2 \theta(\lambda x, \lambda^2 t) \end{cases}$$

and

$$\begin{cases} \rho_{0\lambda}(x) := \rho_0(\lambda x) \\ u_{0\lambda}(x) := \lambda u_0(\lambda x) \\ \theta_{0\lambda}(x) := \lambda^2 \theta_0(\lambda x) \end{cases}$$

# Global small solution in critical spaces: non-vacuum

Kato, Chemin, Koch, Burgain,
Dachin, Paicu, Mucha, Q. L. Chen, C. X. Miao, Z. F. Zhang, P. Zhang, Ting Zhang, C. Wang, D. Y. Fang, J. Xu, R. Z. Zi, $\cdots\cdots$

# Main Result

# Global small: smallness on scaling invariant quantity

**Theorem (JL arXiv:1906.08712)**

Assume $2\mu > \lambda$, $q \in (3,6]$, $\rho_0, \theta_0 \geq 0$, $\rho_0 \leq \bar{\rho}$, and

$$\rho_0 \in H^1 \cap W^{1,q}, \quad \sqrt{\rho_0}\theta_0 \in L^2, \quad (u_0, \theta_0) \in D_0^1 \cap D^2,$$
$$-\mu\Delta u_0 - (\mu+\lambda)\nabla \mathrm{div}\, u_0 + \nabla p_0 = \sqrt{\rho_0} g_1,$$
$$\kappa\Delta\theta_0 + \mathcal{Q}(\nabla u) = \sqrt{\rho_0} g_2,$$

for some $\bar{\rho} > 0$ and $(g_1, g_2) \in L^2$, where $p_0 = R\rho_0\theta_0$.
$\Longrightarrow$ There is

$$\boxed{\varepsilon_0 \sim R, \gamma, \mu, \lambda, \kappa}$$

such that the Cauchy problem of the heat conductive CNS in 3D with initial data $(\rho_0, u_0, \theta_0)$ has a unique global solution provided

$$\mathcal{N}_0 := \boxed{\bar{\rho}(\|\rho_0\|_3 + \bar{\rho}^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2 + \bar{\rho}\|\sqrt{\rho_0}E_0\|_2^2)} \leq \varepsilon_0,$$

where $E_0 = \frac{|u_0|^2}{2} + c_v\theta_0$.

# Remark: comparing with the known results

- The quantity

$$\mathcal{N}_0 := \bar{\rho}(\|\rho_0\|_3 + \bar{\rho}^2\|\sqrt{\rho_0}u_0\|_2^2)(\|\nabla u_0\|_2^2 + \bar{\rho}\|\sqrt{\rho_0}E_0\|_2^2)$$

  is **scaling invariant**, with respect to the natural scaling of full CNS on the initial data.
- The **smallness** assumption isimposed **only on** $\mathcal{N}_0$, and the smallness **depends only on** $R, \gamma, \mu, \lambda, \kappa$, but not on any norms of the initial data.
- The theorem cleans the way to the global entropy-bounded solutions to full CNS in 3D:

$$\boxed{\left\{(\rho_0, u_0, \theta_0)\,\middle|\,\rho_0(x) \geq \frac{K}{1+|x|^2}\right\}} \cap \boxed{\{(\rho_0, u_0, \theta_0)\,|\,\mathcal{N}_0 \leq \varepsilon_0\}} \neq \emptyset$$

# Remarks

- Assumptions in Huang–Li 2018 and Wen–Zhu 2017:

$$\int \left( \frac{\rho_0}{2}|u_0|^2 + R(\rho_0 \log \rho_0 - \rho_0 + 1) + \frac{R}{\gamma - 1}\rho_0(\theta_0 - \log \theta_0 + 1) \right)$$
$$\leq \varepsilon_0 = \varepsilon_0(\|\rho_0\|_\infty, \|\theta_0\|_\infty, \|\nabla u_0\|_2, R, \gamma, \mu, \lambda, \kappa)$$

and

$$\int \rho_0 dx \leq \varepsilon_0 = \varepsilon_0(\|\rho_0\|_\infty, \|\sqrt{\rho_0}\theta_0\|_2, \|\nabla u_0\|_2, R, \gamma, \mu, \lambda, \kappa),$$

respectively.

