Analysis on the model hierarchy for mean field interacting particle system via non Lipschitz and velocity dependent force

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Problem setting and background

Interacting particle system

Particle model

A two-dimensional interacting particle system with position $x_i \in \mathbb{R}^2$ and velocity $v_i \in \mathbb{R}^2$,

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \quad i = 1, \ldots, N \\ \\ \dfrac{dv_i}{dt} = \dfrac{1}{N-1} \sum_{i \neq j} F(x_i - x_j, v_i - v_j) + G(x_i, v_i), \end{cases}$$

- $F(x, v)$, the total interaction force,
- $G(x, v)$ the desired velocity and direction acceleration.

Background: mean field interacting particle models for material flow, pedestrian flow, group behavior for small animals (for example, Cucker-Smale).

Total interaction force and acceleration force

based on the material flow and pedestrian flow model proposed by Göttlich and her collaborators.

- $F(x, v)$ consists of the interaction force $F_{int}(x)$ and the dissipative force $F_{diss}(x, v)$,

$$F(x, v) \quad = \quad (F_{int}(x) + F_{diss}(x, v))\mathcal{H}(x, v)$$

  where $\mathcal{H}(x, v)$ is a smooth cut-off function with compact support B_{2R}.

  - Interaction force, $F_{int}(x) = -\nabla V(|x|) = -\nabla k_n(2R|x| - \frac{|x|^2}{2})$
  - Dissipative force, $F_{diss} = F^n_{diss} + F^t_{diss}$ with

$$F^n_{diss} = -\gamma_n \langle v, \frac{x}{|x|} \rangle \frac{x}{|x|}$$

$$F^t_{diss} = -\gamma_t \left(v - \langle v, \frac{x}{|x|} \rangle \frac{x}{|x|} \right)$$

    where γ_n, γ_t are suitable positive friction constants.

- Velocity and direction acceleration is given by

$$G(x, v) := g(x) - v.$$

Kinetic equation (mesoscopic level)

After taking the mean field limit, one obtains the evolution of the (effective one particle) density $f(t,x,v)$

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] + \nabla_v \cdot (Gf) = 0.$$

Hydrodynamic model (macroscopic level)

- Ansatz $f(t,x,v) = \rho(t,x)\delta_u(t,x)(v)$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t u + u \cdot \nabla u = \int F(x-y, u(x)-u(y))\rho(y)dy + G(x,u)$$

- Ansatz $f(t,x,v) = \rho(t,x)\frac{1}{(2\pi)^{\frac{d}{2}}}e^{-\frac{|v-u(t,x)|^2}{2}}$

$$\partial_t \rho + \nabla \cdot (\rho u) = 0,$$
$$\partial_t u + u \cdot \nabla u + \frac{\nabla \rho}{\rho} = \int F(x-y, u(x)-u(y))\rho(y)dy + G(x,u)$$

Rigorous derivation of the kinetic model (mean field limit)

Rigorous derivation of the kinetic model—mean field limit

Particle model with position $x_i \in \mathbb{R}^2$ and velocity $v_i \in \mathbb{R}^2, i = 1, \ldots, N$.

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N-1} \displaystyle\sum_{i \neq j} F(x_i - x_j, v_i - v_j) + G(x_i, v_i), \end{cases}$$

Particle model with cut-off of order $N^{-\theta}$ with $0 < \theta < \frac{1}{4}$,

$$F^N(x,v) = \begin{cases} \Big(2Rk_n \dfrac{x}{|x|} - k_n x + \dfrac{\langle v, x\rangle}{|x|^2}(\gamma_t - \gamma_n)x \\ \qquad\qquad -\gamma_t v\Big)\mathcal{H}(x,v), \quad |x| \geq N^{-\theta}, \\ \Big((2Rk_n N^{\theta} - k_n)x + N^{2\theta}\langle v, x\rangle(\gamma_t - \gamma_n)x \\ \qquad\qquad -\gamma_t v\Big)\mathcal{H}(x,v), \quad |x| < N^{-\theta}. \end{cases}$$

Cut-off interaction

The interaction force $F^N(x,v)$ with cut-off has the following properties,

(a) $F^N(x,v)$ is bounded, i.e., $|F^N(x,v)| \leq C$.

