Continuous Weak Solutions Of Boussinesq Equations

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Outline

- J. Nash, (1954), the main idea of C^1 isometric embeddings.
- Camillo De Lellis and László Székelyhidi, (2013), the main idea of Dissipative continuous Euler flows
- Our main results.
- Proof of the main results.

Nash's C^1 isometric embedding

In 1954, Nash introduced an iterative scheme for the proof of his famous C^1 isometric embedding.

Let $\Omega \subset R^n$ and $g(x)$ is a given $n \times n$ positive matrix.

$u : \Omega \mapsto R^{n+2}$ is a short embedding, if $(\nabla u)^T(\nabla u) < g(x)$.

Question: How to increase metric such that $g(x) - (\nabla u)^T(\nabla u)$ became more smaller and smaller, and get an isometric embedding finally?

Nash's C^1 isometric embedding

Nash's method: (Using geometric information of codimension 2).

Since $u(\Omega) \subset R^n$ is an n-dimensional submanifold with codimension 2, there exist two normal vectors $\xi(x)$, $\eta(x)$ such that $|\xi| = |\eta| = 1$, $\xi \cdot \eta = 0$ and

$$(\nabla u)^T \xi = (\nabla u)^T \eta = 0.$$

Set

$$v(x) = u(x) + \frac{a(x)}{\lambda}(sin(\lambda x \cdot \xi)\xi + cos(\lambda x \cdot \xi)\eta)$$

Nash's C^1 isometric embedding

A direct computation gives

$$\nabla v(x) = \nabla u(x) + a(x)(cos(\lambda x \cdot \xi)\xi \otimes \xi - sin(\lambda x \cdot \xi)\eta \otimes \xi) + O(\frac{1}{\lambda}),$$

$$(\nabla v(x))^T = (\nabla u(x))^T + a(x)(cos(\lambda x \cdot \xi)\xi \otimes \xi - sin(\lambda x \cdot \xi)\eta \otimes \xi) + O(\frac{1}{\lambda}),$$

where $\xi \otimes \xi = \xi\xi^T$. Hence

$$(\nabla v(x))^T \nabla v(x) = (\nabla u(x))^T \nabla u + a(x)^2 \xi \otimes \xi + O(\frac{1}{\lambda}),$$

the metric induced by v is increased (low frequency part increase metric).

Introduction

Nash's C^1 isometric embedding

Based on this computation, we may decompose the error. In fact, by some convex analysis (geometric lemma), we know that there exist ξ_k and a_k, $k = 0, 1, \cdots M - 1$, for any $x \in \Omega$,

$$g(x) - (\nabla u)^T(\nabla u)) = \sum_{k=1}^{M} a_k(x)^2 \xi_k \otimes \xi_k.$$

Thus, we can increase metric by iterations

$$u_{k+1}(x) = u_k(x) + \frac{\sqrt{1-\delta}a_k(x)}{\lambda_k}(sin(\lambda_k x \cdot \xi_k)\xi_k + cos(\lambda - kx \cdot \xi_k)\eta_k),$$

where $k = 0, 1, \cdots M - 1$.

Incompressible Euler equation

Incompressible Euler equation

$$\begin{cases} v_t + \mathrm{div}(v \otimes v) + \nabla p = 0, & \text{in} \quad R^3 \times [0,1] \\ \mathrm{div} v = 0, \end{cases} \tag{1}$$

where v is the velocity vector, p is the pressure,

• One of the famous problem is the Onsager conjecture on Euler equation as following:

1. $C^{0,\alpha}$ solutions are energy conservative when $\alpha > \frac{1}{3}$.
2. For any $\alpha < \frac{1}{3}$, there exist dissipative solutions in $C^{0,\alpha}$.

Incompressible Euler equation

For Onsager conjecture

• The part (1) was proved by Gregory L. Eyink., and P. Constantin, E Weinan and E. Titi in 1994.

• Slightly weak assumption for energy conservation proved by Constantin etc, in 2008.

• P. Isett and Sung-jin Oh (2015) proved for the Euler equations on manifolds by heat flow method.

Incompressible Euler equation

For Onsager conjecture

• P. Isett (2016) proved Onsager's conjecture, that is for any $\alpha < \frac{1}{3}$, there exist dissipative solutions.

• C. De Lellis, L. Székelyhidy, T. Buckmaster and V.Vicol (2017) give another short proof. They make use of the Mikado flow to construct weak solutions.

Incompressible Euler equation

The part (2) has been treated by many authors.

• For weak solutions, V. Scheffer in 1993, A. Shnirelman in 1997 and Camillo De Lellis, László Székelyhidi (2009).

