The Vlasov-Nordström-Fokker-Planck system

Tong Yang

City University of Hong Kong

Joint work with Renjun Duan and Shuangqian Liu

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Cauchy problem

The Vlasov-Nordstr$\ddot{o}$m-Fokker-Planck (VNFP) system:

$$\begin{cases} \partial_t F + \nabla_p \left(\sqrt{e^{2\phi} + p^2} \right) \cdot \nabla_x F - \nabla_x \left(\sqrt{e^{2\phi} + p^2} \right) \cdot \nabla_p F \\ \qquad = e^{2\phi} \nabla_p \cdot (\Lambda_\phi \nabla_p F + \beta p F), \; t > 0, \; x \in \mathbb{R}^3, \; p \in \mathbb{R}^3, \\ (\partial_t^2 - \Delta)\phi = -e^{2\phi} \int_{\mathbb{R}^3} \frac{F(t,x,p)}{\sqrt{e^{2\phi}+p^2}} dp + \rho(x), \end{cases} \tag{1}$$

with the initial data

$$F(0,x,p) = F_0(x,p), \; \phi(0,x) = \phi_0(x), \; \partial_t \phi(0,x) = \phi_1(x).$$

- $F = F(t,x,p) \geq 0$ is the density distribution of the particles,
- ϕ gravitational field, $\beta \geq 0$ friction coefficient,
- relativistic diffusion matrix

$$\Lambda_\phi = (\Lambda_\phi)_{ij} = \frac{e^{2\phi} I + p \otimes p}{\sqrt{e^{2\phi} + p^2}}.$$

Background of the model

Consider the collision particles under the influence of the self-generated gravitational force governed by the Vlasov-Fokker-Planck system with space-time given by the Lorentzian manifold $(\mathbb{R}^4, g)$*, where*

$$g = e^{2\phi}\eta.$$

Here $\eta = diag(-1,1,1,1)$ *is the canonical Minkowski metric. Felix-Calogero derived the VNFP by applying the Nordström theory instead of the general relativity as a toy model of the very complicated Einstein-Vlasov system.*

Jüttner solution and stationary solution ($\beta = \rho(x) = 1$):

$$J(p) = C_J e^{-p_0},\ p_0 = \sqrt{1+p^2}, \quad [F,\phi] = [J(p), 0],$$

with $\int_{\mathbb{R}^3} J(p) p_0^{-1} dp = 1$. Note that $J_\phi = C_J e^{-\sqrt{e^{2\phi}+p^2}}$ satisfies $\Lambda_\phi \nabla_p J_\phi + p J_\phi = 0$. Set $F = J_\phi + J^{1/2} f$, then VNFP becomes

$$\begin{cases} \partial_t f + \nabla_p\left(\sqrt{e^{2\phi}+p^2}\right)\cdot\nabla_x f - \nabla_x\left(\sqrt{e^{2\phi}+p^2}\right)\cdot\nabla_p f \\ \quad -\frac{e^{2\phi}\partial_t\phi}{\sqrt{e^{2\phi}+p^2}} J_\phi J^{-1/2} + \frac{1}{2}\nabla_x\left(\sqrt{e^{2\phi}+p^2}\right)\cdot\frac{p}{\sqrt{1+p^2}} f \\ \quad = e^{2\phi} J^{-1/2}\nabla_p\cdot\left[\Lambda_\phi\nabla_p(J^{1/2}f) + pJ^{1/2}f\right] \overset{\text{def}}{=} -e^{2\phi} L_\phi f, \\ (\partial_t^2 - \Delta)\phi = -e^{2\phi}\int_{\mathbb{R}^3}\frac{J_\phi}{\sqrt{e^{2\phi}+p^2}}dp - e^{2\phi}\int_{\mathbb{R}^3}\frac{J^{1/2}f}{\sqrt{e^{2\phi}+p^2}}dp + \rho(x). \end{cases} \tag{2}$$

Linearized collision operator L

$$Lf = L_0 f = -J^{-1/2}\nabla\cdot(\Lambda_0\nabla_p(J^{1/2}f)+pJ^{1/2}f), \quad \Lambda_0 = (\Lambda_0)_{ij} = \frac{I+p\otimes p}{p_0}:$$