(b) $F^N(x,v)$ satisfies the following property

$$|F^N(x,v) - F^N(y,v)| \leq q^N(x,v)|x-y|,$$

where q^N has compact support in $B_{2R} \times B_{2\widetilde{R}}$ with

$$q^N(x,v) := \begin{cases} C \cdot \dfrac{1}{|x|} + C, & |x| \geq N^{-\theta}, \\ C \cdot N^{\theta}, & |x| < N^{-\theta}. \end{cases}$$

(c) $F^N(x,v)$ is Lipschitz continuous in v.

Particle model with cut-off interaction

$$\begin{cases} \frac{d}{dt} X_t^N = V_t^N, \\ \frac{d}{dt} V_t^N = \Psi^N(X_t^N, V_t^N) + \Gamma(X_t^N, V_t^N), \end{cases}$$

$\Psi^N(X_t^N, V_t^N)$ denotes the total interaction force with

$$\left(\Psi^N(X_t^N, V_t^N)\right)_i = \frac{1}{N-1} \sum_{i \neq j} F^N(x_i^N - x_j^N, v_i^N - v_j^N),$$

$\Gamma(X_t^N, V_t^N)$ stands for the desired velocity and direction acceleration with

$$\left(\Gamma(X_t^N, V_t^N)\right)_i = G(x_i^N, v_i^N).$$

An intermediate particle model

$$\begin{cases} \frac{d}{dt}\overline{X}_t^N = \overline{V}_t^N, \\ \frac{d}{dt}\overline{V}_t^N = \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) + \Gamma(\overline{X}_t^N, \overline{V}_t^N), \end{cases}$$

where the total interaction force

$$\left(\overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N)\right)_i = \iint F^N(\overline{x}_i^N - y, \overline{v}_i^N - w) f^N(t, y, w)\, dydw$$

and the desired velocity and direction acceleration

$$\left(\Gamma(\overline{X}_t^N, \overline{V}_t^N)\right)_i = G(\overline{x}_i^N, \overline{v}_i^N).$$

It is obvious that $(\overline{X}_t^N, \overline{V}_t^N)_i$ is the trajectory on $\mathbb{R}^4$ which evolves according to the Vlasov equation

$$\partial_t f^N + v \cdot \nabla_x f^N + \nabla_v \cdot [(F^N * f^N) f^N] + \nabla_v \cdot (G f^N) = 0,$$

Theorem 1

For $\theta \in (0, \frac{1}{4})$, $\alpha \in (0, \frac{1}{5})$, $\beta \in (\alpha, \frac{1-\alpha}{4})$, $\gamma \in (0, \frac{1-\alpha-4\theta}{4})$. Let $f^N(t,x,v)$ be the solution to the cut-offed Vlasov equation, with $\|f^N\|_{L^1} + \|f^N\|_{L^\infty}$ bounded uniformly in N. If $G(x,v)$ is Lipschitz continuous, then $\exists C > 0$, it holds with a θ-independent convergence rate that

$$\mathbb{P}_0\left(\sup_{0\le s\le t}\left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_\infty > N^{-\alpha}\right) \le e^{Ct}\cdot r(N),$$

where the convergence rate

$$r(N) = \max\{N^{-(1-\alpha-4\beta)}, N^{\alpha-\beta}, N^{-(1-\alpha-4\gamma)}\ln^2 N\}.$$

Remark The assumption that $\|f^N\|_{L^1} + \|f^N\|_{L^\infty}$ bounded uniformly in N can be obtained in studying the vlasov equation separately (will be presented in the next section).

Idea of the proof. (Inspired by Pickl's work on the derivation of Vlasov-Poission system)
We consider a stochastic process given by

$$S_t = \min\left\{1, N^{\alpha} \sup_{0\le s\le t}\left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_{\infty}\right\}.$$

The proof for the result in the first theorem is actually based on the control of $\mathbb{E}_0\left[S_t\right]$ if one observes the following Markov inequality

$$\mathbb{P}_0\left(\sup_{0\le s\le t}\left|(X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N)\right|_{\infty} > N^{-\alpha}\right) = \mathbb{P}_0(S_t = 1) \le \mathbb{E}_0\left[S_t\right].$$