• The construction of continuous and Hölder solution was made by Camillo De Lellis, László Székelyhidi in 2013,

• T. Buckmaster. Camillo De Lellis and László Székelyhidi (2015) developed an iterative scheme (some kind of convex integration).

• The solution is a superposition of infinitely many (perturbed) and weakly interacting Beltrami flows.

Euler equation

Incompressible Euler equation

For the Onsager critical spatial regularity (Hölder exponent $\theta = \frac{1}{3}$),

• T. Buckmaster (2015) constructed Hölder continuous (with exponent $\theta < \frac{1}{5} - \varepsilon$ in time-space) periodic solutions which for almost every time belongs to C_x^θ, for any $\theta < \frac{1}{3}$.

• T. Buckmaster, Camillo De Lellis and László Székelyhidi (2015) constructed Hölder continuous periodic solution which belongs to $L_t^1 C_x^\theta$, for any $\theta < \frac{1}{3}$, and has compact support in time.

Incompressible Navier-Stokes equation

• Buckmaster and Vicol (2017) established the non-uniqueness of weak solution to the 3D incompressible Navier-Stokes.

• T.Luo and E.S.Titi (2018) construct weak solution with compact support in time for hyperviscous Navier-Stokes equation.

• X. Luo (2018) proved the non-uniqueness of weak solution for high dimension ($d \geq 4$) stationary Navier-Stokes equation.

Camillo De Lellis and László Székelyhidi's main idea

Incompressible Euler equation

Let $\mathcal{S}_0^{3\times 3}$ denotes the vector space of symmetric trace-free 3×3 matrices.
Assume $v, p, \mathring{R}, f$ are smooth functions on $T^3 \times [0, 1]$ taking values, respectively, in $R^3, R, \mathcal{S}_0^{3\times 3}, R^3$.
They solve the Euler-Reynolds system if

$$\begin{cases} \partial_t v + \mathrm{div}(v \otimes v) + \nabla p = \mathrm{div} \mathring{R} \\ \mathrm{div} v = 0. \end{cases} \tag{2}$$

Beltrami flows are the stationary solutions to the 3D Euler equations satisfying *curl* $\quad v = \mu v$, for some constant μ.

Camillo De Lellis and László Székelyhidi's main idea

Incompressible Euler equation

Let $\lambda_0 > 1$ and let $A_k \in R^3$ be such that

$$A_k \cdot k = 0, \quad |A_k| = \frac{1}{\sqrt{2}}, \quad A_{-k} = A_k$$

for some $k \in Z^3$ with $|k| = \lambda_0$. Put

$$B_k = A_k + i\frac{k}{|k|} \times A_k \in C^3.$$

For $a_k \in C$ with $\overline{a}_k = a_{-k}$, $v(\xi) = \sum_{|k|=\lambda_0} a_k B_k e^{i\lambda k\cdot\xi}$ solves

$$\begin{cases} \mathrm{div}(v \otimes v) - \nabla(\frac{|v|^2}{2}) = 0 \\ \mathrm{div}\, v = 0. \end{cases} \tag{3}$$

Incompressible Euler equation

Moreover,

$$v\otimes v=\sum_{k,j}a_ka_jB_k\otimes B_je^{i\lambda(k+j)\cdot\xi}=\sum_{k,j}a_k\overline{a}_jB_k\otimes\overline{B_j}e^{i\lambda(k-j)\cdot\xi}.$$

$$v\otimes v=\sum_k|a_k|^2B_k\otimes\overline{B_k}+\lambda \text{ oscillate terms.}$$

Notice that

$$B_k\otimes\overline{B_k}=2(A_k\otimes A_k+(\frac{k}{|k|}\times A_k)\otimes(\frac{k}{|k|}\times A_k)).$$

And the triple $\sqrt{2}A_k$, $\sqrt{2}\frac{k}{|k|}\times A_k$, $\frac{k}{|k|}$ forms an orthonormal basis of R^3. Thus

$$B_k\otimes\overline{B_k}=Id-\frac{k}{|k|}\otimes\frac{k}{|k|}.$$

Geometric Lemma

Camillo De Lellis and László Székelyhidi's Geometric Lemma

For every $N \in \mathcal{N}$, we can choose $r_0 > 0$ and $\bar{\lambda} > 1$ such that the following property holds:
There exist disjoint subsets

$$\Lambda_j \subseteq \{k \in Z^3 : |k| = \bar{\lambda}\}, \quad j \in \{1, \cdots, N\},$$

smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(B_{r_0}(Id)), j \in \{1, \cdots, N\}, \;\; k \in \Lambda_j,$$

where $B_{r_0}(Id)$ is a small neighborhood of the identity matrix.