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The scaling invariant quantities on which the smallness guarantee the global existence can not be identified there.

# Proof

# Key observation: an equation for $\rho^3$

An equation:

$$\frac{2\mu+\lambda}{2}(\partial_t\rho^3+\mathrm{div}\,(u\rho^3))+\rho^3 p+\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t$$
$$+\rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)=0.$$

$\Longrightarrow$

$$\sup_{0\le t\le T}\|\rho\|_3^3+\int_0^T\int\rho^3 p \lesssim \sup_{0\le t\le T}(\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}\|\rho\|_3^3)$$
$$+\int_0^T\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2dt+\|\rho_0\|_3^3.$$

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- Indicates the smallness of $\boxed{\int_0^\infty\|\nabla u\|_2^2dt}$ may be sufficient.
- Provides extra estimate $\int_0^\infty\int\rho^3 pdxdt$.

# If using continuity equation $\cdots\cdots$

If using the continuity equation

$$\rho_t + \mathrm{div}(\rho u) = 0$$

$\Longrightarrow$

$$\sup_{0\le t\le T} \|\rho\|_3^3 \le \|\rho_0\|_3^3 + 2\int_0^T \int |\mathrm{div} u|\rho^3 dxdt.$$

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Requires

$$\boxed{\int_0^\infty \|\mathrm{div}\, u\|_\infty dt < \infty},$$

and thus requires faster decay on $\nabla u$.

# Where does $\mathcal{N}_0$ come?

$$\sup_{0\le t\le T}\|\rho\|_3^3+\int_0^T\int\rho^3 p \lesssim \sup_{0\le t\le T}\left(\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}\|\rho\|_3^3\right) + \int_0^T\|\rho\|_\infty^2\|\rho\|_3^2\|\nabla u\|_2^2 dt+\|\rho_0\|_3^3.$$

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$$\sup_{0\le t\le T}\|\rho\|_3^3+\cdots\le \sup_{0\le t\le T}\left(\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}\|\rho\|_3^3\right)\cdots$$

$\rightsquigarrow$ smallness on $\|\rho\|_\infty^{\frac{2}{3}}\|\sqrt{\rho}u\|_2^{\frac{1}{3}}\|\sqrt{\rho}|u|^2\|_2^{\frac{1}{3}}$

$\rightsquigarrow$ smallness on $\|\rho_0\|_\infty^{\frac{2}{3}}\|\sqrt{\rho_0}u_0\|_2^{\frac{1}{3}}\|\sqrt{\rho_0}|u_0|^2\|_2^{\frac{1}{3}}$ ←-- (part of $\mathcal{N}_0$)

$\rightsquigarrow$ smallness on $\mathcal{N}_0$ ←-- (due to the coupling)

# Local existence and blow-up criteria

## Local existence: Cho-Kim 2006

Under the conditions in the theorem, there is a unique solution $(\rho, u, \theta)$, on $\mathbb{R}^3 \times (0, T_*)$, satisfying

$$\rho \in C([0, T_*]; H^1 \cap W^{1,q}), \quad \rho_t \in C([0, T_*]L^2 \cap L^q),$$
$$(u, \theta) \in C([0, T_*]; D_0^1 \cap D^2) \cap L^2(0, T_*; D^{2,q}),$$
$$(u_t, \theta_t) \in L^2(0, T_*; D_0^1), \quad (\sqrt{\rho} u_t, \sqrt{\rho} \theta_t) \in L^\infty_{\text{loc}}(0, T_*; L^2).$$

## Blow-up criteria: Huang-Li 2013

Let $T^* < \infty$ be the maximal time of existence of a solution $(\rho, u, \theta)$, with initial data $(\rho_0, u_0, \theta_0)$. Then,

$$\lim_{T \to T^*} \left( \|\rho\|_{L^\infty(0,T;L^\infty)} + \|u\|_{L^s(0,T;L^r)} \right) = \infty,$$

for any $(s, r)$ such that $\frac{2}{s} + \frac{3}{r} \leq 1$ and $3 < r \leq \infty$.