Mean Field Limit

The cut-off particle system and the intermediate cut-off particle system are

$$\begin{aligned}(X^N_{t+dt}, V^N_{t+dt}) &= (X^N_t, V^N_t) + (V^N_t, \Psi^N(X^N_t, V^N_t) + \Gamma(X^N_t, V^N_t))dt + o(dt),\\ (\overline{X}^N_{t+dt}, \overline{V}^N_{t+dt}) &= (\overline{X}^N_t, \overline{V}^N_t) + (\overline{V}^N_t, \overline{\Psi}^N(\overline{X}^N_t, \overline{V}^N_t) + \Gamma(\overline{X}^N_t, \overline{V}^N_t))dt + o(dt).\end{aligned}$$

Taking the difference gives

$$\left|(X^N_{t+dt}, V^N_{t+dt}) - (\overline{X}^N_{t+dt}, \overline{V}^N_{t+dt})\right|_\infty \le \left|(X^N_t, V^N_t) - (\overline{X}^N_t, \overline{V}^N_t)\right|_\infty$$

$$+ \left|\left(V^N_t, \Psi^N(X^N_t, V^N_t) + \Gamma(X^N_t, V^N_t)\right) - \left(\overline{V}^N_t, \overline{\Psi}^N(\overline{X}^N_t, \overline{V}^N_t) + \Gamma(\overline{X}^N_t, \overline{V}^N_t)\right)\right|_\infty dt + o(dt),$$

which is

$$S_{t+dt} - S_t \le \left|\left(V^N_t, \Psi^N(X^N_t, V^N_t) + \Gamma(X^N_t, V^N_t)\right) - \right.$$

$$\left.\left(\overline{V}^N_t, \overline{\Psi}^N(\overline{X}^N_t, \overline{V}^N_t) + \Gamma(\overline{X}^N_t, \overline{V}^N_t)\right)\right|_\infty N^\alpha dt + o(dt)$$

The set, where $|S_t| = 1$, is defined as $\mathcal{N}_\alpha$, i.e.,

$$\mathcal{N}_\alpha := \left\{ (X,V) : \sup_{0\le s\le t} \left| (X_s^N, V_s^N) - (\overline{X}_s^N, \overline{V}_s^N) \right|_\infty > N^{-\alpha} \right\}.$$

In order to handle the difference from the right hand side, we define

$$\mathcal{N}_\beta := \left\{ (X,V) : \left| \Psi^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty > N^{-\beta} \right\},$$

$$\mathcal{N}_\gamma := \left\{ (X,V) : \left| Q^N(\overline{X}_t^N, \overline{V}_t^N) - \overline{Q}^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty > N^{-\gamma} \right\}$$

Key estimates

- $\mathbb{P}_0(\mathcal{N}_\beta) \le CN^{-(1-4\beta)}$,
- $\mathbb{P}_0(\mathcal{N}_\gamma) \le C \cdot N^{-(1-4\gamma)} \ln^2 N$.

Mean Field Limit

Taking the expectation over both sides yields

$$
\begin{aligned}
\mathbb{E}_0\left[\, S_{t+dt} - S_t \right] &= \mathbb{E}_0\left[S_{t+dt} - S_t \mid \mathcal{N}_\alpha \,\right] + \mathbb{E}_0\left[\, S_{t+dt} - S_t \mid \mathcal{N}_\alpha^c \,\right] \\
&\leq \mathbb{E}_0\left[\, S_{t+dt} - S_t \mid (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \,\right] + \mathbb{E}_0\left[\, S_{t+dt} - S_t \mid (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c \,\right] \\
&\leq \mathbb{E}_0\left[\left| V_t^N - \overline{V}_t^N \right|_\infty \,\middle|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha \, dt \\
&\quad + \mathbb{E}_0\left[\left| \Psi^N(X_t^N, V_t^N) - \overline{\Psi}^N(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty \,\middle|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha \, dt \\
&\quad + \mathbb{E}_0\left[\left| \Gamma(X_t^N, V_t^N) - \Gamma(\overline{X}_t^N, \overline{V}_t^N) \right|_\infty \,\middle|\, (\mathcal{N}_\beta \cup \mathcal{N}_\gamma) \setminus \mathcal{N}_\alpha \right] N^\alpha \, dt \\
&\quad + \mathbb{E}_0\left[\, S_{t+dt} - S_t \mid (\mathcal{N}_\alpha \cup \mathcal{N}_\beta \cup \mathcal{N}_\gamma)^c \,\right] + o(dt) \\
&=: J_1 + J_2 + J_3 + J_4 + o(dt),
\end{aligned}
$$

