UNIVERSITAS GADJAH MADA

Dark Matters from Particles with Unusual Statistics

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Outline of the Talk

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Introduction

- Dark matter (DM) comprise around 22% of the energy of the universe
- The energy density of DM around five times the energy density of the visible matter $\Omega_{\rm dm} \simeq 5\Omega_b$.
- There are many model of DM, among others: WIMP, axion, sterile neutrinos, Asymmetric DM.
- On all (except few in the literature) of the model, DM are always considered as either Bose of Fermi particles
- There are other statistics besides Bose and Fermi that is permutation invariant statistics, fulfill cluster decomposition property, and have non negative norm states.
- Some motivation from Quantum Gravity theory, Quantum Foam, and Holography theory — Dark energy and (maybe also) dark matter fulfill infinite statistics.

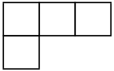
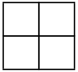
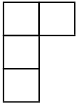
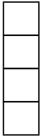
Outline of the Talk

- For n identical particles in a space with the dimension of $d \geq 3$, the Hilbert Space should be invariant under the action of the permutation group S_n.
- The Hilbert space of n-identical particles system with a set of quantum number and its multiplicity (j, ν) can be decomposed according to the IUR's of the permutation group S_n

$$\mathcal{H}_{(n)}^{(j,\nu)} = \oplus_{\lambda \in \Lambda_n} \mathcal{H}_{\lambda}^{(j,\nu)}, \tag{1}$$

  where Λ_n is a subset of $\mathrm{Irr}(S_n) =$ the set of all IUR's of S_n.
- The IUR's of S_n is denoted with the partition of n, denoted by $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_i \geq \lambda_{i+1}$ and $|\lambda| \equiv \sum_i \lambda_i = n$.
- Inside each Irr-space λ there is d_λ vectors. $d_\lambda = n! \prod_{i<j}(l_i - l_j)/l_1! \dots l_n!$, with $l_i = \lambda_i + n - i$.
- Vectors that live inside Irr-space λ are called to have symmetry type λ.
- The λ can be denoted by a Young tableaux: a set of left justified boxes denoting the partition.

Young tableau, example S_4 : (4), (3,1), (2,2), (2,1,1), (1,1,1,1)

(4) :  $(3,1)$:  $(2,2)$:  $(2,1,1)$:  $(1,1,1,1)$: 

Cluster Decomposition

J. B. Hartle, R. H. Stolt and J. R. Taylor, Phys. Rev. **D 2**, 1759 (1970).

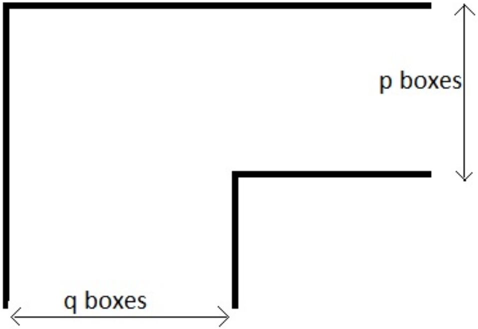
Cluster decomposition: measuring any physical properties of a set of "isolated" particles should not depend on the existence or nonexistence of particles elsewhere.
Hartle, Stolt and Taylor have shown that the allowed Λ_n's under cluster decomposition should contain either:

1. All $\lambda \in \mathrm{Irr}(S_n)$ for each n (infinite statistics)
2. or All λ whose Young tableau are inside the '(p, q)-envelope' for each n, where p and q are finite non-negative integers with $p + q > 0$.

The (p, q)-envelope is the set of all Young tableau with $\lambda_p \leq q$ for the associated partition λ.
Parafermi and parabose statistics of order p are part of the second case.
Bose and Fermi are just special case of Parabose and Parafermi of order 1.

Cluster Decomposition

(p,q)-envelope



Indistinguishability

- Knowing the allowed state symmetry type is not enough to determine the statistics.
- It could happen that certain states inside each $\mathcal{H}_{\lambda,i}^{(j,\nu)}$ are *indistinguishable* from each other — There is no observable which has different expectation values for the states in question.
- In such a case, these states should be considered as physically equivalent and not counted as different.
- Denote by c_λ the number of distinct physical state inside each Irr subspace λ.
- The value of c_λ depends on the observables in the system.
- In general, the set $\mathcal{O}^{(n)}$ of observables for the n-particle states is a subset of all hermitian operators that act on $\mathcal{H}^{(n)}$, denoted by $\mathrm{Herm}(\mathcal{H}^{(n)})$.