Camillo De Lellis and László Székelyhidi's Geometric Lemma

And there exist vectors

$$A_k^j \in R^3, \ \ |A_k^j| = \frac{1}{\sqrt{2}}, \ \ k \cdot A_k^j = 0, \ \ j \in \{1, \cdots, N\}, \ \ k \in \Lambda_j$$

such that

1. $k \in \Lambda_j$ implies $-k \in \Lambda_j$ and $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;
2. for every $R \in B_{r_0}(Id)$, the following identity holds:

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} (\gamma_k^{(j)}(R))^2 \Big(Id - \frac{k}{|k|} \otimes \frac{k}{|k|} \Big).$$

Boussinesq equations

We consider the following Boussinesq equations

$$\begin{cases} v_t + \mathrm{div}(v \otimes v) + \nabla p = \theta e_3, & \text{in} \quad T^3 \times [0,1] \\ \mathrm{div} v = 0, \\ \theta_t + \mathrm{div}(v\theta) = h, & \text{in} \quad T^3 \times [0,1], \end{cases} \tag{4}$$

where $T^3 = S^1 \times S^1 \times S^1$ and $e_3 = (0,0,1)^T$. And v is the velocity vector, p is the pressure, θ is a scalar function denoting the temperature and h is the heat sources.

Boussinesq equations

We also consider the following Boussinesq equations

$$\begin{cases} v_t + \mathrm{div}(v \otimes v) + \nabla p = \theta e_2, & \text{in} \quad T^2 \times [0,1] \\ \mathrm{div} v = 0, \\ \theta_t + \mathrm{div}(v\theta) - \Delta\theta = 0, & \text{in} \quad T^2 \times [0,1], \end{cases} \tag{5}$$

where $T^2 = S^1 \times S^1$ and $e_2 = (0,1)^T$. And v is the velocity vector, p is the pressure, θ is a scalar function denoting the temperature and h is the heat sources.

Boussinesq equations

Boussinesq equations

• The Boussingesq equations was introduced in understanding the coupling nature of the thermodynamics and the fluid dynamics.

• The Boussinesq equations model many geophysical flows, such as atmospheric fronts and ocean circulations.

Boussinesq equations

The pair (v, θ) on $T^3 \times [0,1]$ is called a weak solution of (1) if they solve (1) in the following sense:

$$\int_0^1 \int_{T^3} (\partial_t \varphi \cdot v + \nabla \varphi : v \otimes v + p \mathrm{div} \varphi - \theta e_3 \cdot \varphi) dxdt = 0,$$

for all $\varphi \in C_c^\infty(T^3 \times (0,1); R^3)$,

$$\int_0^1 \int_{T^3} (\partial_t \phi \theta + v \cdot \nabla \phi \theta + h\phi) dxdt = 0,$$

for all $\phi \in C_c^\infty(T^3 \times (0,1); R)$ and

$$\int_0^1 \int_{T^3} v \cdot \nabla \psi dxdt = 0.$$

for all $\psi \in C_c^\infty(T^3 \times (0,1); R)$.

Motivation and difficulty

Motivated by Onsager's conjecture of Euler equation and the above earlier works, we consider the Boussinesq equations and want to know if the similar phenomena can also happen when considering the temperature effects.

• The difference is that there are conversions between internal energy and mechanical energy.

• The difficulty of interactions between velocity and temperature.

Our main results

Some notations

$$\Theta := \{\theta(x) \in C^\infty(T^3) : \theta \text{ only dependent on } x_3, \quad i.e. \quad \theta(x) = \overline{\theta}(x_3)\},$$

and

$$\Xi := \{a(t)b(x_3) : a \in C^\infty([0,1]) \text{ and } b \in C^\infty([0,2\pi])\}.$$

Main results

Theorem 1 (Tao tao, ∼)

Assume that $e(t) : [0,1] \to R$ is a given positive smooth function and $\theta_0 \in \Theta$. Then there exist

$$(v, p, \theta) \in C(T^3 \times [0,1]; R^3 \times R \times R)$$

and a positive number $M = M(e)$ such that they solve the system (4) with $h = 0$ in the sense of distribution and

$$e(t) = \int_{T^3} |v|^2(x,t)dx, \qquad \|\theta - \theta_0\|_0 < 4M,$$

where $\|\theta\|_0 = \sup_{x,t}|\theta(x,t)|$.

Our main results

Remark

In our Theorem 1, if $\theta = 0$, then it's the continuous Euler flow with prescribed kinetic energy and have been constructed by Camillo De Lellis and László Székelyhidi.
In general, for example, if we take $\theta_0 = 10Mcosx_3 \in \Theta$, then we must have $\theta \neq 0$.