Finally,

$$
\frac{d}{dt}\,\mathbb{E}_0[S_t] \leq C \cdot \mathbb{E}_0[S_t] + C \cdot \max\{N^{-(1-4\gamma)} \ln^2 N N^\alpha, N^{-(1-\alpha-4\beta)}, N^{\alpha-\beta}\}.
$$

Gronwall's inequality yields

$$
\mathbb{E}_0\left[S_t\right] \leq e^{Ct} \cdot \max\{\widetilde{r}(N) N^\alpha, N^{-(1-\alpha-4\beta)}), N^{\alpha-\beta}\}.
$$

The convergence from cut-off vlasov to non-cut-off vlasov The trajectory of the non-cut-off vlasov equation is

$$\begin{cases} \frac{d}{dt}\overline{X}_t = \overline{V}_t, \\ \frac{d}{dt}\overline{V}_t = \overline{\Psi}(\overline{X}_t, \overline{V}_t) + \Gamma(\overline{X}_t, \overline{V}_t), \end{cases}$$

where the total interaction force

$$\left(\overline{\Psi}(\overline{X}_t, \overline{V}_t)\right)_i = \iint F(\overline{x}_i - y, \overline{v}_i - w) f(t, y, w)\, dy dw$$

and the desired velocity and direction acceleration $\left(\Gamma(\overline{X}_t, \overline{V}_t)\right)_i = G(\overline{x}_i, \overline{v}_i)$.

Theorem 2

If additionally the initial condition f_0 has the property that ∇f_0 is integrable, then there holds

$$\lim_{N\to\infty} \mathbb{P}_0 \left(\sup_{0\le s\le t} \left| (X_s^N, V_s^N) - (\overline{X}_s, \overline{V}_s) \right|_\infty > N^{-\alpha} \right) = 0.$$

Existence of Weak Solution

Weak Solution (Existence of weak solution)

The mean field kinetic equation is

$$\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot [(F * f)f] + \nabla_v \cdot (Gf) = 0.$$

Definition Let $f_0(x, v) \in L^1(\mathbb{R}^2 \times \mathbb{R}^2) \cap L^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$. A function $f = f(t, x, v)$ is said to be a weak solution to the kinetic mean field equation (1) with initial data f_0, if

$$\begin{aligned}
\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\varphi(x,v)\,dxdv &= \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)\varphi(x,v)\,dxdv \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} vf(s,x,v)\cdot\nabla_x\varphi(x,v)\,dxdvds \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} (F(x,v) * f(s,x,v))\,f(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdvds \\
&+ \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} G(x,v)f(s,x,v)\cdot\nabla_v\varphi(x,v)\,dxdvds
\end{aligned}$$

for all $\varphi(x, v) \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ and $t \in \mathbb{R}_+$.

Weak Solution (Existence)

Theorem 3 Let $f_0(x,v)$ be a nonnegative function in $L^1 \cap L^\infty$ and

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f_0(x,v)\,dxdv =: \mathcal{E}_0, \quad \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|x|^2 f_0(x,v)\,dxdv =: \mathcal{M}_2.$$

Then, there exists a weak solution $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^2 \times \mathbb{R}^2))$ to the mean field equation (1) with initial data f_0. Moreover this solution satisfies

$$0 \le f(t,x,v) \le \|f_0\|_{L^\infty(\mathbb{R}^2\times\mathbb{R}^2)} e^{Ct}, \quad \text{for a.e. } (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2,\ t \ge 0$$

together with the mass conservation

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(t,x,v)\,dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} f_0(x,v)\,dxdv =: \mathcal{M}_0$$

and the kinetic energy and second moment bounds

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|v|^2 f(t,x,v)\,dxdv \le C, \quad \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2}|x|^2 f(t,x,v)\,dxdv \le \mathcal{M}_2 e^{Ct}.$$

Weak Solution (Existence)

Lemma (Golse, Lecture notes, 2013) Assume that $K(z,z') \in C(\mathbb{R}^4 \times \mathbb{R}^4; \mathbb{R}^4)$ is Lipschitz continuous in z, uniformly in z' (and conversely), i.e., there exists a constant $L > 0$ such that

$$\sup_{z' \in \mathbb{R}^4} |K(z_1, z') - K(z_2, z')| \leq L|z_1 - z_2|,$$
$$\sup_{z \in \mathbb{R}^4} |K(z, z_1) - K(z, z_2)| \leq L|z_1 - z_2|.$$

