Averaging for Vlasov and Vlasov-Poisson equations

Florian Méhats

University of Rennes I, INRIA team MINGuS

Joint work with P. Chartier, N. Crouseilles and M. Lemou

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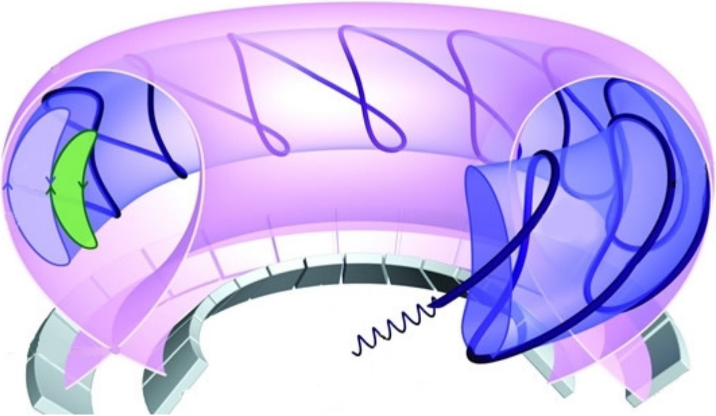
Motivation: simulation of magnetic confinement of plasmas in Tokamaks

In such devices, a **very large external magnetic field** B confines the **plasma** (e.g. gas at very high temperature made of neutral and charged particles) in a torus and induces a **cyclotronic motion**.



Model equations: Vlasov-Poisson (VP)

Vlasov equation describes the time evolution of the distribution function of the plasma:

$$\partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon + (E^\varepsilon + \tfrac{1}{\varepsilon} v \times B) \cdot \nabla_v f^\varepsilon = 0, \quad f^\varepsilon(0,x,v) = f_0^\varepsilon(x,v)$$

$f^\varepsilon(t,x,v)$: distribution of charge at time $t \in \mathbb{R}$, position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$.

$\frac{1}{\varepsilon} B$: external magnetic field assumed to be given and large here.

E^ε: self-consistent electric field given by **Poisson equation** (in resolved form here):

$$E^\varepsilon(t,x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x-x'}{|x-x'|^3} \rho^\varepsilon(t,x') dx', \quad \rho^\varepsilon(t,x) = \int_{\mathbb{R}^3} f^\varepsilon(t,x,v) dv.$$

Intrinsic difficulties

Main numerical and computational challenges come from

- the problem dimension (7D = 3D position + 3D velocity +time);
- the necessity to preserve energy and mass;
- the occurrence of various time scales when $\frac{1}{\varepsilon}B$ is large.

Gyrokinetics theory provides a model for plasmas with large $\frac{1}{\varepsilon}B$.

The trajectory of particles is a helix composed of

- a slow motion along the field line;
- a fast circular motion around the field line, called gyromotion.

For most plasma behavior, gyromotion is irrelevant. **Gyrokinetics** (*Littlejohn 83'* , see also *Brizard* Lecture Notes 13') reduces the equations to $4D + t$

Asymptotic models have been derived in the literature by various authors, in various situations:

1. **for Vlasov-Poisson in $2D$ with constant magnetic field:** Bostan, Frénod, Golse, Miot, Saint-Raymond
2. **for Vlasov in $3D$:** Bostan, Degond, Filbet, Possaner
3. **for Vlasov-Poisson with negligible curvature magnetic field lines:** Bostan
4. **for Vlasov-Poisson with constant intensity magnetic field:** Golse

Main goals

Our ambition is

1. to derive **asymptotic equations** in the regime where ε tends to zero. This work is thus an attempt to (re-)derive rigorously the equations of gyrokinetics;
2. to design **asymptotically preserving** or even **uniformly accurate methods** for solving fast-oscillating kinetic equations, i.e. methods whose cost and accuracy do not depend on ε:

$$error \leq C \, (computational\ cost)^{-p} \, .$$

The main tools used to reach this objective are averaging and PDE techniques. Here, I will focus on the first.

