Local Well-posedness of three-dimensional compressible Full-Navier-Stokes equations with degenerate viscosities and far field vacuum

Qin Duan

Shenzhen University

Joint work with Zhouping Xin (CUHK)

Shengguo Zhu (OXFORD)

Dec.26, 2019

Full-Navier-Stokes equations

The motion of a compressible viscous,heat-conductive, and Newtonian polytropic fluid is governed by the following full compressible Navier-Stokes system:

$$\begin{cases} \rho_t + \text{div}(\rho u) = 0, \\ (\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P = \text{div}\mathbb{T}, \\ (\rho E)_t + \text{div}(\rho E u + P u) = \text{div}(u\mathbb{T}) + \text{div}(\kappa \nabla \theta). \end{cases} \tag{1.1}$$

- $t \geq 0$ is the time, $x \in \mathbb{R}^3$;
- ρ: fluid density;
- $u = (u_1, u_2, u_3)^T$: velocity;
- e: specific internal energy;

Full-Navier-Stokes equations

- θ: absolute temperature;
- $E = e + \frac{1}{2}|u|^2$: specific total energy;
- $P(\rho, e)$: pressure,

$$P = R\rho\theta = (\gamma - 1)\rho e = Ae^{S/c_v}\rho^\gamma, \quad e = c_v\theta, \quad c_v = \frac{R}{\gamma - 1}; \tag{1.2}$$

  R, A: positive constant, $\gamma > 1$: the adiabatic exponent; S: the entropy.

- $\mathbb{T}$: the viscosity stress tensor, $D(u)$: deformation tensor,

$$\mathbb{T} = 2\mu(\theta)D(u) + \lambda(\theta)\mathrm{div}u\mathbb{I}_3, \quad D(u) = \frac{\nabla u + (\nabla u)^T}{2}, \tag{1.3}$$

Full-Navier-Stokes equations

- μ: shear viscosity coefficient;
- $\lambda + \frac{2}{3}\mu$: bulk viscosity coefficient;
- κ: heat conductivity coefficient;
- $$\mu(\theta) = \alpha\theta^{b}, \quad \lambda(\theta) = \beta\theta^{b}, \quad \kappa = \nu\theta^{b}, \tag{1.4}$$

where (α, β, ν, b) are all constants satisfying

$$\alpha > 0, \quad 2\alpha + 3\beta \geq 0, \quad \nu \geq 0, \quad \text{and} \quad b \geq 0. \tag{1.5}$$

Full-Navier-Stokes equations

For example,

- Maxwellian molecules, $b = 1$.
- Rigid elastic spherical molecules, $b = \frac{1}{2}$.
- Ionized gas, $b = \frac{3}{2}$.

Introduction

Some known results when μ, λ, κ are all constants for Cauchy problem:

- Serrin,J.(1959)and Nash,J.(1962) Local existence and uniqueness of classical solutions in the absence of vacuum (multi-dimensional case).
- Kazhikhov,A.V.and Shelukhin,V.V.(1977) Well-posedness for the one-dimensional problem with strictly positive initial density and temperature.
- Z.Xin (1998) Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density.
- E.Feireisl(2004) Global existence of weak solution with finite energy for $\gamma > \frac{3}{2}$.

Introduction

- Y.Choe and H.Kim(2006) Local well-posedness of smooth solutions with non-negative density.
- X.Huang and J.Li(2017) Global well-posedness of classical solutions with smooth initial data which are of small energy but possibly large oscillations where the initial density is allowed to vanish.
- Huan.Wen and Chang,Zhu(2017) Global well-posedness of classical solutions when density vanishes at infinity.
- Jinkai, Li and Zhouping,Xin(2019) Entropy bounded solutions to the one-dimensional compressible Navier-Stokes equations with zero heat conduction and far field vacuum.

Introduction

Some known results when $(\mu,\ \lambda,\ \kappa)$ are dependent of θ:

- H.K.Jenssen and T.Karper (2010) Global existence of weak solution in 1-D under the assumption $\mu = \alpha$, $\kappa(\theta) = \nu\theta^b$ for $b \in [0, \frac{3}{2})$.
- H. Liu, T.Yang, H.Zhao and Q.Zou(2014) Global existence of a smooth nonvacuum solution is obtained in 1-D with $\mu = \mu(\theta) > 0$, $\kappa = \kappa(\theta) > 0$ for $\theta > 0$.
- W.Zhang (2014) Global existence of variational solution with finite energy.