For any given $z_0 = (x_0, v_0) \in \mathbb{R}^2 \times \mathbb{R}^2$ and Borel probability measure $\mu_0 \in \mathcal{P}_1(\mathbb{R}^4)$, there exists a unique C^1-solution $Z(t, z_0, \mu_0)$ to the problem

$$\begin{cases} \dfrac{d}{dt} Z(t, z_0, \mu_0) = \displaystyle\int_{\mathbb{R}^4} K\left(Z(t, z_0), z'\right) \mu(t, dz'), \\ Z(0, z_0, \mu_0) = z_0, \end{cases}$$

where $\mu(t, \cdot)$ is the push-forward of μ_0, i.e., $\mu(t, \cdot) = Z(t, \cdot, \mu_0) \# \mu_0$.

Weak Solution (Existence)

Idea of the proof By taking the interaction kernel K as

$$K^N(z,z') = K^N(x,v,x',v') := \left(v, F^N(x-x', v-v') + G(x,v)\right),$$

the mean field cut-off equation can be put into the form

$$\partial_t f^N(t,z) + \mathrm{div}_z\left(f^N(t,z)\iint_{\mathbb{R}^2\times\mathbb{R}^2} K^N(z,z')f^N(t,z')dz'\right) = 0.$$

For the given initial data f_0^N, we have

$$f^N(t,z) := f_0^N\left(Z^N(t,\cdot)^{-1}(z)\right)J(0,t,z), \quad \forall\, t \geq 0,$$

where $J(0,t,z)$ is the Jacobian, i.e.,

$$J(0,t,z) = \exp\left(\int_t^0 \mathrm{div}_v\left(F^N * f^N(s,Z^N(s,z)) + G^N(Z^N(s,z))\right)ds\right).$$

Weak Solution (Existence)

- Conservation of mass $\|f^N\|_{L^1} = \mathcal{M}_0$
- L^∞ estimate comes from the boundedness of the force

$$
\begin{aligned}
&|f^N(t,z)| \le |f_0^N\left(Z^N(t,\cdot)^{-1}(z)\right) J(0,t,z)| \\
\le\quad & \|f_0^N\|_{L^\infty} \exp\left(\int_0^t \|\nabla_v F^N * f^N\|_{L^\infty}\, ds + Ct\right) \\
\le\quad & \|f_0^N\|_{L^\infty(} \exp\left(\int_0^t \|\nabla_v F^N\|_{L^\infty} \|f^N\|_{L^1}\, ds + Ct\right) \le \|f_0^N\|_{L^\infty} e^{Ct},
\end{aligned}
$$

- Kinetic energy estimate comes from the contribution from damping

$$
\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2} v^2 f^N(t,x,v)\, dxdv \le C - \iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{2} v^2 f^N(t,x,v)\, dxdv,
$$

- Bound for second moment

$$
\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 f^N(t,x,v)\, dxdv = \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 \partial_t f^N(t,x,v)\, dxdv \\
=\quad & \iint_{\mathbb{R}^2\times\mathbb{R}^2} x\cdot v f^N(t,x,v)\, dxdv \le \iint_{\mathbb{R}^2\times\mathbb{R}^2} |x|^2 f^N(t,x,v)\, dxdv + C.
\end{aligned}
$$

Weak Solution (Existence)

To take the limit in the approximated problem for the existence of weak solution, we need to care about the nonlinear term

$$\nabla_v[(F^N * f^N)f^N]$$

Instead of trying to get a strong convergence of f^N, we obtain additional estimates for $F^N * f^N$ in the following

$$\|F^N * f^N\|_{L^\infty(L^1)} \le C\left(\|F\|_{L^1}, \mathcal{M}_0, \bar{R}\right)$$
$$\|F^N * f^N\|_{L^\infty(L^\infty)} \le C\left(\|F\|_{L^\infty}, \mathcal{M}_0\right)$$
$$\|\nabla_v\left(F^N * f^N\right)\|_{L^\infty(L^1)} \le C\left(\|\nabla_v F\|_{L^1}, \mathcal{M}_0, \bar{R}\right)$$
$$\|\nabla_v\left(F^N * f^N\right)\|_{L^\infty(L^\infty)} \le C\left(\|\nabla_v F\|_{L^\infty}, \mathcal{M}_0\right)$$
$$\|\nabla_x\left(F^N * f^N\right)\|_{L^\infty(L^2)} \le C \cdot \left\|\left(\chi_{\bar{R}} \cdot \frac{1}{|x|}\right) * f^N\right\|_{L^\infty(L^2)} \le \|f^N\|_{L^\infty(L^p)}, \forall p > 1.$$