- The null space: $\mathrm{span}\{J^{1/2}\}$ (only mass is conserved);
- Positivity. Define $\{\mathbf{I}-\mathbf{P}_0\}f = f-\mathbf{P}_0 f$ with $\mathbf{P}_0 f = \frac{\langle f, J^{1/2}\rangle}{\langle J,1\rangle}J^{1/2} \stackrel{\text{def}}{=} a(t,x)J^{1/2}$, then

$$\langle Lf, f\rangle = \int_{\mathbb{R}^3} J\nabla_p\left(\frac{\{\mathbf{I}-\mathbf{P}_0\}f}{\sqrt{J}}\right)\Lambda_0\nabla_p\left(\frac{\{\mathbf{I}-\mathbf{P}_0\}f}{\sqrt{J}}\right)dp \geq 0;$$

- Coercivity estimate.

$$\langle Lf, f\rangle \gtrsim |(\mathbf{I}-\mathbf{P}_0)f|_D^2.$$

L_ϕ has similar properties as L if ϕ is small.

Norms

Here

$$|f|_D^2 = \int_{\mathbb{R}^3} \left\{ \Lambda_0^{ij} \partial_{p_i} f \partial_{p_j} f + \frac{1}{4}\frac{p^2}{p_0} f^2 \right\} dp$$
$$= \int_{\mathbb{R}^3} \frac{1}{p_0} \left| \frac{p}{|p|} \times \nabla_p f \right|^2 + p_0 \left| \frac{p}{|p|} \cdot \nabla_p f \right|^2 dp + \int_{\mathbb{R}^3} \frac{1}{4}\frac{p^2}{p_0} f^2 dp.$$

Let $w(p) = p_0$ and $l \in \mathbb{R}$, define

$$\|w^l f\|^2 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} w^{2l} |f|^2 dx dp, \quad \|w^l f\|_D^2 = \int_{\mathbb{R}^3 \times \mathbb{R}^3} |w^l f|_D^2 dx,$$

and

$$|w^l f|_D^2 = \int_{\mathbb{R}^3} w^{2l} \left\{ \Lambda_0^{ij} \partial_{p_i} f \partial_{p_j} f + \frac{1}{4}\frac{p^2}{p_0} f^2 \right\} dp.$$

Norms

The energy functionals:

$$\mathcal{E}_{N,l}(t) \sim \sum_{|\alpha|\leq N} \left\{ \|\partial^\alpha f\|^2 + \|\partial_t \partial^\alpha \phi\|^2 + \|\nabla_x \partial^\alpha \phi\|^2 + \|\partial^\alpha \phi\|^2 \right\}$$
$$+ \sum_{|\alpha|+|\beta|\leq N, |\alpha|\leq N-1} \left\| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}_0\} f \right\|^2,$$

$$\mathcal{E}^h_{N,l}(t) \sim \sum_{1\leq|\alpha|\leq N} \left\{ \|\partial^\alpha a\|^2 + \|\partial_t \partial^\alpha \phi\|^2 + \|\nabla_x \partial^\alpha \phi\|^2 \right\}$$
$$+ \sum_{|\alpha|\leq N} \|\partial^\alpha \{\mathbf{I}-\mathbf{P}_0\} f\|^2 + \sum_{|\alpha|+|\beta|\leq N, |\alpha|\leq N-1} \left\| w^l \partial^\alpha_\beta \{\mathbf{I}-\mathbf{P}_0\} f \right\|^2,$$

and dissipation rate

$$\mathscr{D}_{N,l}(t) = \sum_{|\alpha|\leq N} \|\partial^\alpha\{\mathbf{I}-\mathbf{P}_0\}f\|_D^2 + \sum_{|\alpha|\leq N-2} \|\partial^\alpha \nabla_x \partial_t \phi\|^2 + \sum_{|\alpha|\leq N-1} \|\nabla_x \partial^\alpha [a,\phi]\|^2 + \sum_{|\alpha|+|\beta|\leq N, |\alpha|\leq N-1} \|w^l \partial_\beta^\alpha \{\mathbf{I}-\mathbf{P}_0\}f\|_D^2,$$

where $l \geq 0$.