Outline of the talk

The Vlasov equation (VE) with an external electric field

Assuming for the time-being that $E^\varepsilon \equiv E^\varepsilon(t,x) = -\nabla\phi^\varepsilon(t,x)$ is given, we have

$$\boxed{\partial_t f^\varepsilon + v\cdot\nabla_x f^\varepsilon + (E^\varepsilon + \tfrac{v}{\varepsilon}\times B)\cdot\nabla_v f^\varepsilon = \partial_t f^\varepsilon + F^\varepsilon\cdot\nabla_y f^\varepsilon = 0}$$

where $y=(x,v)^T$. Its solution is of the form

$$f^\varepsilon(t,y) = f_0^\varepsilon(\varphi_{-t}^\varepsilon(y))$$

where $t\mapsto\varphi_t^\varepsilon(y)$ is the flow of **the characteristics**

$$\begin{pmatrix}\dot x\\ \dot v\end{pmatrix} = \begin{pmatrix} v\\ \frac{1}{\varepsilon}v\times B + E^\varepsilon\end{pmatrix} = \begin{pmatrix}0 & I_3\\ -I_3 & \frac{1}{\varepsilon}\mathcal{J}_B\end{pmatrix}\nabla\left(\frac{1}{2}|v|^2+\phi^\varepsilon\right) = \Omega\nabla H$$

with

$$\mathcal{J}_B(x) = \begin{pmatrix} 0 & B_3(x) & -B_2(x)\\ -B_3(x) & 0 & B_1(x)\\ B_2(x) & -B_1(x) & 0\end{pmatrix}$$

Averaging for the characteristics

Theorem (e.g. C., Murua and Sanz-Serna, FOCM 15')

Consider a vector field

$$F^\varepsilon = \frac{1}{\varepsilon} G + K$$

*such that the flow associated with G is 2π-**periodic**, regardless of the specific trajectory. There exist two formal vector fields G^ε and K^ε such that*

1. *both vector fields commute, i.e. $[G^\varepsilon, K^\varepsilon] = 0$;*
2. *vector field G^ε generates a 2π-periodic flow;*
3. $F^\varepsilon = \frac{1}{\varepsilon} G^\varepsilon + K^\varepsilon$

Note that:

- the flows associated with G^ε and K^ε **commute**
- if G and K share the same structure, so do G^ε and K^ε.

Main statement for Vlasov equation

Corollary

Consider a vector field F^ε *as in previous theorem and its associated "averaged" form* $\frac{1}{\varepsilon}G^\varepsilon + K^\varepsilon$*. Then the solution of the transport equation*

$$\partial_t f(t,y) + F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0$$

may be obtained as the **diagonal value (i.e. for** $\tau = t/\varepsilon$**)** *of the two-scale function* $\tilde{f}(t,\tau,y)$*, periodic in* τ *and satisfying* $\tilde{f}(0,0,y) = f_0(y)$ *and the two equations*

$$(i) \quad \forall (t,\tau,y), \quad \partial_\tau \tilde{f}(t,\tau,y) + G^\varepsilon(y) \cdot \nabla_y \tilde{f}(t,\tau,y) = 0,$$
$$(ii) \quad \forall (t,\tau,y), \quad \partial_t \tilde{f}(t,\tau,y) + K^\varepsilon(y) \cdot \nabla_y \tilde{f}(t,\tau,y) = 0.$$

Equation (ii) *is the so-called averaged equation. It is formal.*

See Chartier, Crouseilles, Lemou, M., Zhao, for an application of this result for $|B(x)| \equiv B_0$ and uniformly accurate approximations.

Change of variables in non-canonical Hamiltonian systems

Some basic properties of non-canonical Hamiltonian systems

- if $\Omega(y)$ is **skew-symmetric** and satisfies **Jacobi identity**, then

$$\{H_1, H_2\}_\Omega = (\nabla_y H_1)^T \Omega(y) \nabla_y H_2$$

  generalizes canonical brackets to non-canonical brackets. If $B \equiv \nabla \cdot A$ then Ω in the characteristics satisfies Jacobi.