Further estimates shows that

$$\|\partial_t(F^N * f^N)\|_{L^\infty([0,T];W^{-1,2}(\mathbb{R}^2\times\mathbb{R}^2))} \le C.$$

Weak Solution (Existence)

Therefore, for any test function $\varphi \in C_0^\infty$ with $\Omega = \mathrm{supp}\varphi$, according to Aubin-Lions compact embedding theorem there exists a subsequence such that

$$F^N * f^N \to F * f \quad \text{in } L^\infty([0,T]; L^2(\Omega)).$$

We then get $\forall\, t \in (0,T)$ the following estimates:

$$\begin{aligned}
& \Big| \int_0^t \iint_{\mathbb{R}^2\times\mathbb{R}^2} \Big(\big((F^N * f^N) f^N\big)(s,x,v)\nabla_v\varphi(x,v) - \big((F*f)f\big)(s,x,v)\nabla_v\varphi(x,v)\Big)\, dxdvds \Big| \\
\leq\ & \Big| \int_0^t \iint_\Omega \Big(\big((F^N * f^N) f^N\big)(s,x,v)\nabla_v\varphi(x,v) - \big((F*f)f^N\big)(s,x,v)\nabla_v\varphi(x,v)\Big)\, dxdvds \Big| \\
& \quad + \Big| \int_0^t \iint_\Omega \Big(\big((F*f)f^N\big)(s,x,v)\nabla_v\varphi(x,v) - \big((F*f)f\big)(s,x,v)\nabla_v\varphi(x,v)\Big)\, dxdvds \Big| \\
=:\ & J_1 + J_2 .
\end{aligned}$$

For the first term J_1, we have

$$\lim_{N\to\infty} J_1 \leq \lim_{N\to\infty} \|F^N * f^N - F * f\|_{L^\infty(L^2(\Omega))} \|f^N\|_{L^\infty(L^\infty)} \|\nabla_v \varphi\|_{L^2} = 0$$

while for the second term J_2 we use the fact that $f^N \overset{*}{\rightharpoonup} f$ in $L^\infty(\mathbb{R}_+; L^\infty(\mathbb{R}^2\times\mathbb{R}^2))$ for $F * f \cdot \nabla_v\varphi \in L^1(L^1)$, namely

$$\lim_{N\to\infty} J_2 = 0.$$

Global classical solution of compressible Euler system with velocity alignment

Formal Understanding of the Model

After taking the mean field limit, one obtains the evolution of the (effective one particle) density $f(t,x,v)$

$$\partial_t f + v\cdot\nabla_x f + \nabla_v\cdot[(F*f)f] + \nabla_v\cdot(Gf) = 0.$$

Hydrodynamic model (macroscopic level)

Ansatz $f(t,x,v) = I_{|v-u(t,x)|^2\le\rho^{\frac{2}{d}}(t,x)}(x,v)$.

This is the minimizer of the kinetic energy $\displaystyle\iint \frac{v^2}{2} f(t,x,v)\,dxdv$ under the constraint $\|f(t,x,v)\|_\infty \le 1$.

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot(\rho u) = 0,\\
&\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla\rho^{\frac{d+2}{d}}\\
&\qquad\qquad = \rho(x)\int F(x-y,u(x)-u(y))\rho(y)dy + \rho G(x,u)
\end{aligned}$$

In particular, $F(x,u) = \dfrac{x}{|x|}\otimes\dfrac{x}{|x|}u\mathcal{H}(x,u)$.