Introduction

- R.Pan and W.Zhang (2015) Global strong solution away from vacuum in 1-D for $b \in (0, \infty)$.
- T, Wang and H.Zhao (2016) Global existence and uniqueness non-vacuum solution in 1-d to its Cauchy problem with viscosity coefficient depends on both the density and the temperature.

Introduction

Recently, the local and global well-posedness of classical solutions with vacuum to the isentropic compressible Navier-Stokes with degenerate viscosities has been obtained by Y.Li, R.Pan, Z.Xin and S.Zhu.

Qn: How about the well-posedness for classical solutions of Full-Navier-Stokes equations in 3-D with the degenerated viscosities and far field vacuum?

Main difficulties:

- Strong degeneracy near the vacuum.
- Strong nonlinearity in viscosities.

Introduction

We consider the following case

$$\mu(\theta) = \alpha\theta^b, \quad \lambda(\theta) = \beta\theta^b, \quad \kappa(\theta) = 0, \tag{2.6}$$

where $b > 0$. By using of the relation of $\theta = \frac{A}{R}\rho^{\gamma-1}e^{\frac{S}{c_v}}$, we can rewrite the equation with respected to the (ρ, u, S)

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ \rho u_t + \rho u \cdot \nabla u + A\rho^\gamma \nabla e^{\frac{S}{c_v}} + Ae^{\frac{S}{c_v}}\nabla\rho^\gamma = \frac{A^b}{R^b}\rho^{(\gamma-1)b}\mathrm{div}(e^{\frac{S}{c_v}b}Q(u)) \\ + \nabla\rho^{(\gamma-1)b}e^{\frac{S}{c_v}b}Q(u), \\ A\rho^\gamma[(e^{\frac{S}{c_v}})_t + u \cdot \nabla e^{\frac{S}{c_v}}] = \frac{A^b}{R^b}(\gamma-1)\rho^{(\gamma-1)b}e^{\frac{S}{c_v}b}H(u), \end{cases} \tag{2.7}$$

Introduction

where

$$\begin{cases} Q(u) = \alpha(\nabla u + (\nabla u^T) + \beta \mathrm{div} u \mathbb{I}_3, \\ Lu = -\mathrm{div} Q(u) = -\alpha \triangle u - (\alpha+\beta)\nabla \mathrm{div} u, \\ H(u) = \frac{\alpha}{2}|\nabla u + (\nabla u)^T|^2 + \beta |\mathrm{div} u|^2 = 2\alpha |Du|^2 + \beta |\mathrm{div} u|^2, \end{cases}$$

and $0 < \delta = (\gamma - 1)b < 1$.

We look for smooth solutions (ρ, u, S) with finite energy to the Cauchy problem for (2.7) with the following initial data and far field behavior:

$$(\rho, u, S)|_{t=0} = (\rho_0(x) > 0, u_0(x), S_0(x)) \quad \text{for} \quad x \in \mathbb{R}^3, \tag{2.8}$$

$$(\rho, u, S)(t, x) \to (0, 0, \bar{S}) \quad as \quad |x| \to \infty \quad \text{for} \quad t \geq 0, \tag{2.9}$$

where $\bar{S}$ is a constant.

Introduction

Definition 1: Let $T > 0$ be a finite constant. A solution (ρ, u, S) to the Cauchy problem $(2.7) - (2.9)$ is called a regular solution in $[0, T] \times \mathbb{R}^3$ if (ρ, u, S) satisfies this problem in the sense of distribution and:

$$\begin{aligned}
&\rho > 0, \rho^{\gamma-1} \in C([0,T]; D^1 \cap D^3), \nabla \rho^{\delta-1} \in C([0,T]; L^q \cap D^{1,3} \cap D^2),\\
&u \in C([0,T]; H^3) \cap L^2([0,T]; H^4), \quad \rho^{\delta-1} \nabla u \in L^\infty([0,T]; D^1),\\
&\rho^{\frac{\delta-1}{2}} \nabla u \in C([0,T]; L^2), \rho^{\delta-1} \nabla^2 u \in L^\infty([0,T]; H^1) \cap L^2([0,T]; D^2),\\
&u_t \in C([0,T]; H^1) \cap L^2([0,T]; D^2),\\
&S - \bar{S} \in C([0,T]; D^1 \cap D^3).
\end{aligned} \tag{2.10}$$

where $q > 3$.

Remark 1: It follows from the Definition 1 that $\nabla\rho^{\delta-1} \in L^\infty$, which means that the vacuum occurs only in the far field.

Remark 2: Note that the conservation of momentum is not clear for the strong solution with vacuum to the flows of constant viscosities (see Y.Choe, H.Choe and H.Kim[2004]). In the sense, the definition of the regular solutions here is consistent with the physical background.