Consider the Boussinesq equations (5).

Theorem 2 (Luo, Tianwen, Tao tao, ∼)(2018)

Assume that $e(t) : [0,1] \to [1,+\infty)$. Then there exist

$$v \in C([0,1]L^2(T^2)), \quad \theta \in \cap_{p>2} C([0,1]L^p(T^2)) \cap L^2([0,1]H^1(T^2)).$$

such that they solve the system (5) in the sense of distribution and

$$e(t) = \int_{T^3} |v|^2(x,t)dx.$$

and

$$||\theta(t,\cdot)||_{L^2}^2 + 2\int_0^t ||\nabla\theta(s,\cdot)||_{L^2}^2 ds = ||\theta(0,\cdot)||_{L^2}^2$$

Our main results

Consider the effect of temperature on the velocity field.

Theorem 3 (Tao tao, ∼)

For a given positive constant M and any positive number λ, there exist

$$(v, p, \theta) \in C(T^3 \times [0,1]; R^3 \times R \times R)$$

such that they solve the system (4) in the sense of distribution and

$$\|v(x,0)\|_0 \leq 4M, \qquad \int_0^1 \int_{T^3} |\theta|^2(x,t)dxdt \geq \lambda^2,$$

$$\sup_{x\in T^3}|v(x,t)| \geq \lambda, \qquad \inf_{x\in T^3}|v(x,t)| \leq 4M, \qquad \forall t \in [\frac{1}{2}, 1].$$

Our main results

Consider the effect of temperature on the velocity field.

Theorem 4 (Tao tao, $\sim$)

Assume that $e(t) : [0, T] \to R$ is a given positive smooth function. For any positive number $\alpha < \frac{1}{5}$, there exist

$$(v, \theta) \in C^{\alpha}(T^3 \times [0, 1]; R^3 \times R)$$

such that they solve the system (4) in the sense of distribution and

$$e(t) = \int_{T^3} |v|^2(x, t)dx.$$

Remark

For the Boussinesq system on T^3, even the initial velocity is small, the oscillation of velocity after sometime could be as large as possible if we have enough thermos in the systems.

Remark

The above theorems also hold for the two-dimensional Boussinesq system on T^2.

Proof of main result

Theorems will be proved through an iteration procedure. $\mathcal{S}_0^{3\times 3}$ denotes the vector space of symmetric trace-free 3×3 matrices.

Definition

Assume $v, p, \theta, \mathring{R}, f$ are smooth functions on $T^3 \times [0,1]$ taking values, respectively, in $R^3, R, R, \mathcal{S}_0^{3\times 3}, R^3$. We say that they solve the Boussinesq-Reynolds system (with or without heat source) if

$$\begin{cases} \partial_t v + \mathrm{div}(v \otimes v) + \nabla p = \theta e_3 + \mathrm{div}\mathring{R} \\ \mathrm{div} v = 0 \\ \theta_t + \mathrm{div}(v\theta) = h + \mathrm{div} f. \end{cases} \tag{6}$$

Proof of the main result

The main proposition

Proposition 1

Let $e(t)$ be as in Theorem 1 and Theorem 2. Then we can choose two positive constants η and M only dependent of $e(t)$, such that the following properties hold:
For any $0 < \delta \leq 1$, if $(v, p, \theta, \mathring{R}, f) \in C^\infty([0,1] \times T^3)$ solve Boussinesq-Reynolds system (6) and

$$\frac{3\delta}{4} e(t) \leq e(t) - \int_{T^3} |v|^2(x,t)dx \leq \frac{5\delta}{4} e(t), \quad \forall t \in [0,1], \tag{7}$$

$$\sup_{x,t} |\mathring{R}(x,t)| \leq \eta\delta, \tag{8}$$

$$\sup_{x,t} |f(x,t)| \leq \eta\delta, \tag{9}$$

Proof of the main result

then we can construct new functions $(\tilde{v},\ \tilde{p},\ \tilde{\theta},\ \tilde{\mathring{R}},\ \tilde{f}) \in C^\infty([0,1]\times T^3)$, they also solve Boussinesq-Reynolds system (6) and satisfy

$$\frac{3\delta}{8}e(t) \le e(t) - \int_{T^3} |\tilde{v}|^2(x,t)dx \le \frac{5\delta}{8}e(t), \quad \forall t \in [0,1], \tag{10}$$