- from a non-canonical Hamiltonian equation

$$\dot{y} = \Omega(y) \nabla_y H(y),$$

  $y = \varphi(Y)$ gives another non-canonical Hamiltonian equation

$$\dot{Y} = \tilde{\Omega} \nabla_Y \tilde{H}, \; \tilde{\Omega} = (\varphi')^{-1} \left(\Omega \circ \varphi\right) (\varphi')^{-T}, \; \tilde{H} = H \circ \varphi.$$

See Hairer, Lubich, Wanner, *Geometric numerical integration*

What about structures? (the case of constant B)

If B is constant, one may consider the equations in 2 dimensions

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & \frac{1}{\varepsilon} J_2 \end{pmatrix} \begin{pmatrix} \nabla \phi^\varepsilon(t,x) \\ v \end{pmatrix}, \quad J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Though the stiff part of the system generates a $2\pi\varepsilon$-periodic flow, it amounts to a splitting of the **structure matrix**

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{\varepsilon} J_2 \end{pmatrix} \begin{pmatrix} \nabla \phi^\varepsilon(t,x) \\ v \end{pmatrix} + \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} \nabla \phi^\varepsilon(t,x) \\ v \end{pmatrix}$$

which is **broken** by the averaging procedure. Here, an Hamiltonian splitting is possible

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & I_2 \\ -I_2 & \frac{1}{\varepsilon} J_2 \end{pmatrix} \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} 0 & I_2 \\ -I_2 & \frac{1}{\varepsilon} J_2 \end{pmatrix} \begin{pmatrix} \nabla \phi^\varepsilon(t,x) \\ 0 \end{pmatrix}$$

and leads to geometric averaging.

What about structures? (the case of constant B)

Going back to $3D$ with

$$J_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

we have to modify the Hamiltonian splitting accordingly by taking special care of the component of v that is parallel to $B = (1, 0, 0)$

$$\begin{pmatrix} \dot{x} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & I_3 \\ -I_3 & \frac{1}{\varepsilon} J_3 \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_2 \\ v_3 \end{pmatrix} + \begin{pmatrix} 0 & I_3 \\ -I_3 & \frac{1}{\varepsilon} J_3 \end{pmatrix} \begin{pmatrix} \nabla \Phi^{\varepsilon}(t, x) \\ v_1 \\ 0 \\ 0 \end{pmatrix}$$

in order to retain the periodicity of the stiff part solution.

Characteristics in the Littlejohn's variables (i)

In order to treat the general case (variable B), Littlejohn introduces the following changes of variables:

$$\begin{pmatrix} x \\ v \end{pmatrix} \underbrace{\xrightarrow{\varphi_1}}_{v=v_{\parallel}b(x)+v_{\perp}c(\theta,x)} \begin{pmatrix} x \\ v_{\parallel} \\ v_{\perp} \\ \theta \end{pmatrix} \underbrace{\xrightarrow{\varphi_2}}_{v_{\perp}=\sqrt{2|B(x)|\mu}} \begin{pmatrix} x \\ v_{\parallel} \\ \mu \\ \theta \end{pmatrix}$$

where $(a(\theta,x), b(x), c(\theta,x))$ is the Littlejohn's triplet defined as follows: given two **smooth** unit vectors $(e_1(x), e_2(x))$ in the orthogonal plane to $b(x) = \frac{B(x)}{|B(x)|}$, we set

$$c(\theta,x) = -\sin(\theta)e_1(x) - \cos(\theta)e_2(x),$$
$$a(\theta,x) = c(\theta,x) \times b(x) = -\partial_\theta c(\theta,x)$$

so that (a,b,c) is a direct orthonormal basis.