Main theorem

Main Theorem: Let parameters $(\gamma, \delta, \alpha, \beta)$ satisfy

$$\gamma > 1, \quad 0 < \delta < 1, \quad \alpha > 0, \quad 2\alpha + 3\beta \geq 0, \quad \gamma + \delta \leq 2. \tag{2.11}$$

If the initial data (ρ_0, u_0, S_0)satisfy

$$\begin{aligned} &\rho_0 > 0, \quad \rho_0^{\gamma-1} \in L^\infty \cap D^1 \cap D^3, \quad \nabla \rho_0^{\delta-1} \in L^q \cap D^{1,3} \cap D^2, \\ &\nabla \rho_0^{\frac{\delta-1}{2}} \in L^6, \quad u_0 \in H^3, \quad S_0 - \bar{S} \in D^1 \cap D^3, \end{aligned} \tag{2.12}$$

for some $q \in (3, \infty)$,

Main theorem

and the initial layer compatibility conditions:

$$\nabla u_0 = \rho_0^{\frac{1-\delta}{2}} g_1, \quad \operatorname{div}(e^{\frac{S_0}{c_v} b} Q(u_0)) = \rho_0^{1-\delta} g_2,$$
$$\nabla(\rho_0^{\delta-1} \operatorname{div}(e^{\frac{S_0}{c_v} b} Q(u_0))) = \rho_0^{\frac{1-\delta}{2}} g_3, \quad \rho_0^{\gamma-1} \nabla^2 e^{\frac{S_0}{c_v}} = \rho_0^{\frac{1-\delta}{2}} g_4, \tag{2.13}$$

for some $(g_1, g_2, g_3, g_4) \in L^2$, then there exists a time $T^* > 0$ and a unique regular solution (ρ, u, S) in $[0, T^*] \times \mathbb{R}^3$ to the Cauchy problem $(2.7) - (2.9)$. Moreover, (ρ, u, S) is a classical solution to $(2.7) - (2.9)$ in $(0, T^*] \times \mathbb{R}^3$.

Main theorem

Remark 3: We remark that (2.12)-(2.13) identifies a class of admissible initial data that provide unique solvability to our problem (2.16)-(2.18), for example,

$$\rho_0(x) = \frac{1}{(1+|x|)^{2a}}, \quad u_0(x) \in C_0^3(\mathbb{R}^3),$$
$$\frac{1}{4(\gamma-1)} < a < \frac{1-3/q}{2(1-\delta)}, \quad \frac{3}{2} + \frac{\delta}{2} < \gamma + \delta \leq 2,$$
$$S_0 - \bar{S} \in C_0^3(\mathbb{R}^3).$$

Particularly, when ∇u_0 and $\nabla e^{\frac{S_0}{c_v}}$ compactly supported, the compatibility conditions (2.13) are satisfied automatically.

Main theorem

Remark 4: The compatibility conditions (2.13) are also necessary for the existence of regular solution. It plays a key role in the derivation of $u_t \in L^\infty([0,T^*];L^2)$, $\rho^{\frac{\delta-1}{2}}\nabla u_t \in L^\infty([0,T^*];L^2)$, which will be used in the uniform estimates for $|u|_{D^2}$, $|u|_{D^3}$.

Reformation

Introducing some new variables

$$\phi = \frac{A\gamma}{\gamma - 1}\rho^{\gamma-1}, \quad e^{\frac{S}{c_v}} = l, \quad h = \frac{1}{a}\rho^{\delta-1} = \phi^{2\iota}, \quad n = \rho^{2-\delta-\gamma},$$
$$\psi = \frac{\delta}{\delta - 1}\nabla\rho^{\delta-1}, \quad \varphi = h^{-1} = \phi^{-2\iota}, \quad f = \phi\psi = \frac{a\delta}{\delta - 1}\frac{\nabla h}{h}$$
$$a_1 = \frac{\gamma - 1}{\gamma}, \quad a_2 = a\left(\frac{A}{R}\right)^b, \quad a_3 = \left(\frac{A}{R}\right)^b, \quad a_4 = \frac{A^{b-1}a^2(\gamma - 1)}{R^b} \tag{2.14}$$

where

$$\iota = \frac{\delta - 1}{2(\gamma - 1)}, \quad a = \left(\frac{A\gamma}{\gamma - 1}\right)^{\frac{1-\delta}{\gamma-1}}, \tag{2.15}$$