$$\sup_{x,t} |\tilde{\mathring{R}}(x,t)| \le \frac{\eta\delta}{2}, \tag{11}$$

$$\sup_{x,t} |\tilde{f}(x,t)| \le \frac{\eta\delta}{2}, \tag{12}$$

$$\sup_{x,t} |\tilde{v}(x,t) - v(x,t)| \le M\sqrt{\delta}, \tag{13}$$

$$\sup_{x,t} |\tilde{\theta}(x,t) - \theta(x,t)| \le M\sqrt{\delta}, \tag{14}$$

$$\sup_{x,t} |\tilde{p}(x,t) - p(x,t)| \le M\sqrt{\delta}. \tag{15}$$

Proof of the main result

Proposition 2

There exist two absolute constants M and η such that
For any $0 < \delta \leq 1$, if $(v, p, \theta, \mathring{R}, f) \in C^\infty([0,1] \times T^3)$ solve Boussinesq-Reynolds system (6) and

$$\sup_{x,t} |\mathring{R}(x,t)| \leq \eta\delta, \tag{16}$$

$$\sup_{x,t} |f(x,t)| \leq \eta\delta, \tag{17}$$

then we can construct new functions $(\tilde{v}, \tilde{p}, \tilde{\theta}, \tilde{\mathring{R}}, \tilde{f}) \in C^\infty([0,1] \times T^3)$, they also solve (6) and satisfy

$$\sup_{x,t} |\tilde{\mathring{R}}(x,t)| \leq \frac{\eta}{2}\delta, \tag{18}$$

$$\sup_{x,t} |\tilde{f}(x,t)| \leq \frac{\eta}{2}\delta, \tag{19}$$

$$\sup_{x,t} |\tilde{v}(x,t) - v(x,t)| \leq M\sqrt{\delta}, \tag{20}$$

$$\sup_{x,t} |\tilde{\theta}(x,t) - \theta(x,t)| \leq M\sqrt{\delta}, \tag{21}$$

$$\sup_{x,t} |\tilde{p}(x,t) - p(x,t)| \leq M\sqrt{\delta}. \tag{22}$$

Proof of the main result

Proof of Theorem 1. In this case, $h = 0$. We first set

$$v_0 := 0,\ \theta_0 \in \Theta,\ p_0 := \int_0^{x_3} \theta_0(y)dy,\ \mathring{R}_0 := 0,\ f_0 := 0$$

and $\delta = 1$. Obviously, they solve Boussinesq-Reynolds system (6) and satisfy the following estimates

$$\frac{3\delta}{4}e(t) \leq e(t) - \int_{T^3} |v_0|^2(x,t)dx \leq \frac{5\delta}{4}e(t), \qquad \forall t \in [0,1]$$

$$\sup_{x,t} |\mathring{R}_0(x,t)| = 0 \quad (\leq \eta\delta),$$

$$\sup_{x,t} |f_0(x,t)| = 0 \quad (\leq \eta\delta).$$

Proof of the main result

By using Proposition 1, we can construct a sequence $(v_n,\ p_n,\ \theta_n,\ \mathring{R}_n,\ f_n)$, which solve (6) and satisfy

$$\frac{3}{4}\frac{e(t)}{2^n} \leq e(t) - \int_{T^3} |v_n|^2(x,t)dx \leq \frac{5}{4}\frac{e(t)}{2^n}, \quad \forall t \in [0,1] \tag{23}$$

$$\sup_{x,t} |\mathring{R}_n(x,t)| \leq \frac{\eta}{2^n}, \tag{24}$$

$$\sup_{x,t} |f_n(x,t)| \leq \frac{\eta}{2^n}, \tag{25}$$

$$\sup_{x,t} |v_{n+1}(x,t) - v_n(x,t)| \leq M\sqrt{\frac{1}{2^n}}, \tag{26}$$

$$\sup_{x,t} |\theta_{n+1}(x,t) - \theta_n(x,t)| \leq M\sqrt{\frac{1}{2^n}}, \tag{27}$$

$$\sup_{x,t} |p_{n+1}(x,t) - p_n(x,t)| \leq M\sqrt{\frac{1}{2^n}}. \tag{28}$$

Proof of the main result

Therefore from (24)-(28), we know that $(v_n,\ p_n,\ \theta_n,\ \mathring{R}_n,\ f_n)$ are Cauchy sequence in $C(T^3 \times [0,1])$, therefore there exist

$$(v, p, \theta) \in C(T^3 \times [0,1])$$

such that

$$v_n \to v, \qquad p_n \to p, \qquad \theta_n \to \theta, \qquad \mathring{R}_n \to 0, \qquad f_n \to 0.$$

in $C(T^3 \times [0,1])$ as $n \to \infty$. Moreover, by (23) and (27),

$$e(t) = \int_{T^3} |v|^2(x,t)dx \qquad \forall t \in [0,1].$$

$$\|\theta - \theta_0\|_0 \le M \sum_{n=0}^{\infty} \sqrt{\frac{1}{2^n}} < 4M.$$

Passing into the limit in (6), we conclude that $v,\ p,\ \theta$ solve (5) in the sense of distribution.