# Main difficulties: absence of dissipation estimates

- The basic energy identity does not provide any dissipation estimates

$$\int \rho \left( \frac{|u|^2}{2} + c_v \theta \right) dx = \int \rho_0 \left( \frac{|u_0|^2}{2} + c_v \theta_0 \right) dx.$$

- The entropy inequality does not hold for the far field vacuum case.

# Unconditional $\|u\|_{L_t^\infty(L^2)}$ inequality

$$\int \left\{ \rho(\partial_t u + (u\cdot\nabla)u) - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \nabla p = 0 \right\} \cdot udx$$

$\Longrightarrow$

$$\begin{aligned} & \frac{d}{dt}\|\sqrt{\rho}u\|_2^2 + \mu\|\nabla u\|_2^2 + (\mu+\lambda)\|\mathrm{div}\, u\|_2^2 \\ = \quad & \int p\mathrm{div} u dx = R\int \rho\theta \mathrm{div} u dx \le R\|\rho\|_3\|\theta\|_6\|\mathrm{div} u\|_2 \\ \le \quad & \|\rho\|_3\|\nabla\theta\|_2\|\mathrm{div}\, u\|_2 \le \eta\|\mathrm{div}\, u\|_2^2 + C_\eta\|\rho\|_3^2\|\nabla\theta\|_2^2 \end{aligned}$$

$\Longrightarrow$

$$\sup_{0\le t\le T}\|\sqrt{\rho}u\|_2^2 + \int_0^T \|\nabla u\|_2^2 \lesssim \left(\sup_{0\le t\le T}\|\rho\|_3^2\right)\int_0^T \|\nabla\theta\|_2^2 + \|\sqrt{\rho_0}u_0\|_2^2$$

# Unconditional $\|\sqrt{\rho}E\|_{L^\infty_t(L^2)}$ inequality — Step 1

$$E = \frac{|u|^2}{2} + c_v\theta$$

$$\left\{\rho(\partial_t E + u\cdot\nabla E) + \mathrm{div}\,(up) - \kappa\Delta\theta = \mathrm{div}\,(\mathcal{S}(\nabla u)\cdot u)\right\}\cdot E dx$$

---

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}E\|_2^2 + \kappa c_v\|\nabla\theta\|_2^2 \\
= &\int\left[-\frac{\kappa}{2}\nabla\theta\cdot\nabla|u|^2 + (up - \mathcal{S}(\nabla u)\cdot u)\cdot\left(c_v\nabla\theta + \frac{\nabla|u|^2}{2}\right)\right]dx \\
\leq &\frac{c_v\kappa}{2}\|\nabla\theta\|_2^2 + C\||u|\nabla u\|_2^2 + C\int\rho^2\theta^2|u|^2dx,
\end{aligned}$$

---

$$\boxed{\frac{d}{dt}\|\sqrt{\rho}E\|_2^2 + \kappa c_v\|\nabla\theta\|_2^2 \lesssim \||u|\nabla u\|_2^2 + \int\rho^2\theta^2|u|^2dx}$$

# Unconditional $\|\sqrt{\rho}E\|_{L_t^\infty(L^2)}$ inequality — Step 2

$$\int \left\{ \rho(\partial_t u + (u\cdot\nabla)u) - \mu\Delta u - (\mu+\lambda)\nabla \mathrm{div}\, u + \nabla p = 0 \right\} \cdot |u|^2 u dx$$