Characteristics in the Littlejohn's variables (ii)

Inserting $(\varphi_2 \circ \varphi_1)'$ in previous formula gives the structure matrix

$$\Omega = \begin{pmatrix} 0 & I_3 \\ -I_3 & \frac{1}{\varepsilon}\mathcal{J}_B(x) \end{pmatrix} \to \begin{pmatrix} 0 & (DQ^T)^{-1} \\ -(QD)^{-1} & \frac{1}{\varepsilon}J_3 + R^T - R \end{pmatrix} = \tilde{\Omega}$$

for $Q \equiv Q(x, v_\parallel, \mu, \theta)$, $R \equiv R(x, v_\parallel, \mu, \theta)$, $D = \mathrm{diag}(1, \frac{B(x)}{v_\perp}, 1)$, while the new Hamiltonian is now

$$H = \frac{1}{2}|v|^2 + \phi^\varepsilon(t,x) \longrightarrow \frac{1}{2}v_\parallel^2 + |B(x)|\mu + \phi^\varepsilon(t,x) = \tilde{H}$$

In these variables, the system is non-canonically Hamiltonian of the form

$$\begin{pmatrix} \dot{x} \\ \dot{v}_\parallel \\ \dot{\mu} \\ \dot{\theta} \end{pmatrix} = \tilde{\Omega} \begin{pmatrix} \nabla_x \tilde{H} \\ v_\parallel \\ |B| \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{|B|}{\varepsilon} \end{pmatrix} + k = \frac{1}{\varepsilon} g + k$$

Littlejohn's reduction (i): elimination of θ

Illustration for $|B(x)| = B_0 > 0$

Normal form theory asserts that there exists a change of variables

$$\boxed{(x, v_\parallel, \mu, \theta) = \varphi^\varepsilon(X, V_\parallel, M, \Theta) = (X, V_\parallel, M, \Theta) + \mathcal{O}(\varepsilon)}$$

transforming $\frac{1}{\varepsilon} g + k$ into $\frac{1}{\varepsilon} g + k^\varepsilon$ such that

$$[k^\varepsilon, g] = \frac{\partial k^\varepsilon}{\partial(X, V_\parallel, M, \Theta)} g - \frac{\partial g}{\partial(X, V_\parallel, M, \Theta)} k = B_0 \frac{\partial k^\varepsilon}{\partial \Theta} = 0$$

Hence, in the new variables $(X, V_\parallel, M, \Theta)$, the vector field

$$\boxed{\tfrac{1}{\varepsilon} g + k^\epsilon = \left(\varphi^{\varepsilon\prime}\right)^{-1} \left(\tilde{\Omega} \circ \varphi^\varepsilon\right) \left(\varphi^{\varepsilon\prime}\right)^{-T} \nabla\left(\tilde{H} \circ \varphi^\varepsilon\right)}$$

is again non-canonically Hamiltonian and **does not depend on** Θ.

Littlejohn's reduction (ii)

In Littlejohn 83' the author carries on a transformation of this sort for general magnetic fields with varying intensity $|B(x)|$.

More precisely

- Littlejohn constructs φ^{ε}, the change of variables which eliminates θ, by working on the Lagrangian formulation of the characteristics for general B.
- In his construction, the magnetic moment μ becomes an invariant.
- If one is not interested by the gyro-angle θ, the system has reduced dimension 4 (this is fundamental for the discretisation of the PDE).

Averaging of a single particle for varying $|B(x)|$ (i)

Characteristics in Littlejohn's variable are

$$(\mathbf{C}) : \begin{pmatrix} \dot{x} \\ \dot{v}_{\|} \\ \dot{\mu} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v_{\|} b + v_{\perp}\, c \\ -v_{\|} v_{\perp} (\partial_x c) b \cdot b - v_{\perp}^2 (\partial_x c) c \cdot b + E^{\varepsilon} \cdot b \\ \frac{v_{\perp}(E^{\varepsilon} \cdot c)}{|B|} - \frac{v_{\perp} v_{\|}^2 (\partial_x b) b \cdot c}{|B|} - \frac{\mu \nabla |B| \cdot (v_{\|} b + v_{\perp} c)}{|B|} - 2\mu v_{\|} (\partial_x b) c \cdot c \\ \frac{|B|}{\varepsilon} + \frac{v_{\|}^2 (\partial_x b) b \cdot a}{v_{\perp}} - \frac{E^{\varepsilon} \cdot a}{v_{\perp}} + \cdots \end{pmatrix}$$

with $v_{\perp} = \sqrt{2|B|\mu}$. With $y = (x, v_{\|}, \mu)$, this system is of the form

$$\begin{pmatrix} \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v_{\perp} f_{\theta}(t,y) + h(t,y) \\ \frac{B(x)}{\varepsilon} + \frac{1}{v_{\perp}} g_{\theta}(t,y) + k_{\theta}(t,y) \end{pmatrix}$$

where the average of $\dot{\mu}$ w.r.t. θ vanishes.