Proof of the main result

Proof of Theorem 3. We set

$$v_0 = \begin{pmatrix} tN sin(N^2 x_2) \\ 0 \\ 0 \end{pmatrix}, \quad \mathring{R}_0 = \begin{pmatrix} 0 & -\frac{cos(N^2 x_2)}{N} & 0 \\ -\frac{cos(N^2 x_2)}{N} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{29}$$

$$f_0 = \begin{pmatrix} 0 \\ 0 \\ \frac{cos(N^2 x_3)}{N} \end{pmatrix},$$

$$p_0 = -(1-t)\frac{cos(N^2 x_3)}{N}, \qquad \theta_0 = (1-t)N sin(N^2 x_3),$$

and $\delta = 1$.

Proof of the main result

Then they solve (6). If we take $N \geq \frac{2}{\eta}$, then they satisfy the following estimates

$$\sup_{x,t} |\mathring{R}_0(x,t)| \leq \eta\delta,$$

$$\sup_{x,t} |f_0(x,t)| \leq \eta\delta.$$

By Proposition 2, we can construct $(v_n,\ p_n,\ \theta_n,\ \mathring{R}_n,\ f_n)$ satisfying

$$\sup_{x,t} |\mathring{R}_n(x,t)| \leq \frac{\eta}{2^n}, \tag{30}$$

$$\sup_{x,t} |f_n(x,t)| \leq \frac{\eta}{2^n}, \tag{31}$$

$$\sup_{x,t} |v_{n+1}(x,t) - v_n(x,t)| \leq M\sqrt{\frac{1}{2^n}}, \tag{32}$$

$$\sup_{x,t} |\theta_{n+1}(x,t) - \theta_n(x,t)| \leq M\sqrt{\frac{1}{2^n}}, \tag{33}$$

Proof of the main result

Then we know that $(v_n,\ p_n,\ \theta_n,\ \mathring{R}_n,\ f_n)$ are Cauchy sequence in $C(T^3 \times [0,1])$, there exist

$$(v, p, \theta) \in C(T^3 \times [0,1])$$

such that

$$v_n \to v, \qquad p_n \to p, \qquad \theta_n \to \theta, \qquad \mathring{R}_n \to 0, \qquad f_n \to 0,$$

in $C(T^3 \times [0,1])$ as $n \to \infty$.
By (32) and (33), we have

$$\|v - v_0\|_0 \leq M \sum_{n=0}^{\infty} \sqrt{\frac{1}{2^n}} < 4M,$$

and

$$\|\theta - \theta_0\|_0 \leq M \sum_{n=0}^{\infty} \sqrt{\frac{1}{2^n}} < 4M.$$

Finally, let λ be as in Theorem 3 and take $N = \max\{\frac{2}{\eta}, 4\lambda, 16M\}$, then for $t \in [\frac{1}{2}, 1]$

$$\sup_{x \in T^3} |v(x,t)| \geq \sup_{x \in T^3} |v_0(x,t)| - 4M \geq \frac{N}{4} \geq \lambda,$$

$$\inf_{x \in T^3} |v(x,t)| \leq \inf_{x \in T^3} |v_0(x,t)| + 4M \leq 4M.$$

Moreover, since $v_0(x, 0) = 0$, we have

$$\|v(x, 0)\|_0 \leq 4M.$$

A direct calculation gives,

$$\int_0^1 \int_{T^3} |\theta_0|^2(x,t)dxdt = \frac{4\pi^3}{3}N^2,$$

therefore

$$\begin{aligned}
&\int_0^1 \int_{T^3} |\theta|^2(x,t)dxdt \\
\geq &\frac{1}{2}\int_0^1 \int_{T^3} |\theta_0|^2(x,t)dxdt - \int_0^1 \int_{T^3} |\theta - \theta_0|^2(x,t)dxdt \\
\geq &\frac{2\pi^3}{3}N^2 - (2\pi)^3(4M)^2 \\
\geq &\lambda^2.
\end{aligned}$$

Passing into the limit in (6) we conclude that $v,\ p,\ \theta$ solve (5) in the sense of distribution.

Outline of the proof of propositions

The constructions of the functions $\tilde{v}, \tilde{\theta}$ consist of several steps.