Indistinguishable Statistics

- The first case: For each observable $O \in \mathcal{O}^{(n)}$ we have $[U(\pi), O] = 0$ for all $\pi \in S_n$ and all n. In this case the observables don't change under the action of $U(\pi)$, — the *symmetric observables*.
- Any two vectors in $\mathcal{H}_{\lambda,i}^{(j,\nu)}$ will have the same expectation values for any observable and hence cannot be distinguished from each other.
- All d_λ linearly independent vectors inside each $\mathcal{H}_{\lambda,i}^{(j,\nu)}$ should be considered physically equivalent and be counted as one state ($c_\lambda = 1$).
- The whole subspace $\mathcal{H}_{\lambda,i}^{(j,\nu)}$ is often called a *generalized ray* (A. L. M. Messiah, O. W. Greenberg, Phys. Rev. 136, B248 (1964).)
- We will refer to this case as *indistinguishable statistics*.

Distinguishable Statistics

- The second case is when $\mathcal{O}^{(n)} = \mathrm{Herm}(\mathcal{H}^{(n)})$ for all n.
- There is always some observable that has different expectation values for any two states (even inside the same irreducible subspace).
- For example, the (hermitian) projection operator constructed from one particular state in any basis of $\mathcal{H}_{\lambda,i}^{(j,\nu)}$ will distinguish that state from all other $d_\lambda - 1$ linearly independent states in that basis.
- Thus, inside each $\mathcal{H}_{\lambda,i}^{(j,\nu)}$ we have d_λ physically distinct linearly independent states ($c_\lambda = d_\lambda$).
- We will refer to this case as *distinguishable statistics*. (note: it is still identical particles!!)

The Grand Canonical Partition Functions

The Grand Canonical Partition Function (GCPF) for m energy levels, can be written as

$$Z(x_1, \ldots, x_m) = \mathrm{Tr}\;\; e^{\,\beta(\mu N - H)} = \sum_{n=1}^{\infty} \sum_{\pi,(j,\nu)} \langle \pi(j,\nu) | e^{\,\beta(\mu N - H)} | \pi(j,\nu) \rangle, \tag{2}$$

where $x_i = e^{\beta(\mu - E_i)}$, $\beta = 1/kT$, E_i is the energy of one particle state, and μ is the chemical potential. We can decompose each state into it's components inside the $\mathcal{H}_\lambda$'s. After a long manipulation, one has (I. G. Macdonald, Symmetric Functions and Hall Polynomials)

$$Z(x_1, \ldots, x_m) = \sum_{\lambda \in \Lambda} c_\lambda \; s_\lambda(x_1, \ldots, x_m). \tag{3}$$

where $s_\lambda(x_1, \ldots, x_m)$ is the Schur polynomial in m variables given by

$$s_\lambda(x_1, \ldots, x_m) = \begin{cases} |x_j^{m+\lambda_i - i}| / \prod_{i<j}^{m} (x_i - x_j) & \text{for } \lambda_{m+1} = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

The Grand Canonical Partition Functions

Some of the GCPF of the permutation invariant statistics are known:

- Distinguishable Infinite Statistics

$$Z_{\infty}^{\text{dis}}(x_1, \ldots, x_m) = \sum_{\text{all } \lambda} d_\lambda s_\lambda(x_1, \ldots, x_m) = \frac{1}{1 - (x_1 + \cdots + x_m)}. \quad (5)$$

- Indistinguishable Infinite Statistics (I. G. Macdonald, Symmetric Functions and Hall Polynomials)

$$Z_{\infty}^{\text{ind}}(x_1, \ldots, x_m) = \sum_{\text{all } \lambda} s_\lambda(x_1, \ldots, x_m) = \frac{1}{\prod_i^m (1 - x_i) \prod_{i<j}(1 - x_i x_j)}, \quad (6)$$

- Indistinguishable Parefermion Statistics (I. G. Macdonald, Symmetric Functions and Hall Polynomials)

$$Z_{\text{parafermi}-\text{p}}^{\text{ind}}(x_1, \ldots, x_m) = \sum_{\Lambda_{(0,p)}} s_\lambda(x_1, \ldots, x_m) = \frac{|x_j^{m-i} - x_j^{m+p+i-1}|}{|x_j^{m-i} - x_j^{m+i-1}|}, \quad (7)$$

Creation-Annihilation Operator Realization

1. Green parastatistics creation and annihilation operator (CAO) algebra

$$[[a_i, a_j^\dagger]_\pm, a_k^\dagger] = \frac{2}{p}\delta_{ik}a_j^\dagger, \tag{8}$$

where p is a nonzero integer. Correspond to indistinguishable parabose and parafermi of order p
2. Govorkov parastatistics CAO algebra

$$[a_i a_j^\dagger, a_k^\dagger] = \frac{1}{p}\delta_{ik}a_j^\dagger, \tag{9}$$

where p is a nonzero integer. Correspond to distinguishable parabose or parafermi of order p.
3. Greenberg's "q-mutator" CAO algebra, interpolates between Bose ($q = 1$) and Fermi ($q = -1$)

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij}. \tag{10}$$

This q-mutator CAO algebra correspond to distinguishable infinite statistics.