Reformation

the system (2.7) can be rewritten as

$$\begin{cases} \phi_t + u \cdot \nabla \phi + (\gamma - 1)\phi \mathrm{div} u = 0, \\ u_t + u \cdot \nabla u + a_1 \phi \nabla l + l \nabla \phi - a_2 \phi^{2\iota} \mathrm{div}(l^b Q(u)) = a_3 \psi l^b Q(u), \\ l_t + u \cdot \nabla l = a_4 \phi^{4\iota} n l^b H(u), \\ \psi_t + \nabla(u \cdot \psi) + (\delta - 1)\psi \mathrm{div} u + \delta a \phi^{2\iota} \nabla \mathrm{div} u = 0. \end{cases} \tag{2.16}$$

The initial data are given by

$$\begin{aligned} &(\phi, u, l, \psi)|_{t=0} = (\phi_0, u_0, l_0, \psi_0) \\ &= \Big(\frac{A\gamma}{\gamma - 1}\rho_0^{\gamma-1}(x), u_0(x), e^{S_0(x)/c_v}, \frac{\delta}{\delta - 1}\nabla \rho_0^{\delta-1}(x)\Big), \quad x \in \mathbb{R}^3. \end{aligned} \tag{2.17}$$

Reformation

(ϕ, ψ, l, u) also need to satisfy the far filed behavior:

$$(\phi, u, l, \psi) \to (0, 0, \bar{l}, 0), \quad as \quad |x| \to +\infty, \quad t \geq 0. \tag{2.18}$$

where $\bar{l} = e^{\bar{S}/c_v}$.

To prove the Main Theorem, our first step is to establish the following well-posedness to the reformulated problem (2.16) − (2.18).

Reformation

Main difficulties:

- For equation of ψ: Singularity of the source term;
- For equation of u: Singularity of $\phi^{2\iota}$.(Note $\phi^{2\iota} \to \infty$ as $\rho \to 0$).

Reformation

Observation:

- For equation of ψ: Symmetric hyperbolic system + Singularity $a\phi^{2\iota}\nabla \mathrm{div} u = \rho^{\delta-1}\nabla \mathrm{div} u$.
- Regularity estimate for elliptic equation:

$$\begin{cases} \mathrm{div}(\rho^{\delta-1} l^b Q(u)) = f, \\ \rho^{\delta-1} u \to 0, \quad as \quad |x| \to \infty. \end{cases}$$

Reformation

Theorem: Let (2.11) hold. If the initial data $(\phi_0, u_0, l_0, \psi_0)$ satisfies:

$$\begin{aligned} &\phi_0 > 0, \quad (\phi_0, u_0) \in H^3, \quad l_0 - \bar{l} \in D^1 \cap D^3, \quad l_0^{-1} \in L^\infty, \\ &\psi_0 \in L^q \cap D^{1,3} \cap D^2, \quad \nabla \phi_0^\iota \in L^6, \end{aligned} \tag{2.19}$$

and the initial layer compatibility conditions:

$$\begin{aligned} &\nabla u_0 = \phi_0^{-\iota} g_1, \qquad \operatorname{div}(l_0^b Q(u_0)) = \phi_0^{-2\iota} g_2, \\ &\nabla(\phi_0^{2\iota} \operatorname{div}(l_0^b Q(u_0))) = \phi_0^{-\iota} g_3, \quad \phi_0 \nabla^2 l_0 = \phi_0^{-\iota} g_4, \end{aligned} \tag{2.20}$$

for some $(g_1, g_2, g_3, g_4) \in L^2$, then there exists a time $T^* > 0$ and a unique classical solution $(\phi, u, l, \psi = \frac{a\delta}{\delta - 1} \nabla \phi^{2\iota})$ to the Cauchy problem $(2.16) - (2.18)$,

Reformation

satisfying

$$
\begin{aligned}
&\phi \in C([0,T^*];H^3), \quad \nabla\phi/\phi \in L^\infty([0,T^*];L^\infty \cap L^6 \cap D^{1,3} \cap D^2),\\
&\psi \in C([0,T^*];L^q \cap D^{1,3} \cap D^2), \phi^{-2\iota} \in L^\infty([0,T^*];L^\infty \cap L^6 \cap D^{2,3} \cap D^3),\\
&u \in C([0,T^*];H^3) \cap L^2([0,T^*];H^4), \quad \phi^{2\iota}\nabla u \in L^\infty([0,T^*];D^1),\\
&l-\bar{l} \in C([0,T^*];D^1 \cap D^3),
\end{aligned}
\tag{2.21}
$$

Linearized problem for uniformly positive initial density and artificial viscosity