• Adding perturbations to v_0, θ_0 and functions v_{01}, θ_{01} as

$$\begin{aligned} v_{01} =& v_0 + w_{1o} + w_{1oc} := v_0 + w_1, \\ \theta_{01} =& \theta_0 + \chi_1. \end{aligned}$$

where w_{1o}, w_{1oc}, χ_1 are highly oscillated functions. Let parameters $\mu_1,\ \lambda_1$ in the construction satisfy $\mu_1,\ \lambda_1,\ \frac{\lambda_1}{\mu_1} \in \mathrm{N}$.

• Finding functions R_{01}, p_{01} and f_{01} which satisfies the desired estimate and solves the system (6).

Outline of the proof of propositions

The stress becomes smaller in the sense, if

$$\rho(t)Id - \mathring{R}_0 = \sum_{i=1}^{L} a_i^2 \left(Id - \frac{k_i}{|k_i|} \otimes \frac{k_i}{|k_i|} \right),$$

$$f_0 = \sum_{i=1}^{3} b_i A_{k_i},$$

then

$$R_{01} = \sum_{i=2}^{L} a_i^2 \left(Id - \frac{k_i}{|k_i|} \otimes \frac{k_i}{|k_i|} \right) + \delta R_{01},$$

$$f_{01} = \sum_{i=2}^{3} b_i A_{k_i} + \delta f_{01}.$$

where $\delta R_{01}, \delta f_0$ can be small by the appropriate choice on μ_1 and λ_1. We can obtain the needed functions $(\tilde{v},\ \tilde{p},\ \tilde{\theta},\ \tilde{\mathring{R}},\ \tilde{f})$.

Geometric Lemma

For every $N \in \mathcal{N}$, we can choose $r_0 > 0$ and $\bar{\lambda} > 1$ such that the following property holds:
There exist disjoint subsets

$$\Lambda_j \subseteq \{k \in Z^3 : |k| = \bar{\lambda}\}, \quad j \in \{1, \cdots, N\},$$

smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(B_{r_0}(Id)), j \in \{1, \cdots, N\}, \ \ k \in \Lambda_j,$$

vectors

$$A_k^j \in R^3, \ \ |A_k^j| = \frac{1}{\sqrt{2}}, \ \ k \cdot A_k^j = 0, \ \ j \in \{1, \cdots, N\}, \ \ k \in \Lambda_j$$

Geometric Lemma

and smooth functions

$$g_k^{(j)} \in C^\infty(R^3), j \in \{1, \cdots, N\}, \quad k \in \Lambda_j,$$

such that

1. $k \in \Lambda_j$ implies $-k \in \Lambda_j$ and $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;
2. for every $R \in B_{r_0}(Id)$, the following identity holds:

$$R = \frac{1}{2} \sum_{k \in \Lambda_j} (\gamma_k^{(j)}(R))^2 \Big(Id - \frac{k}{|k|} \otimes \frac{k}{|k|} \Big);$$

3. for every $f \in C^\infty(R^3)$, we have the identity

$$f = \sum_{k \in \Lambda_j} g_k^{(j)}(f) A_k^j.$$

The main construction

By Geometric Lemma, there exist $\bar{\lambda} > 1$, $r_0 > 0$, subset $\Lambda = \{\pm k_1, ..., \pm k_L\}$ and vectors $\{A_{\pm k_j}, j = 1, \cdots, L\}$ together with corresponding functions

$$\gamma_{k_i} \in C^{(\infty)}(B_{r_0}(Id)), \qquad g_{k_i} \in C^{(\infty)}(R^3), \qquad i = 1, \cdots, L.$$

where L is a fixed integer. Thus the result can be restated as following: For any $R \in B_{r_0}(Id)$, we have the identity

$$R = \sum_{i=1}^{L} \gamma_{k_i}^2(R)\Big(Id - \frac{k_i}{|k_i|} \otimes \frac{k_i}{|k_i|}\Big). \tag{35}$$

and for any $f \in C^{(\infty)}(R^3)$, we have

$$f = \sum_{i=1}^{L} g_{k_i}(f) A_{k_i}.$$

The main construction

The proof of the our theorem relies on the following two propositions. We set

$$\bar{\rho}(t) := \frac{1}{(2\pi)^3}\Big(e(t)\Big(1-\frac{\delta}{2}\Big) - \int_{T^3} |v_0|^2(x,t)dx\Big), \tag{36}$$

and

$$R_0(x,t) := \bar{\rho}(t)Id - \mathring{R}_0(x,t). \tag{37}$$

then for any $l \in Z^3$, we denote b_{1l} by

$$b_{1l}(x,t) := \sqrt{\bar{\rho}(t)}\alpha_l(\mu_1 v_0)\gamma_{k_1}\Big(\frac{R_0(x,t)}{\bar{\rho}(t)}\Big), \tag{38}$$