Outline of the Talk

Distinguishable Infinite Statistics

The GCPF for distinguishable infinite statistics (where $x_i = e^{\beta(\mu - E_i)}$, and $\beta = 1/kT$.)

$$Z_{\infty}^{\mathrm{dis}}(x_1, \ldots, x_m) = \sum_{\mathrm{all}\ \ \lambda} d_\lambda s_\lambda(x_1, \ldots, x_m) = \frac{1}{1 - (x_1 + x_2 + \cdots + x_m)} \tag{11}$$

The total number of particles and the energy is given by

$$N = \sum_{k=1}^{m} \frac{x_k}{\left(1 - \sum_{i=1}^{m} x_i\right)}; \quad U = \sum_{k=1}^{m} \frac{x_k E_k}{\left(1 - \sum_{i=1}^{m} x_i\right)} \tag{12}$$

In the continuum limit, the total number of particles and the energy are given by

$$N(T,V,\mu) = \frac{2\pi^{d/2}V}{h^d\Gamma(d/2)d}(mc)^d u\int_1^\infty dt\ (t^2-1)^{\frac{d}{2}}\frac{ze^{-ut}}{(1-VLa)} = \frac{VLa}{(1-VLa)}$$

$$U = \frac{2\pi^{d/2}V}{h^d\Gamma(d/2)}c(mc)^{d+1}\int_1^\infty t^2 dt\ (t^2-1)^{\frac{d-2}{2}}ze^{-ut}\frac{1}{(1-VLa)} = \left(\frac{d}{u}+\frac{K_{\frac{d-1}{2}}(u)}{K_{\frac{d+1}{2}}(u)}\right)\frac{VLamc^2}{(1-VLa)} \tag{13}$$

where d is the space dimension, $a = ze^{-\beta mc^2}$, $K_n(u)$ is the modified Bessel function of the second kind, and L is given by,

$$L \equiv \beta\frac{2c}{\hbar^d}\left(\frac{m}{2\pi\beta}\right)^{\frac{d+1}{2}} e^u K_{\frac{d+1}{2}}(u). \tag{14}$$

In non relativistic case $L \to \lambda_T^{-d}$, where $\lambda_T = \sqrt{2\pi\hbar^2\beta/m}$ is the thermal wavelength.

Infinite Statistics Dark Matter

C. M. Ho, D. Minic, and Y. J. Ng, ‘Dark matter, infinite statistics, and quantum gravity’, Phys. Rev. **D 85**, 104033 (2012)

- MONDian dark matter which behaves like cold dark matter at cluster and cosmological scales but emulates modified Newtonian dynamics (MOND) at the galactic scale
- Quanta of the MONDian dark matter obey infinite statistics, and the theory must be fundamentally non local.

Z. Ebadia, B. Mirza and H. Mohammadzadeh, ‘Infinite statistics condensate as a model of dark matter’, Journal of Cosmology and Astroparticle Physics, (2013)

- Extension of an idea that dark matter is a condensate bosonic system. They claim that condensation is also possible for particles that obey infinite statistics and they derive the critical condensation temperature.
- Start with the Greenberg quon’s statistics but then used the Medvedev Ambiguous Statistics, where one has both Bose and Fermi statistics with some probability p_b and p_f respectively.
- They avoid the problem with distinguishable infinite statistics thermodynamics.

M. V. Medvedev, 'Properties of Particles Obeying Ambiguous Statistics', Phys. Rev. Lett.**78**, 4147 (1997) A system with N_j particles, where in each realization, the system has k bosons and $N_j - k$ fermions with the probability of $p_b^k p_f^{N_j - k}$. The total number of states of the system is

$$W = \prod_j \sum_{k=0}^{N_j} \frac{N_j!}{k!(N_j - k)!} w_b(k) w_f(N_j - k) p_b^k p_f^{N_j - k} \tag{15}$$

where $w_b(k)$ and $w_f(k)$ are the Bose and Fermi counting function for m energy level

$$w_b(k) = \frac{(m+k-1)!}{k!(m-1)!} \quad w_f(k) = \frac{m!}{k!(m-k)!} \tag{16}$$