---

$$\begin{aligned} & \frac{1}{4}\frac{d}{dt}\|\sqrt{\rho}|u|^2\|_2^2 - \int (\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div}\, u)\cdot |u|^2 u dx \\ = & -\int p\mathrm{div}\,(|u|^2 u) dx \leq \eta \int |u|^2|\nabla u|^2 dx + C_\eta \int \rho^2\theta^2|u|^2 dx \end{aligned}$$

and

$$-\int (\mu\Delta u + (\mu+\lambda)\nabla \mathrm{div}\, u)\cdot |u|^2 u \geq (2\mu-\lambda)\int |u|^2|\nabla u|^2$$

---

$$\boxed{\frac{d}{dt}\|\sqrt{\rho}|u|^2\|_2^2 + 2(2\mu-\lambda)\||u|\nabla u\|_2^2 \lesssim \int \rho^2\theta^2|u|^2 dx}$$

# Unconditional $\|\sqrt{\rho}E\|_{L_t^\infty(L^2)}$ inequality — Conclusion

$$\frac{d}{dt}\|\sqrt{\rho}E\|_2^2 + \kappa c_v\|\nabla\theta\|_2^2 \lesssim \||u|\nabla u\|_2^2 + \int \rho^2\theta^2|u|^2 dx$$

$$\frac{d}{dt}\|\sqrt{\rho}|u|^2\|_2^2 + 2(2\mu-\lambda)\||u|\nabla u\|_2^2 \lesssim \int \rho^2\theta^2|u|^2 dx$$

$\Longrightarrow$

$$\frac{d}{dt}\|\sqrt{\rho}E\|_2^2 + \|(\nabla\theta, |u|\nabla u)\|_2^2 \le C\int \rho^2\theta^2|u|^2 dx,$$

---

$$\int \rho^2\theta^2|u|^2 dx \le \cdots \lesssim \|\sqrt{\rho}\theta\|_2\|\nabla\theta\|_2\|\nabla|u|^2\|_2\|\rho\|_\infty\|\rho\|_3^{\frac{1}{2}}$$

---

$$\boxed{\frac{d}{dt}\|\sqrt{\rho}E\|_2^2 + \|(\nabla\theta, |u|\nabla u)\|_2^2 \lesssim \left(\|\rho\|_\infty^2\|\rho\|_3\|\sqrt{\rho}\theta\|_2^2\right)^{\frac{1}{2}}\|(\nabla\theta, |u|\nabla u)\|_2^2}$$

# An equation for $\rho^3$

$$\rho^3\Delta^{-1}\mathrm{div}\left\{\partial_t(\rho u)+\mathrm{div}(\rho u\otimes u)-\mu\Delta u-(\mu+\lambda)\nabla\mathrm{div}\, u+\nabla p=0\right\}$$

$\Longrightarrow$

$$\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t+\rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)-(2\mu+\lambda)\mathrm{div}\, u\rho^3+\rho^3 p=0.$$

---

Replace $\mathrm{div} u\rho^3$ by using

$$\partial_t\rho^3+\mathrm{div}\,(u\rho^3)+2\mathrm{div}\, u\rho^3=0$$

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$$\frac{2\mu+\lambda}{2}(\partial_t\rho^3+\mathrm{div}\,(u\rho^3))+\rho^3 p+\rho^3\Delta^{-1}\mathrm{div}\,(\rho u)_t$$
$$+\rho^3\Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u\otimes u)=0.$$

# Unconditional $\|\rho\|_{L_t^\infty(L^3)}$ inequality

$$\int \left\{ \frac{2\mu+\lambda}{2}(\partial_t \rho^3 + \mathrm{div}\,(u\rho^3)) + \rho^3 p + \rho^3 \Delta^{-1}\mathrm{div}\,(\rho u)_t \right.$$
$$\left. + \rho^3 \Delta^{-1}\mathrm{div}\,\mathrm{div}\,(\rho u \otimes u) = 0 \right\} dx$$