and

$$B_{k_1} := A_{k_1} + i\frac{k_1}{|k_1|} \times A_{k_1}. \tag{39}$$

The main construction

Then we let l-perturbation

$$w_{1ol} := b_{1l}(x,t)\Big(B_{k_1} e^{i\lambda_1 2^{|l|} k_1\cdot(x-\frac{l}{\mu_1}t)} + B_{-k_1} e^{-i\lambda_1 2^{|l|} k_1\cdot(x-\frac{l}{\mu_1}t)}\Big). \qquad (40)$$

where we set $A_{-k_1} = A_{k_1}$.
Finally, we let 1-th perturbation

$$w_{1o} := \sum_{l\in Z^3} w_{1ol}. \qquad (41)$$

Obviously, w_{1ol}, w_{1o} are all real 3-dimensional vector functions. We have supp$\alpha_l \cap$ supp$\alpha_{l'} = \emptyset$ if $|l - l'| \geq 2$.

The main construction

We denote the l-correction

$$w_{1ocl} := \frac{1}{\lambda_1\lambda_0}\Big(\frac{\nabla b_{1l}(x,t)\times B_{k_1}}{2^{|l|}}e^{i\lambda_1 2^{|l|}k_1\cdot(x-\frac{l}{\mu_1}t)} \tag{42}$$

$$+\frac{\nabla b_{1l}(x,t)\times B_{-k_1}}{2^{|l|}}e^{-i\lambda_1 2^{|l|}k_1\cdot(x-\frac{l}{\mu_1}t)}\Big), \tag{43}$$

then denote 1-th correction

$$w_{1oc} := \sum_{l\in Z^3} w_{1ocl}. \tag{44}$$

Finally, we denote 1-th perturbation

$$w_1 := w_{1o} + w_{1oc}. \tag{45}$$

Thus, if we denote w_{1l} by

$$w_{1l} := w_{1ol} + w_{1ocl}$$

$$= \frac{1}{\lambda_1 \lambda_0} \mathrm{curl}\Big(\frac{b_{1l}(x,t) B_{k_1}}{2^{|l|}} e^{i\lambda_1 2^{|l|} k_1 \cdot (x - \frac{l}{\mu_1} t)} \tag{46}$$

$$+ \frac{b_{1l}(x,t) B_{-k_1}}{2^{|l|}} e^{-i\lambda_1 2^{|l|} k_1 \cdot (x - \frac{l}{\mu_1} t)} \Big), \tag{47}$$

then

$$w_1 = \sum_{l \in Z^3} w_{1l}, \qquad \mathrm{div}\, w_{1l} = 0,$$

and

$$\mathrm{div} w_1 = 0.$$

Moreover, if we set

$$B_{1lk_1} := b_{1l}(x,t)B_{k_1} + \frac{1}{\lambda_1\lambda_0}\frac{\nabla b_{1l}(x,t)\times B_{k_1}}{2^{|l|}},$$
$$B_{-1lk_1} := b_{1l}(x,t)B_{-k_1} + \frac{1}{\lambda_1\lambda_0}\frac{\nabla b_{1l}(x,t)\times B_{-k_1}}{2^{|l|}},$$

then

$$w_{1l} = B_{1lk_1} e^{i\lambda_1 2^{|l|}k_1\cdot(x-\frac{l}{\mu_1}t)} + B_{-1lk_1} e^{-i\lambda_1 2^{|l|}k_1\cdot(x-\frac{l}{\mu_1}t)}.$$

Thus we complete the construction of perturbation w_1.

The main construction

To construct χ_1, we first denote β_{1l} by

$$\beta_{1l}(x,t) := \frac{\alpha_l(\mu_1 v_0)}{2\sqrt{\rho(t)}} \frac{g_{k_1}(-f_0(x,t))}{\gamma_{k_1}\left(\frac{R_0(x,t)}{\rho(t)}\right)}, \tag{48}$$

then denote the l-perturbation

$$\chi_{1l}(x,t) := \beta_{1l}(x,t)\Big(e^{i\lambda_1 2^{|l|} k_1\cdot(x-\frac{l}{\mu_1}t)} + e^{-i\lambda_1 2^{|l|} k_1\cdot(x-\frac{l}{\mu_1}t)}\Big). \tag{49}$$

We set the perturbation

$$\chi_1(x,t) := \sum_{l\in Z^3} \chi_{1l}. \tag{50}$$

Both χ_{1l} and χ_1 are real scalar functions.
Finally, by some estimates, we prove that functions R_{01}, p_{01} and f_{01} satisfy the desired estimate and solve the system (6).

Thank You !