- Using the usual Lagrange multiplier method, they got the occupation number. For the case $p_f = p_b = p$ (the case of $q = 0$ quon statistics) they got the total energy and the total number of particles (in a d dimensional spatial space)

$$U = \frac{A}{\beta^{d/2+1}} \int_0^\infty \frac{4pze^x x^{d/2}}{e^{2x} - p^2 z^2} dx; \quad N = \frac{A}{\beta^{d/2}} \int_0^\infty \frac{4pze^x x^{d/2-1}}{e^{2x} - p^2 z^2} dx \tag{17}$$

where $A = \frac{V}{\Gamma(d/2)} \frac{(2m\pi)^{d/2}}{h^d}$, $z = \exp(\beta\mu)$ and $x = \beta E$

- They showed that condensation will occur at $z = 1/p$ or at $\mu = -kT \ln p$. They got the critical temperature

$$kT_{cr} = \frac{2\pi\hbar^2}{m_q} \left(\frac{n}{\zeta(d/2)} \right)^{2/d} \frac{2^{1-4/d}}{(2^{d/2} - 1)^{2/d}} \tag{18}$$

where m_q is the quon's mass.

- The Medvedev's Ambiguous Statistics is not a realization of infinite statistics!
- One can start with the following GCPF for m energy levels (where $x_i = \exp(\beta(\mu - E_i))$).

$$Z(x_1, \ldots, x_m) = \prod_{i=1}^{m} \frac{1 + px_i}{1 - px_i} \tag{19}$$

From which one can have the total number of particles and the total energy, given by

$$N = \sum_{k=1}^{m} \frac{2px_k}{1 - p^2 x_k^2}; \quad U = \sum_{k=1}^{m} \frac{2px_k E_k}{1 - p^2 x_k^2} \tag{20}$$

For the continuum limit, one will get their result.

Using the following Cauchy identity

$$\sum_{\lambda} s_\lambda(x_1,\dots)s_\lambda(y_1,\dots) = \prod_{i,j}(1-x_iy_j)^{-1}; \quad \sum_{\lambda} s_\lambda(x_1,\dots)s_{\lambda'}(y_1,\dots) = \prod_{i,j}(1+x_iy_j) \tag{21}$$

The GCPF can be written as

$$\prod_{i=1}^{m}\frac{1+px_i}{1-px_i} = \sum_{\lambda} s_\lambda(p)s_\lambda(x_1,\dots,x_m)\sum_{\mu} s_{\mu'}(p)s_\mu(x_1,\dots,x_m) \tag{22}$$

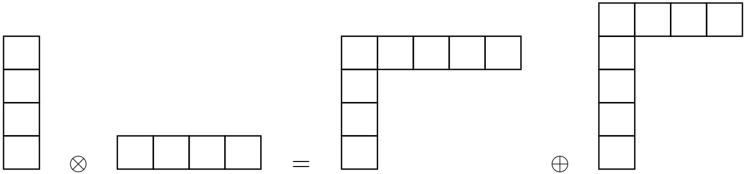
where $\lambda = (|\lambda|, 0, \dots)$ and $\mu = (1, 1, \dots)$. Using Littlewood Richardson rule

$$\prod_{i=1}^{m}\frac{1+px_i}{1-px_i} = \sum_{\nu}\sum_{\lambda\mu} c^{\nu}_{\lambda\mu} s_\lambda(p)s_{\mu'}(p)s_\nu(x_1,\dots,x_m) \tag{23}$$

where $c^{\nu}_{\lambda\mu}$ is the Littlewood Richardson coefficient.

Infinite statistics condensate as a model of dark matter
Comment

The Littlewood Richardson coefficient: For Irr-λ of $S_{|\lambda|}$ and Irr-μ of $S_{|\mu|}$, $c^{\nu}_{\lambda\mu}$ gives the multiplicities of the Irr-ν in the tensor product $\lambda \otimes \mu$ when restrict down to $S_{|\lambda|+|\mu|}$.
For our case, we have $\lambda = (|\lambda|, 0, 0, \ldots)$ and $\mu = (1, 1, \ldots)$.



Thus ν should be only in the $(1,1)$-envelope. Most of the symmetry type of infinite statistics is not there.
The Medvedev statistics is more like a semi Bose semi Fermi statistics. It is not the realization of quon distinguishable infinite statistics.
But it is extensive, and the norm of the states are non negative. But it has 'probabilistic' state counting c_λ.