In order to proceed with the nonlinear problem (2.16)-(2.18), we need to consider the corresponding linearized problem.

$$\begin{cases} \phi_t + v \cdot \nabla \phi + (\gamma - 1)\phi \mathrm{div} v = 0, \\ u_t + v \cdot \nabla v + a_1 \phi \nabla l + l \nabla \phi - a_2 \sqrt{h^2 + \epsilon^2} \mathrm{div}(l^b Q(u)) = a_3 \psi l^b Q(v), \\ l_t + v \cdot \nabla l = a_4 l^b g^2 n (2\alpha |Dv|^2 + \beta |\mathrm{div} v|^2), \\ h_t + v \cdot \nabla h + (\delta - 1) g \mathrm{div} v = 0, \\ (\phi, u, l, h)|_{t=0} = (\phi_0, u_0, l_0, h_0), \quad x \in \mathbb{R}^3, \\ (\phi, u, l, h) \to (\phi^\infty, 0, \bar{l}, h^\infty), \quad \text{as } |x| \to +\infty, \quad t > 0, \end{cases} \tag{3.22}$$

Linearized problem for uniformly positive initial density and artificial viscosity

where ϵ is a positive constants

$$h^{\infty} = (\phi^{\infty})^{2\iota}, \psi = \frac{a\delta}{\delta - 1} \nabla h, \tag{3.23}$$

$v = (v^1, v^2, v^3) \in \mathbb{R}^3$ is a known vector and g is a known real function satisfying $(v(0,x), g(0,x)) = (u_0(x), h_0(x))$ and:

$$\begin{aligned}
& g \in L^{\infty} \cap C([0,T] \times \mathbb{R}^3), \quad \nabla g \in C([0,T]; H^2), \quad g_t \in C([0,T]; H^2), \\
& \nabla g_{tt} \in L^2([0,T]; L^2), \quad v \in C([0,T]; H^3) \cap L^2([0,T]; H^4), \\
& t^{\frac{1}{2}} v \in L^{\infty}([0,T]; D^4), \quad v_t \in C([0,T]; H^1) \cap L^2([0,T]; D^2), \\
& v_{tt} \in L^2([0,T]; L^2), \quad t^{\frac{1}{2}} v_t \in L^{\infty}([0,T]; D^2) \cap L^2([0,T]; D^3), \\
& t^{\frac{1}{2}} v_{tt} \in L^{\infty}([0,T]; L^2) \cap L^2([0,T]; D^1).
\end{aligned} \tag{3.24}$$

Linearized problem for uniformly positive initial density and artificial viscosity

Remark 5: Here, we need to linearize (ϕ, u, l, h) instead of (ϕ, u, l, ψ). Suppose we use the equation of ψ in the linearized system, we can have

$$\psi_t + \nabla(u \cdot \psi) + (\delta - 1)\psi \mathrm{div} u + \delta a g \nabla \mathrm{div} v = 0.$$

Then the relationship

$$\psi = \frac{a\delta}{\delta - 1} \nabla \phi^{2\iota}$$

will be destroyed.

Linearized problem for uniformly positive initial density and artificial viscosity

Then the L^2- estimate for u will encounter an obvious difficulty:

$$\frac{d}{dt}|u|_2^2 + a_2\alpha|\phi^\iota l^b \nabla u|_2^2 + a_2(\alpha+\beta)|\phi^\iota l^b \mathrm{div} u|_2^2$$
$$= -\int (v\cdot\nabla v + \ldots + a_2 l^b \underbrace{\nabla \phi^{2\iota}}_{\neq \psi} \cdot Q(u)) \cdot u.$$

That is, there is no way to control the term $\nabla\phi^{2\iota}\cdot Q(u)$!

Linearized problem for uniformly positive initial density and artificial viscosity

Lemma

Let (2.11) *hold and* $\epsilon > 0$*. Assume that* $(\phi_0, u_0, l_0, h_0 = (\phi_0)^{2\iota})$ *satisfies*

$$\begin{aligned} &\phi_0 > \eta, \quad \phi_0 - \phi^\infty \in H^3, \quad u_0 \in H^3, \\ &l_0 - \bar{l} \in D^1 \cap D^3, \quad l_0^{-1} \in L^\infty \end{aligned} \tag{3.25}$$

for some constant $\eta > 0$*. Then for any* $T > 0$*, there exists a unique classical solution* (ϕ, u, l, h) *to* (3.22) *such that*

Linearized problem for uniformly positive initial density and artificial viscosity

$$
\begin{aligned}
&\phi - \phi^{\infty} \in C([0,T]; D^1 \cap D^3), \quad h \in L^{\infty} \cap C([0,T] \times \mathbb{R}^3),\\
&\nabla h \in C([0,T]; H^2), \quad h_t \in C([0,T]; H^2),\\
&u \in C([0,T]; H^3) \cap L^2([0,T]; H^4), \quad l - \bar{l} \in C([0,T]; D^1 \cap D^2),\\
&l_t \in C([0,T^*]; D^1), \quad l_{tt} \in L^2([0,T^*]; D^1).
\end{aligned}
\tag{3.26}
$$

A priori estimates independent of (ϵ, η).