Averaging of a single particle for varying $|B(x)|$ (ii)

Introducing

$$\langle f\rangle(t,y) = \frac{1}{2\pi}\int_0^{2\pi} f_\tau(t,y)d\tau \text{ and } F_\theta(t,y) = \int_{\theta_0}^{\theta}\Big(f_\tau(t,y) - \langle f\rangle(t,y)\Big)d\tau$$

we can write (with $\theta \equiv \theta(t)$ and $y \equiv y(t)$)

$$\boxed{f_\theta(t,y) - \langle f\rangle(t,y) = \tfrac{1}{\dot\theta}\Big(\tfrac{d}{dt}F_\theta(t,y) - \partial_t F_\theta(t,y) - \partial_y F_\theta(t,y)\dot y\Big)}$$

and compute the component $y(t)$ as follows

$$\begin{aligned} y(t) =& y_0 + \int_0^t h ds + \int_0^t v_\perp \langle f\rangle ds + \int_0^t \frac{v_\perp}{\dot\theta}\frac{d}{ds}F_\theta ds \\ & - \int_0^t \frac{v_\perp}{\dot\theta}\Big(\partial_t F_\theta + \partial_y(v_\perp f + h)\Big)ds \end{aligned}$$

Averaging of a single particle (iii)

Integration by parts in the third integral leads to

$$y(t) = y_0 + \int_0^t h ds + \int_0^t v_\perp \langle f \rangle ds + \frac{v_\perp}{\dot\theta} F_\theta + \int_0^t \left(\frac{v_\perp \ddot\theta}{\dot\theta^2} - \frac{\dot v_\perp}{\dot\theta} \right) F_\theta ds$$
$$- \int_0^t \frac{v_\perp}{\dot\theta} \Big(\partial_t F_\theta + \partial_y (v_\perp f + h) \Big) ds$$

Now, for ε small enough, as long as $\mu \geq C\varepsilon$, we can have bounds of the form

$$\dot\theta \geq \frac{C}{\varepsilon} \text{ and } |\ddot\theta| \leq \frac{C}{\varepsilon}.$$

and conclude that (on an interval of length independent of ε)

$$\boxed{y(t) = y_0 + \int_0^t h ds + \int_0^t v_\perp \langle f \rangle ds + \mathcal{O}(\varepsilon)}$$

Averaging of a single particle (iv)

The asymptotic system is finally obtained by noticing that

$$\langle a \rangle = \langle c \rangle = 0 \quad \text{and} \quad \langle (\partial_x b) c \cdot c \rangle = \frac{1}{2} \text{div } b = -\frac{\nabla |B| \cdot b}{2|B|}$$

and dropping the variable θ

$$(\mathbf{A}) \quad \begin{pmatrix} \underline{\dot{x}} \\ \underline{\dot{v}_\parallel} \\ \underline{\dot{\mu}} \end{pmatrix} = \begin{pmatrix} \underline{v_\parallel} b \\ b \cdot (E^\varepsilon - \underline{\mu} \nabla |B|) \\ 0 \end{pmatrix}.$$

The system is non-canonical Hamiltonian with Hamiltonian

$$\underline{H}(x, v_\parallel, \mu) = \frac{1}{2} v_\parallel^2 + \Phi^\varepsilon + \mu |B|.$$

Averaging of a single particle (v)