Modification of Medvedev idea

One may extend the Medvedev idea. Include a number of n bosons and fermions each with probability p such that $\sum_i^n 2p = 1$, and take $n \to \infty$.
In this way one will include all the symmetry types of the permutation group.
The GCPF in this case will be

$$Z(x_1, \ldots, x_m) = \lim_{n\to\infty} \prod_{i=1}^{n} \prod_{i=1}^{m} \frac{1 + px_i}{1 - px_i} = \lim_{n\to\infty} \prod_{i=1}^{m} \left(\frac{1 + \frac{1}{2n}x_i}{1 - \frac{1}{2n}x_i} \right)^n = \prod_{i=1}^{m} \exp(x_i) \tag{24}$$

We get the classical Maxwell Boltzman statistics (as a realization of a quantum statistics).
Unfortunately as we know, the Maxwell Boltzmann statistics doesn't have Bose-like condensation.

Outline of the Talk

The GCPF for indistinguishable infinite statistics

$$
\begin{aligned}
Z_{\infty}^{\mathrm{ind}}(x_1,\ldots,x_m) &= \sum_{\text{all } \lambda} s_\lambda(x_1,\ldots,x_m) = \prod_{i=1}^{m} \frac{1}{(1-x_i)} \prod_{i<j}^{m} \frac{1}{(1-x_i x_j)} \\
&= \left(\prod_{i=1}^{m} \frac{(1+x_i)}{(1-x_i)} \prod_{i,j=1}^{m} \frac{1}{(1-x_i x_j)} \right)^{1/2}
\end{aligned} \tag{25}
$$

The total number of particle and the total energy are given by

$$
N = \sum_{k=1}^{m} \frac{x_k}{1-x_k^2} + \sum_{i,k=1}^{m} \frac{x_i x_k}{(1-x_i x_k)}.; \quad U = \sum_{k=1}^{m} E_k \frac{x_k}{1-x_k^2} + \sum_{i,k=1}^{m} E_k \frac{x_i x_k}{(1-x_i x_k)}. \tag{26}
$$

We have a restriction $0 \leq a \leq 1$ (Note: $a = ze^{-\beta mc^2}$).

Taking the continuum limit means

$$\sum_i \to \int \frac{d^d r \; d^d p}{h^d} = \frac{2\pi^{d/2} V}{h^d \Gamma(d/2)} \int_0^\infty p^{d-1} \;\; dp = \frac{2\pi^{d/2} V}{h^d \Gamma(d/2)} (mc)^d \int_1^\infty tdt \; (t^2 - 1)^{\frac{d-2}{2}} \tag{27}$$

where $t = E/mc^2$, $p = \sqrt{(E/c)^2 - m^2 c^2}$ is the single-particle state momentum.
The total energy U

$$\begin{aligned} U = {} & \frac{2Vc^3\beta}{\hbar^d} \left(\frac{m}{2\pi\beta}\right)^{\frac{d+1}{2}} \sum_{k \text{ odd}} \frac{z^k}{k^{\frac{d+1}{2}}} \left(\frac{d}{u} K_{\frac{d+1}{2}}(ku) - K_{\frac{d-1}{2}}(ku)\right) \\ & + \frac{4\beta^2 V^2 c^4}{\hbar^{2d}} \left(\frac{m}{2\pi\beta}\right)^{d+1} \sum_{k=1}^{\infty} \frac{z^{2k}}{k^d} \left(\frac{d}{u} K^2_{\frac{d+1}{2}}(ku) - K_{\frac{d+1}{2}}(ku) K_{\frac{d-1}{2}}(ku)\right) \end{aligned} \tag{28}$$

The total number of particle is given by

$$N = VL \sum_{k \ \text{odd}}^{\infty} \frac{z^k e^{-u}}{k^{\frac{d-1}{2}}} \frac{K_{\frac{d+1}{2}}(ku)}{K_{\frac{d+1}{2}}(u)} + V^2 L^2 \sum_{k=1}^{\infty} \frac{z^{2k} e^{-2u}}{k^{d-1}} \frac{K^2_{\frac{d+1}{2}}(ku)}{K^2_{\frac{d+1}{2}}(u)} + \frac{a}{1-a} \equiv N_e + N_0 \tag{29}$$

In the non relativistic limit we have

$$N_e = VL \sum_{k \ \text{odd}}^{\infty} \frac{a^k}{k^{\frac{d}{2}}} + V^2 L^2 \sum_{k=1}^{\infty} \frac{a^{2k}}{k^d} = VL \left(g_{\frac{d}{2}}(a) + f_{\frac{d}{2}}(a) \right) + V^2 L^2 g_d(a^2), \tag{30}$$

while in the ultrarelativistic limit we have

$$N_e = VL \sum_{k \ \text{odd}}^{\infty} \frac{a^k}{k^d} + V^2 L^2 \sum_{k=1}^{\infty} \frac{a^{2k}}{k^{2d}} = VL \left(g_d(a) + f_d(a) \right) + V^2 L^2 g_{2d}(a^2) \tag{31}$$