Now we fix $T > 0$ and a positive constant c_0 large enough such that

$$
\begin{aligned}
&2 + \phi^\infty + \bar{l} + \|\phi_0\|_{D^1\cap D^2} + \|u_0\|_3 + \|h_0^{-1}\|_{L^\infty\cap D^{1,6}\cap D^{2,3}\cap D^3} \\
&+ \|\nabla \log h_0\|_{L^\infty\cap L^6\cap D^{1,3}\cap D^2} + \|\nabla h_0\|_{L^q\cap D^{1,3}\cap D^2} + |\nabla \sqrt{h_0}|_6 \\
&+ |g_1|_2 + |g_2|_2 + |g_3|_2 + |g_4|_2 + \|l_0 - \bar{l}\|_{D^1\cap D^3} + |l_0^{-1}|_\infty \le c_0.
\end{aligned} \tag{3.27}
$$

We assume that there exist some time $T^* \in (0, T]$ and constants $c_i (i = 1, \cdots, 5)$ such that

$$
1 < c_0 \le c_1 \le c_2 \le c_3 \le c_4 \le c_5, \tag{3.28}
$$

and

A priori estimates independent of (ϵ, η).

$$
\begin{aligned}
&\sup_{0\le t\le T^*} \|\nabla g(t)\|^2_{L^q\cap D^{1,3}\cap D^2} \le c_1^2,\\
&\sup_{0\le t\le T^*} |v(t)|^2_{D^1} + \int_0^{T^*} (|v|^2_{D^2} + |v_t|^2_6)dt \le c_2^2,\\
&\sup_{0\le t\le T^*} (|v|^2_{D^2} + |v_t|^2_6 + |g\nabla^2 v|^2_2) + \int_0^{T^*} (|v|^2_{D^3} + |v_t|^2_{D^1})dt \le c_3^2,\\
&\sup_{0\le t\le T^*} (|v|^2_{D^3} + |v_t|^2_{D^1} + |g_t|^2_{D^1}) + \int_0^{T^*} (|v|^2_{D^4} + |v_t|^2_{D^2} + |v_{tt}|^2_2)dt \le c_4^2,\\
&\sup_{0\le t\le T^*} (|g\nabla^2 v|^2_{D^1} + |g_t|^2_\infty)(t) + \int_0^{T^*} (|(g\nabla^2 v)_t|^2_2 + |g\nabla^2 v|^2_{D^2})dt \le c_4^2,\\
&\sup_{0\le t\le T^*} t(|v_t|^2_{D^2} + |v|^2_{D^4} + |v_{tt}|^2_2)(t) + \int_0^{T^*} t(|v_{tt}|^2_{D^1} + |v_t|^2_{D^3})dt \le c_5^2.
\end{aligned}
\tag{3.29}
$$

A priori estimates independent of (ϵ, η).

T^* and $c_i (i = 1, \cdots, 5)$ will be determined later, and depend only on c_0 and the fixed constants $(\alpha, \beta, \gamma, \delta, T)$.

Under the condition $(3.27) - (3.30)$, we can obtain that

$$
\begin{aligned}
&(\|\phi - \phi^\infty\|_3^2 + \|\phi_t\|_2^2 + |\phi_{tt}|_2^2)(t) + \int_0^t \|\phi_{tt}\|_1^2 ds \leq c_5^2, \\
&\|\psi(t)\|^2_{L^q \cap D^{1,3} \cap D^2} \leq c_1^2, \quad |h_t|_\infty^2 + |\psi_t(t)|_2 \leq c_4^2, \\
&|\psi_t(t)|^2_{D^1} + \int_0^t (|\psi_{tt}|_2^2 + |h_{tt}|_6^2) ds \leq c_5^2, \quad h > \frac{1}{2c_0}, \\
&\frac{2}{3}\eta^{-2\iota} < \varphi, \quad (\|\varphi\|^2_{L^\infty \cap D^{1,6} \cap D^{2,3} \cap D^3} + \|f\|^2_{L^\infty \cap L^6 \cap D^{1,3} \cap D^2})(t) \leq c_2^2, \\
&(\|\varphi_t\|^2_{L^6 \cap D^{1,3} \cap D^2} + \|f_t\|_{L^3 \cap D^1})(t) \leq c_5^2, \\
&|l|_\infty \geq \frac{1}{c_0}, \quad \|l - \bar{l}\|^2_{D^1 \cap D^3}(t) \leq c_1^2, \quad |l_t|^2_{L^\infty \cap L^3} \leq c_5^2,
\end{aligned}
$$