Lemma

Assume $E^\varepsilon \in W^{1,\infty}([0,T]\times\mathbb{R}^3)$ *for some* $T>0$ *and that* $\|E^\varepsilon\|_{W^{1,\infty}} \leq M,\ \forall\varepsilon$. *Given* $R_2>0$, *there exist* $\varepsilon_1>0$, $R_1\geq 1$ *and* $R>0$ *such that for* $0<\varepsilon\leq\varepsilon_1$ *and for any initial condition*

$$(x_0,(v_\parallel)_0,\mu_0)\in B_{R_2}(0)\cap\{\mu\geq R_1\varepsilon\}$$

the solutions of (C) *and* (A) *exist on* $[0,T_0^\varepsilon]$ *and remain in*

$$B_R(0)\cap\{\mu\geq\frac{1}{2}\mu_0\}$$

Moreover, we have

$$\forall t\in[0,T_0^\varepsilon],\ |x(t)-\underline{x}(t)|+|v_\parallel(t)-\underline{v_\parallel}(t)|+|\mu(t)-\underline{\mu}(t)|\leq C\varepsilon$$

for some positive constant (independent of ε*).*

Averaging of the Vlasov-Poisson equations

Assumption

Initial data $f_0^\varepsilon \in W^{1,\infty}(\mathbb{R}^6)$ is positive, has compact support and

$$\mathrm{Supp}(f_0^\varepsilon) \subset \{(x,v) : |(x,v)| \leq M\} \text{ and } \|f_0^\varepsilon - f_0\|_{W^{1,\infty}} \leq C\varepsilon$$

where $f_0 = f_0(x, v_\parallel, v_\perp)$, i.e. f_0 does not depend on θ.

First-order asymptotics of Vlasov-Poisson

$$(AVP) \qquad \begin{aligned} &\partial_t f + v_\parallel b \cdot \nabla_x f + b \cdot (E - \mu \nabla |B|) \partial_{v_\parallel} f = 0; \\ &E(t,x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{x - x'}{|x - x'|^3} \rho(t,x') dx' \\ &\rho(t,x) = 2\pi |B(x)| \int_{\mathbb{R}^3 \times \mathbb{R}_+} f(t,x,v_\parallel,\mu) dv_\parallel d\mu \\ &f(0,x,v_\parallel,\mu) = f_0(x, v_\parallel, \sqrt{2|B|\mu}) \end{aligned}$$

Averaging of the Vlasov-Poisson equations

Theorem (Main result)

Assume that $B \in C_b^2(\mathbb{R}^3)$ *derives from a potential vector* A *and is such that* $|B(x)| \geq B_0 > 0$ *for all* $x \in \mathbb{R}^3$. *Suppose that previous assumption is furthermore satisfied and let* $T > 0$. *There exists* $\underline{\varepsilon}$ *such that for any* $0 < \varepsilon \leq \underline{\varepsilon}$, *the solutions* f^ε *and* f *of* (VP) *and* (AVP) *exist on* $[0, T]$ *and satisfy*

$$\int \left| f^\varepsilon(t, x, v_\| b + v_\perp c) - f\left(t, x, v_\|, \frac{v_\perp^2}{2|B|}\right) \right| v_\perp dx dv_\| dv_\perp d\theta \leq C_T \varepsilon$$

where the constant C_T *does not depend on* ε.

Remark: This result is derived in [7] *Gyrokinetic approximations of the Vlasov-Poisson system with a strong magnetic field in dimension* 3, by Chartier, Crouseilles, Lemou, M. (in preparation). The next-order model is also derived therein: this is the so-called gyrokinetic model.

Comments and perspectives

1. the derivation of the next term of the asymptotic expansion of the characteristics is quite intricate. This is completed in our paper [7].
2. In collaboration with P. Chartier, M. Lemou and A. Murua, we aim at using formal tools such as word-series to derive systematic and explicit expansions in the spirit of Littlejohn.
3. uniformly accurate numerical methods for the full Vlasov-Poisson system exist in the case of magnetic fields with constant intensity $|B(x)| \equiv B_0$ (see [8] next slide). The case of a varying intensity remains a challenge.

References

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Thank you for your attention