The critical temperature can be obtained as the limit when $a \to 1$ (or $\mu \to mc^2$).

$$\frac{N_{\mathrm{crit}}}{V} = L \sum_{k \text{ odd}}^{\infty} \frac{e^{(k-1)u}}{k^{\frac{d-1}{2}}} \frac{K_{\frac{d+1}{2}}(ku)}{K_{\frac{d+1}{2}}(u)} + VL^2 \sum_{k=1}^{\infty} \frac{e^{2(k-1)u}}{k^{d-1}} \frac{K^2_{\frac{d+1}{2}}(ku)}{K^2_{\frac{d+1}{2}}(u)} \tag{32}$$

By solving this equation for the temperature T at a fixed N, we can obtain the critical temperature T_c. In the non-relativistic limit

$$\begin{aligned} N &= VL \sum_{k \text{ odd}}^{\infty} \frac{1}{k^{\frac{d}{2}}} + V^2 L^2 \sum_{k=1}^{\infty} \frac{1}{k^d} = \lambda_T^{-d} \zeta(d/2) 2(1 - \frac{1}{2^{d/2}}) + V \lambda_T^{-2d} \zeta(d), \\ T_c &= \frac{2\pi\hbar^2}{kmV^{2/d}} \left[\left(\frac{\zeta^2(d/2)}{\zeta^2(d) 2^d} \left(2^{d/2} - 1\right)^2 + \frac{N}{\zeta(d)} \right)^{1/2} - \frac{\zeta(d/2)}{\zeta(d)} \left(1 - \frac{1}{2^{d/2}}\right) \right]^{2/d} \end{aligned} \tag{33}$$

In the termodynamic limit $(N, V \to \infty)$ we have $T_c = (2\pi\hbar^2/km)(N/\zeta(d)V^2)^{1/d}$

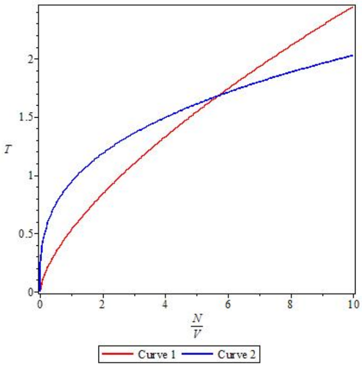


Figure: Fig. T in unit of $kmV^{2/d}/2\pi\hbar^2$ vs density N/V, for a unit Volume. curve 1 for Bose statistics, curve 2 for Indistinguishable Infinite Statistics.

Outline of the Talk

Can we use the Gross Pitaevskii Equation for Infinite Statistics condensate?
For this, lets look at the usual derivation of Gross Pitaevskii Eq. in second quantized non relativistics formulation.
One can start from the definition of the number operators in terms of creation and annihilation operator, where in the case of (scalar) Boson one have $n_i = a_i^\dagger a_i$. The energy operator can be written as

$$H = \sum_i E_i n_i = \sum_i E_i a_i^\dagger a_i = \int \frac{d^3k}{(2\pi)^3}(\frac{k^2}{2m} + V(k)_{\mathrm{ext}})a(\vec{k})^\dagger a(\vec{k}) \tag{34}$$

For interaction, one adds two bodies particle-particle interaction $\sum_{i<j} n_i n_j V_{ij}$

$$H = \int \frac{d^3k}{(2\pi)^3}(\frac{k^2}{2m} + V(k)_{\mathrm{ext}})a(\vec{k})^\dagger a(\vec{k}) + \frac{1}{2}\int\int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}a(\vec{k})^\dagger a(\vec{k'})^\dagger V(k,k')a(\vec{k'})a(\vec{k}) \tag{35}$$

Taking the fourier transform to go to the position basis, one have

$$H = \int d^3 r \Psi^\dagger(r) \left(-\frac{\nabla^2}{2m} + V(r)_{\text{ext}} \right) \Psi(r) + \frac{1}{2} \int \int d^3 r d^3 r' \Psi^\dagger(r) \Psi^\dagger(r') V(r - r') \Psi(r') \Psi(r). \quad (36)$$

Going to time dependent Heisenberg representation, and write the field as

$$\Psi(r,t) = \psi(r,t) + \Psi'(r,t) \quad (37)$$

where $\psi(r,t) = \langle \Psi(r,t) \rangle$ is a classical field, that will become the condensate wave function, constrained by the normalization condition

$$N = \int d^3 r \rho(r,t); \quad \rho(r,t) = |\psi(r,t)|^2. \quad (38)$$

The equation of motion of the condensate can be obtained from the Heisenberg equation of motion for the field

$$i \frac{\partial}{\partial t} \Psi(r,t) = [\Psi(r,t), H] = \left(-\frac{\nabla^2}{2m} + V_{\text{ext}}(r) + \int d^3 r' \Psi^\dagger(r',t) V(r' - r) \Psi(r',t) \right) \Psi(r,t) \quad (39)$$

Gross Pitaevskii Equation for Infinite Statistics?