A priori estimates independent of (ϵ, η).

$$
\begin{aligned}
& |l_t|^2_{D^1\cap D^2} + \int_0^t |\nabla l_{tt}|_2^2 ds \leq c_5^2, \\
& |\sqrt{h}\nabla u(t)|_2^2 + \|u(t)\|_1^2 + \int_0^t (\|\nabla u\|_1^2 + |u_t|_2^2) ds \leq c_2^2, \\
& (|u|^2_{D^2} + |h\nabla^2 u|_2^2 + |u_t|_2^2)(t) + \int_0^t (|u|^2_{D^3} + |h\nabla^2 u|^2_{D^1} + |u_t|^2_{D^1}) ds \leq c_3^2, \\
& (|\sqrt{h}\nabla u_t|_2^2 + |u_t|^2_{D^1} + |u|^2_{D^3} + |h\nabla u|^2_{D^1} + |h\nabla^2 u|^2_{D^1})(t) \leq c_4^2, \\
& \int_0^t (|u_t|^2_{D^2} + |u_{tt}|_2^2 + |u|^2_{D^4} + |h\nabla^2 u|^2_{D^2} + |(h\nabla^2 u)_t|_2^2) ds \leq c_4^2, \\
& t(|u_t|^2_{D^2} + |u_{tt}|_2^2 + |u|^2_{D^4})(t) + \int_0^t (s|u_{tt}|^2_{D^1} + s|u_t|^2_{D^3}) ds \leq c_5^2,
\end{aligned}
\tag{3.30}
$$

for $0 \leq t \leq T_*$.

passing to the limit $\epsilon \to 0$.

With the help of the (ϵ, η)-independent estimates established in (3.30), we can established the local existence result for the following linearized problem without artificial viscosity (i.e.,$\epsilon \to 0$) under the assumption $\phi_0 \geq \eta$,

$$\begin{cases} \phi_t + v \cdot \nabla \phi + (\gamma - 1)\phi \mathrm{div} v = 0, \\ u_t + v \cdot \nabla v + a_1 \phi \nabla l + l \nabla \phi - a_2 h \mathrm{div}(l^b Q(u)) = a_3 \psi l^b Q(v), \\ l_t + v \cdot \nabla l = a_4 l^b g^2 n(2\alpha |Dv|^2 + \beta |\mathrm{div} v|^2), \\ h_t + v \cdot \nabla h + (\delta - 1) g \mathrm{div} v = 0, \\ (\phi, u, l, h)|_{t=0} = (\phi_0, u_0, l_0, h_0) = (\phi_0, u_0, l_0, (\phi_0)^{2\iota}), \quad x \in \mathbb{R}^3, \\ (\phi, u, l, h) \to (\phi^\infty, 0, \bar{l}, h^\infty = (\phi_\infty)^{2\iota}), \quad \text{as} \quad |x| \to +\infty, \quad t > 0. \end{cases} \tag{3.31}$$

Also, (ϕ, u, l, h) satisfies the estimates in (3.30) independent of η.

Construction of the nonlinear approximation solutions away from vacuum.

Based on the assumption that $\phi_0 > \eta$, we can prove the local-in-time well-posedness of the classical solution to the following Cauchy problem

$$\begin{cases} \phi_t + u \cdot \nabla \phi + (\gamma - 1)\phi \mathrm{div} u = 0, \\ u_t + u \cdot \nabla u + a_1 \phi \nabla l + l \nabla \phi - a_2 \phi^{2\iota} \mathrm{div}(l^b Q(u)) = a_3 \psi l^b Q(u), \\ l_t + u \cdot \nabla l = a_4 \phi^{4\iota} n l^b H(u), \\ \psi_t + \nabla (u \cdot \psi) + (\delta - 1)\psi \mathrm{div} u + \delta a \phi^{2\iota} \nabla \mathrm{div} u = 0. \\ (\phi, u, l, \psi)|_{t=0} = (\phi_0, u_0, l_0, \psi_0) = (\phi_0, u_0, l_0, \frac{a\delta}{\delta - 1} \nabla (\phi_0)^{2\iota}), \quad x \in \mathbb{R}^3 \\ (\phi, u, l, \psi) \longrightarrow (\phi^\infty, 0, \bar{l}, 0), \quad as \quad |x| \longrightarrow \infty, \quad t \geq 0, \end{cases} \tag{3.32}$$

Construction of the nonlinear approximation solutions away from vacuum.