General Cauchy Problem

We consider the following general system

$$
\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
&\partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p(\rho) \\
&\qquad\qquad = -\frac{1}{\tau}\rho u - a\rho(x) \int_{\mathbb{R}^N} \mathbf{\Gamma}(x-y)(u(x)-u(y))\rho(y)dy,, \\
&\rho|_{t=0} = \rho_0(x), \quad u|_{t=0} = u_0(x), \quad x \in \mathbb{R}^N.
\end{aligned}
$$

where the pressure $P(\rho) = \rho^\gamma$ The matrix is $\mathbf{\Gamma}(x) \in L^1(\mathbb{R}^N)$. The constants $\gamma \geq 1, \tau > 0, a > 0$ are given.

Known results (for the case that $\mathbf{\Gamma}(x) = \phi(|x|)\mathbf{I}_{N\times N}$, $\phi(|x|) \geq c > 0$)

- 1-D, weak entropy solution, Ha,Huang and Wang, 2014
- 1-D, regular solution, Kieselev, Tan, 2018
- M-D, global smooth solution, Carrillo, Choi, Tadmor, Tan, Kang, Kwon, Kiselev... 2014-2018
- M-D, infinite global weak solution, Carrillo, Feireisl, Gwiazda, Swierczewska-Gwiazda, 2017

Reformulation into a symmetric system

Idea comes from Sideris, Thomases, and Wang's work in 2003.
Let $\kappa(\rho) = \sqrt{P'(\rho)}$, $\bar{\kappa} = \kappa(\bar{\rho})$ the sound speed at a background density $\bar{\rho} > 0$.
The symmetrization in the case of $\gamma = 1$ can be done similarly with a new variable $\ln \rho$.
Define$\sigma(\rho) = \nu(\kappa(\rho) - \bar{\kappa})$ with $\nu = \dfrac{2}{\gamma - 1}$, then the system is transformed into

$$
\begin{aligned}
&\partial_t \sigma + \bar{\kappa} \nabla \cdot u = -u \cdot \nabla \sigma - \frac{1}{\nu} \sigma \nabla \cdot u, \\
&\partial_t u + \bar{\kappa} \nabla \sigma + \frac{1}{\tau} u = -u \cdot \nabla u - \frac{1}{\nu} \sigma \nabla \cdot \sigma \\
&\qquad\qquad - a \int \boldsymbol{\Gamma}(x-y)(u(x) - u(y))(\frac{1}{\nu}\sigma(y) + \bar{\kappa})^{\nu} \mathrm{d}y,
\end{aligned}
$$

where the constants are $a = \gamma^{-1/(\gamma-1)} > 0$. The initial condition becomes

$$(\sigma, u)|_{t=0} = (\sigma_0(x), u_0(x)) = (\nu(\kappa(\rho_0) - \bar{\kappa}), u_0).$$

Main results

Theorem (Local-in-time existence) For $s > \frac{N}{2} + 1$, assume the initial values $(\sigma_0(x), u_0(x)) \in H^s(\mathbb{R}^N)$. Then there exist a unique classical solution (σ, u) satisfying $(\sigma, u) \in C([0, T], H^s(\mathbb{R}^N)) \cap C^1([0, T], H^{s-1}(\mathbb{R}^N))$ for some finite $T > 0$.

Theorem(Global-in-time existence) Suppose background sound speed $\bar{\kappa}$ satisfying $2a\bar{\kappa}^{\nu}\|\mathbf{\Gamma}\|_{L^1} < \frac{1}{\tau}$. If $\|\sigma_0\|_{H^s} + \|u_0\|_{H^s} \leq \delta_0$ with sufficiently small $\delta_0 > 0$, then the Cauchy problem has a unique global classical solution.

Theorem(Decay convergence in torus) Assume that $\mathbf{\Gamma} \in L^2(\mathbb{T}^N \times \mathbb{T}^N)$, $\mathbf{\Gamma}(x, y) = \mathbf{\Gamma}(y, x)$, then the following estimate holds,

$$\int_{\mathbb{T}^N} \left(\rho|u|^2 + (\rho - m_0)^2\right)\mathrm{d}x \leq C \cdot \int_{\mathbb{T}^N} \left(\rho_0 u_0^2 + (\rho_0 - m_0)^2\right)\mathrm{d}x e^{-Ct}.$$

Remark Since the matrix $\mathbf{\Gamma}(x)$ is not positive definite, the damping coefficient needs to be large enough to make the damping term restrain the self-acceleration effect caused by velocity alignment to get the global well-posedness.

THANK YOU!