One can then approximate the dynamics using the condensate wave function $\psi(r,t)$ and neglect the $\Psi'(r,t)$. One use $V(r-r') = g\delta^3(r-r')$ where $g = 4\pi a/m$ where a is the scattering length, then

$$i\frac{\partial}{\partial t}\psi(r,t) = \left(-\frac{\nabla^2}{2m} + V_{\text{ext}}(r) + g|\psi(r,t)|^2\right)\psi(r,t) \tag{40}$$

How about infinite statistics?
The only infinite statistics that may have condensation is the indistinguishable statistics, but there is no known CAO algebra for this statistics. The only known CAO algebra for infinite statistics is the quon's algebra (but it is for distinguishable statistics).
The number operator for quon's algebra is of infinite term (Greenberg, 1990)

$$n_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i^\dagger a_i a_k + \sum_{k_1,k_2} a_{k_1}^\dagger a_{k_2}^\dagger a_i^\dagger a_i a_{k_2} a_{k_1} + \cdots + \sum_{k_1,k_2,\ldots,k_s} a_{k_1}^\dagger a_{k_2}^\dagger \ldots a_{k_s}^\dagger a_i^\dagger a_i a_{k_s} \ldots a_{k_2} a_{k_1} + \cdots . \tag{41}$$

But it still obey $[n_i, a_j] = -a_j\delta_{ij}$

Using $H = \sum_i E_i n_i + \sum_{i<j} n_i n_j V_{ij}$ the dynamics of the creation/annihilation operator is given by

$$\begin{aligned} i\frac{\partial}{\partial t}a(k,t) &= \left[a(k,t), \int d^3k' E(k')n(k',t) + \frac{1}{2}\int \frac{d^3k'}{(2\pi)^3}\int \frac{d^3k'}{(2\pi)^3} V(k',k'')n(k',t)n(k'',t)\right] \\ &= \left(E(k) + \int \frac{d^3k'}{(2\pi)^3} V(k',k)n(k',t)\right) a(k,t) \end{aligned} \tag{42}$$

But putting this into position basis will not be easy, because of the infinite terms in occupation number operator.
But if there is condensation (doubtful for the case of quon's infinite statistics), we can use the condensate field $\psi(r,t)$ (which is a classical field!!). After taking fourier transform into position basis, neglecting the fluctuating field $\Psi'(r,t)$, and assuming contact interaction

$$V(r-r') = g\delta^3(r-r'), \tag{43}$$

Gross Pitaevskii Equation for Infinite Statistics?

One have

$$i\frac{\partial}{\partial t}\psi(r,t) = \left(-\frac{\nabla^2}{2m} + V_{\mathrm{ext}}(r) + gn(r,t)\right)\psi(r,t) \tag{44}$$

where

$$n(r,t) = |\psi(r,t)|^2 \left(1 + \int d^3r' |\psi(r',t)|^2 + \left(\int d^3r' |\psi(r',t)|^2\right)^2 + \left(\int d^3r' |\psi(r',t)|^2\right)^3 \dots\right) \tag{45}$$

and constrained by the following normalization

$$N = \int d^3r |\psi(r,t)|^2 + \left(\int d^3r |\psi(r,t)|^2\right)^2 + \left(\int d^3r |\psi(r,t)|^2\right)^3 + \left(\int d^3r |\psi(r,t)|^2\right)^4 \dots \tag{46}$$

Outline of the Talk

The GCPF for parafermi of order p only known in the discreate energy level case

$$Z^{\mathrm{ind}}_{\mathrm{parafermi-p}}(x_1,\ldots,x_m)=\sum_{\Lambda_{(0,p)}} s_\lambda(x_1,\ldots,x_m)=\frac{|x_j^{m-i}-x_j^{m+p+i-1}|}{|x_j^{m-i}-x_j^{m+i-1}|}, \tag{47}$$

But it is known that the dominator (I. G. Macdonald, Symmetric Functions and Hall Polynomials)

$$|x_j^{m-i}-x_j^{m+i-1}|=\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)(x_i-x_j) \tag{48}$$

While it can be shown that the numerator

$$|x_j^{m-i}-x_j^{m+p+i-1}|=\prod_i(1-x_i)\prod_{i<j}(1-x_ix_j)(x_i-x_j)F(x_1,\ldots,x_m) \tag{49}$$

$F(x_1,\ldots,x_m)$ is a symmetric polynomial of finite order. Thus no condensation in parafermi!!