Theorem: Let (2.11) hold and ϕ^∞ be a positive constant. Assume that the initial data $(\phi_0, u_0, l_0, h_0, \psi_0 = \frac{a\delta}{\delta-1}\nabla(\phi_0)^{2\iota})$ satisfies (3.25), and there exists a positive constant c_0 independent of η such that (3.27) holds. Then there exist a time $T^* > 0$ and a unique classical solution $(\phi, u, l, h = \phi^{2\iota}, \psi = \frac{a\delta}{\delta-1}\nabla h)$ in $[0, T^*] \times \mathbb{R}^3$ to the Cauchy problem (3.32) satisfying (3.26) and

$$\phi^{-2\iota} \in L^\infty([0, T^*]; L^\infty \cap D^{1,6} \cap D^{2,3} \cap D^3),$$

$$\nabla\phi/\phi \in L^\infty([0, T^*]; L^\infty \cap L^6 \cap D^{1,3} \cap D^2),$$

Construction of the nonlinear approximation solutions away from vacuum.

where T^* is independent of η. Moreover, if the initial data satisfies (3.25), then the estimates (3.30) hold for (ϕ, u, l, h) with T_* replaced by T^*, and are independent of η.

Main idea: The proof of this theorem is given by an iteration scheme based on the estimates for the linearized problem (3.31).

Taking limit from the non-vacuum flows to the flow with far field vacuum.

Based on the local (in time) estimates in (3.30), now we can prove our Main Theorem. In order to obtain this result, we need the following property for the local uniform positivity of ϕ.

Lemma: For any $R_0 > 0$ and $\eta \in (0, 1]$, there exists a constant a_{R_0} such that

$$\phi^\eta(t, x) \geq a_{R_0}, \quad \forall \quad (t, x) \in [0, T^*] \times B_{R_0}, \tag{3.33}$$

where a_{R_0} is independent of η.

Non-existence of global solution with L^∞ decay on u.

Qn: Whether the local solutions obtained in Main Theorem can be extended to the globally in time and what the large time behavior is.

In contract to the classical theory phenomenon for constant viscosity case that such an extension is impossible if the velocity field decay to zero as $t \longrightarrow +\infty$.

Non-existence of global solution with L^∞ decay on u.

Definition 2: Let $T > 0$ be any constant. For the Cauchy problem $(2.7) - (2.9)$, a classical solution (ρ, u, S) is said to be in $D(T)$ if (ρ, u, S) satisfies the following conditions:

- (B) Conservation of total mass:$0 < m(0) = m(t) < \infty$ for any $t \in [0, T]$;
- (C) Conservation of momentum: $0 < |\mathbb{P}(0)| = |\mathbb{P}(t)| < \infty$ for any $t \in [0, T]$;
- (D)Conservation of momentum: $0 < E(0) = E(t) < \infty$ for any $t \in [0, T]$.

Then we have

Non-existence of global solution with L^∞ decay on u.

Theorem

Let parameters $(\gamma, \delta, \alpha, \beta)$ *satisfy*

$$\gamma > 1, \quad \delta \geq 0, \quad \alpha > 0, \quad 2\alpha + 3\beta \geq 0.$$

Then for Cauchy problem $(2.7) - (2.9)$, *there is no classical solution* $(\rho, u, S) \in D(\infty)$ *with*

$$\lim_{t\to\infty} \sup |u(t,x)|_\infty = 0.$$

Non-existence of global solution with L^∞ decay on u.

Corollary

Let (2.11) *hold and*

$$1 < \gamma \le \frac{7}{6}, \tag{4.34}$$

assume that $m(0) > 0$ *and* $|\mathbb{P}(0)| > 0$. *Then for the Cauchy problem* $(2.7) - (2.9)$, *there is no global regular solution* (ρ, u, S) *defined in Definition 1 satisfying the following decay*

$$\lim_{t\to\infty} \sup |u(t,x)|_\infty = 0.$$

Non-existence of global solution with L^∞ decay on u.

Remark 6: Also, we show that if γ satisfied (4.34), the solution we obtained satisfied the conservation of mass, momentum and energy.

Thank You!