---

$$\sup_{0\le t\le T} \|\rho\|_3^3 + \int_0^T \int \rho^3 p dx dt$$
$$\lesssim \left( \sup_{0\le t\le T} \|\rho\|_\infty^4 \|\sqrt{\rho} u\|_2^2 \|\sqrt{\rho}|u|^2\|_2^2 \right)^{\frac{1}{6}} \sup_{0\le t\le T} \|\rho\|_3^3$$
$$+ \left( \sup_{0\le t\le T} \|\rho\|_\infty^2 \|\rho\|_3^2 \right) \int_0^T \|\nabla u\|_2^2 + \cdots$$

## Conditional $\|u\|_{L_t^\infty(H^1)}$ inequality — Testing with $u_t$

$$\int \left\{ \rho(\partial_t u + (u \cdot \nabla)u) - \mu\Delta u - (\mu+\lambda)\nabla \operatorname{div} u + \nabla p = 0 \right\} \cdot u_t dx$$

$\Longrightarrow$

$$\frac{1}{2}\frac{d}{dt}(\mu\|\nabla u\|_2^2 + (\mu+\lambda)\|\operatorname{div} u\|_2^2) + \|\sqrt{\rho}u_t\|_2^2 - \int p \operatorname{div} u_t dx = -\int \rho(u\cdot\nabla)u \cdot u_t dx.$$

## Conditional $\|u\|_{L_t^\infty(H^1)}$ — about $-\int p\mathrm{div}\, u_t dx$ and $\int p_t G dx$

Note that $\mathrm{div}\, u = \frac{G+p}{2\mu+\lambda}$, it follows

$$
\begin{aligned}
-\int p\mathrm{div}\, u_t &= -\frac{d}{dt}\int p\mathrm{div}\, u dx + \int p_t \mathrm{div}\, u dx \\
&= -\frac{d}{dt}\int p\mathrm{div}\, u + \frac{1}{2(2\mu+\lambda)}\frac{d}{dt}\|p\|_2^2 + \frac{1}{2\mu+\lambda}\int p_t G
\end{aligned}
$$

Note that

$$
p_t = (\gamma-1)(\mathcal{Q}(\nabla u) - p\mathrm{div}\, u + \kappa\Delta\theta) - \mathrm{div}\,(up),
$$

and, thus, integration by parts gives

$$
\int p_t G dx = \int [(\gamma-1)(\mathcal{Q}(\nabla u) - p\mathrm{div}\, u)G + (up - \kappa(\gamma-1)\nabla\theta)\cdot\nabla G]
$$

# Conditional $\|u\|_{L_t^\infty(H^1)}$ inequality — Estimate for $\nabla G$

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\mu\|\omega\|_2^2+\frac{\|G\|_2^2}{2\mu+\lambda}\right)+\|\sqrt{\rho}u_t\|_2^2\\
= \quad &-\int \rho(u\cdot\nabla)u\cdot u_t dx+\frac{1}{2\mu+\lambda}\int(\kappa(\gamma-1)\nabla\theta-up)\cdot\nabla G dx\\
&-\frac{\gamma-1}{2\mu+\lambda}\int(\mathcal{Q}(\nabla u)-p\mathrm{div}\, u)G dx.
\end{aligned}
$$

---

$$
\int\left\{\rho(u_t+u\cdot\nabla u)=\nabla G-\mu\nabla\times\omega\right\}\cdot(\nabla G-\mu\nabla\times\omega)dx
$$

$\Longrightarrow$

$$
\|\nabla G\|_2^2+\|\nabla\times\omega\|_2^2\le C\|\sqrt{\rho}u_t\|_2^2+\cdots
$$

# Conditional $\|u\|_{L_t^\infty(H^1)}$ inequality — Conclusion

- Conditional $\|u\|_{L_t^\infty(H^1)}$:

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\nabla u\|_2^2+\int_0^T\left\|\left(\sqrt{\rho}u_t,\frac{\nabla G}{\sqrt{\bar\rho}},\frac{\nabla\omega}{\sqrt{\bar\rho}}\right)\right\|_2^2dt\\
\lesssim\ &\left[\bar\rho+\bar\rho^2\left(\sup_{0\le t\le T}\|\rho\|_3\|\sqrt{\rho}\theta\|_2^2\right)^{\frac12}\right]\int_0^T\|(\nabla\theta,|u|\nabla u)\|_2^2\\
&+\bar\rho^3\left(\sup_{0\le t\le T}\|(\nabla u,\sqrt{\bar\rho}\sqrt{\rho}\theta)\|_2^2\|\nabla u\|_2^2\right)\int_0^T\|\nabla u\|_2^2+\cdots,
\end{aligned}
$$

**as long as**

$$
\boxed{\sup_{0\le t\le T}\|\rho\|_\infty\le 4\bar\rho},
$$

where $G=(2\mu+\lambda)\mathrm{div}\,u-p$ and $\omega=\nabla\times u$.

# Conditional $L^\infty$ inequality for $\rho$

- $L^\infty$ of $\rho$:

$$\sup_{0\le t\le T}\|\rho\|_\infty \le \bar\rho\exp\left\{C\bar\rho\int_0^T\|\nabla u\|_2\|(\nabla G,\nabla\omega,\bar\rho\nabla\theta)\|_2dt\right\}\times\exp\left\{C\bar\rho^{\frac23}\left(\sup_{0\le t\le T}\|\sqrt{\rho}u\|_2\|\sqrt{\rho}|u|^2\|_2\right)^{\frac13}\right\},$$

**as long as**

$$\boxed{\sup_{0\le t\le T}\|\rho\|_\infty\le 4\bar\rho},$$

---

$\Delta^{-1}\mathrm{div}\,(\text{momentum eqn}) \rightsquigarrow \mathrm{div}u \curvearrowright (\text{continuity eqn}) \Longrightarrow$

$$\frac{d}{dt}\Big((2\mu+\lambda)\log\rho+(\Delta^{-1}\mathrm{div}\,(\rho u))\Big)+p=[u,\mathcal{R}\otimes\mathcal{R}](\rho u).$$

# A priori estimates

## Key a priori estimates

Denote

$$\mathcal{N}_T = \bar{\rho} \sup_{0\leq t\leq T} (\|\rho\|_3 + \bar{\rho}^2\|\sqrt{\rho}u\|_2^2)(\|\nabla u\|_2^2 + \bar{\rho}\|\sqrt{\rho}E\|_2^2).$$

Then, there is a positive number $\varepsilon_0 \sim R, \gamma, \mu, \lambda, \kappa$, such that

$$\sup_{0\leq t\leq T} \|\rho\|_\infty \leq 2\bar{\rho} \quad \text{and} \quad \mathcal{N}_T \leq \frac{\sqrt{\varepsilon_0}}{2}$$

as long as

$$\sup_{0\leq t\leq T} \|\rho\|_\infty \leq 4\bar{\rho}, \quad \mathcal{N}_T \leq \sqrt{\varepsilon_0}, \quad \textit{and} \quad \mathcal{N}_0 \leq \varepsilon_0.$$

# Conclusion

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- Global existence of small strong solutions to the full CNS was established, and **a scaling invariant quantity was identified**, on which the smallness condition ensures the global existence.
- Different from the existing works, the smallness **depends only on** $R, \gamma, \mu, \lambda, \kappa$, but not on any norms of the initial data.
- The theorem cleans the way to the global entropy-bounded solutions to full CNS in 3D:

$$\left\{(\rho_0, u_0, \theta_0)\,\middle|\,\rho_0(x) \geq \frac{K}{1+|x|^2}\right\} \cap \{(\rho_0, u_0, \theta_0)\,|\,\mathcal{N}_0 \text{ small }\} \neq \emptyset$$

# Thank You!