Parafermionic Dark Matter

T. Kitabayashi and M. Yasue, Phys Rev **D 98**, 043504 (2018)

- They assume that parafermionic dark matter is a cold dark matter particle consisting of nonrelativistic particles, which is described by the standard freeze-out scenario of nonrelativistic particles.
- Started from the usual definition of the creation and annihilation algebra of parafermion
- They used the following Grand Canonical Partition Function (GCPF) which is based on the assumption that the GCPF is the sum of all possible quantum numbers - counting function (parafermi of order q)

$$Z(x_1, \ldots, x_m) = \prod_i \sum_{n_i=1}^{q_i} e^{-\beta(E_i-\mu)n_i} = \prod_i^m (1 + x_i + \cdots + x_i^q)$$

- From which one get the following distribution function for parafermions

$$f(x) = x\frac{d}{dx}\ln(1 + x + \cdots x^q) = \frac{x + 2x + 3x + \cdots + qx^q}{1 + x + x^2 + \cdots + x^q}$$

where $x = e^{\beta(E-\mu)}$

- They generalized the simple-formed Boltzmann equation to estimate the relic abundance of parafermionic dark matter.
- They extend the simple Boltzman equation for the $1+2 \leftrightarrow 3+4$ process, and obtain for the parafermions χ annihilates into fermionics particles

$$\frac{dn_\chi}{dt} + 3Hn_\chi = -\int d\Pi_\chi d\Pi_{\bar{\chi}} d\Pi_f d\Pi_{\bar{f}} (2\pi)^4 \delta^4 (p_\chi + p_{\bar{\chi}} - p_f - p_{\bar{f}}) |\mathcal{M}|^2 \\ \times \{ f_\chi f_{\bar{\chi}} (1 - f_f)(1 - f_{\bar{f}}) - f_f f_{\bar{f}} (q - f_\chi)(q - f_{\bar{\chi}}) \} \tag{50}$$

where q is the order of parafermions, and f is the particle distribution function.

- Assuming that at non relativistic condition $(m >> kT)$ the fermionic and parafermionic distribution functions approximately obey the Maxwell Boltzmann distribution, they obtained

$$\frac{dY_\chi}{dx} = -\frac{\lambda \langle \sigma v \rangle}{x^2} \left[Y_\chi^2 - r^2 (Y_\chi^{EQ})^2 \right] \tag{51}$$

where $x = m_\chi / T$, $Y_\chi = n_\chi / s_\chi$, and $\lambda = \frac{4\pi}{\sqrt{90}} M_{pl} m_\chi \sqrt{g_*}$

The GCPF that they used is not the correct GCPF for parafermion.
Infact, their GCPF is problematic (It was known as the extensive Gentile statistics). This can be seen easily:

$$(1 + x + x^2 + \cdots + x^q) = \prod_{k=1}^{q}(1 - \xi^k x) \tag{52}$$

where $\xi = \exp(2\pi i/(q+1))$.
Thus

$$Z(x_1, \ldots, x_m) = \prod_{j=1}^{q}\prod_{i=1}^{m}(1 - \xi_j x_i) = \prod_{i=1}^{q}\sum_{\mu_i} s_{\mu_i'}(\xi_i) s_{\mu_i}(x_1, \ldots, x_m) \tag{53}$$

Using the same method as in the case of Medvedev statistics above, since we have Schur function with non positive variable, one can show that state of certain symmetry type will have negative (or even imaginary) norms.

The GCPF for parafermi for a finite energy level is given by the symmetric polynomial $F(x_1, \ldots, x_m)$ above.

Even though we don't have a closed form formulation for $F(x_1, \ldots, x_m)$, but we know the highest order of its terms is $(x_1 + x_2 + \cdots + x_m)^q$.

From which we find that the maximum occupancy in each energy level is q.

Thus the Pauli blocking term that they proposed is correct, and as long as one consider the case of non relativistic condition ($m >> kT$), their last result should be ok.

Conclusion

- Some of the proposal about the dark matter being particles obeying infinite statistics or parafermions, have some mistakes (using semi bose and semi fermi statistics instead of infinite statistics and using statistics with negative norm states)
- Dark matter using infinite statistics particles have some problem regarding their thermodynamics being not extensive, (and non local). But if one can accept probabilistics counting of Medvedev, we just get the Maxwell Boltzmann statistics.
- The indistinguishable infinite statistics can have Bose-like condensation, and thus can be a model for a condensed particle dark matter, but the thermodynamics is non extensive.
- The Gross Pitaevskii equation cannot be used directly for infinite quon's statitistics. But some modification of it can be derive, if condensation do occur.

THANK YOU