Main results

Theorem (Global Existence)

Let $\beta = \rho(x) = 1$, *and* $[f_0, \phi_0, \phi_1]$ *satisfy* $F(0,x,p) = J_{\phi_0}(p) + J^{1/2}(p)f_0(x,p) \geq 0$, *there exists* $\mathscr{E}_{N,l}(t)$ *with* $N \geq 4$ *and* $l \geq 0$ *such that if* $\mathscr{E}_{N,l}(0)$ *is sufficiently small, then there exists a unique global classical solution* $\left[f(t,x,p), \phi(t,x)\right]$ *to the Cauchy problem* (1) *and* (2) *with* $F(t,x,p) = J_\phi(p) + J^{1/2}(p)f(t,x,p) \geq 0$ *and*

$$\mathscr{E}_{N,l}(t) + \int_0^t \mathscr{D}_{N,l}(s)ds \lesssim \mathscr{E}_{N,l}(0),$$

for all time $t \geq 0$.

Main results

Theorem (Time decay rates)

Denote $l \wedge 1 = \max\{l, 1\}$ *with* $l \geq 0$. *Let*

$$Y_0 = \sqrt{\mathscr{E}_{N,l\wedge 1}(0)} + \|[\phi_0, \nabla_x \phi_0, \phi_1]\|_{H^N \cap L^1} + \|f_0\|_{Z_1}$$

be small enough, if $N \geq 8$, *then*

$$\mathscr{E}_{N-2,l}(t) \lesssim (1+t)^{-3/2} Y_0^2,$$

and if $N \geq 11$, *then*

$$\mathscr{E}^h_{N-7,l}(t) \lesssim (1+t)^{-5/2} Y_0^2.$$

Related works

(Work for the Vlasov-Nordstrom system)

- *Calogero-Rein [2004], Global weak solution to VN.*
- *Calogero [2005], Global classical solution to VN.*
- *Fajman-Joudioux-Smulevici [2017] and X. Wang [2018], Global classical solutions to VN via vector field method.*

Related works

(Previous results for VNFP)

- *Alcántar-Calogero [2011], derivation of the VNFP*
- *Alcántar-Calogero-Pankavich [2014], global strong solution for spatial homogeneous VNFP system via some theory of stochastic differential equations and diffusion processes.*

Vanishing diffusion

(Spatially homogeneous system, Felix-Calogero-Pankavich, 2014)

Under the assumption of $\beta = \rho(x) = 0$, *they show:*

- *Global existence and uniqueness when initial data is in* $L^1_1 \cap H^1$;
- *Long time asymptotically to a non trivial profile;*
- $\phi \sim -ct$.

Wave structure

$$(\partial_t^2 - \Delta)\phi = -e^{2\phi}\int_{\mathbb{R}^3} \frac{J_\phi}{\sqrt{e^{2\phi}+p^2}}dp - e^{2\phi}\int_{\mathbb{R}^3} \frac{J^{1/2}f}{\sqrt{e^{2\phi}+p^2}}dp + 1.$$

Expand

$$\frac{J_\phi}{\sqrt{e^{2\phi}+p^2}} - \frac{J}{\sqrt{1+p^2}} = -\frac{1+p_0}{p_0^3}J\phi + \frac{1}{2}\partial_{\phi\phi}\left(\frac{J_\phi}{\sqrt{e^{2\phi}+p^2}}\right)\bigg|_{\phi=\theta_3\phi}\phi^2$$

$\Rightarrow$ Klein-Gordon type equation.

Regularity loss type decay structure

Consider the linearized equations:

$$\begin{cases} \partial_t f + \frac{p}{p_0}\cdot\nabla_x f - \frac{\partial_t\phi}{p_0}J^{1/2} + Lf = h, \\ (\partial_t^2 - \Delta + 2 - C_4)\phi = -\int_{\mathbb{R}^3}\frac{J^{1/2}f}{p_0}dp + g, \end{cases} \tag{3}$$

where $C_4 = \int_{\mathbb{R}^3}\frac{p_0+1}{p_0^3}Jdp < 2$.

(Estimate in Fourier space)

$$\partial_t E(f,\phi)(t,\xi) + \lambda\frac{|\xi|^2}{(1+|\xi|^2)^2}E(f,\phi)(t,\xi) \le 0,$$

where

$$E(f,\phi)(t,\xi) \sim |\hat{f}|^2 + |\partial_t\hat{\phi}|^2 + |\xi|^2|\hat{\phi}|^2 + |\hat{\phi}|^2.$$

Sketch of the proof

Inner products of $(3)_1$ with $\psi_0 = J^{1/2}$, $\psi_1 = (|p|^2 - \frac{C_3}{C_1})J^{1/2}$ and $\psi_2 = pJ^{1/2}$ give

$$\begin{cases} C_1\partial_t a - \partial_t\phi + \nabla_x \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}f, \psi_0 \right\rangle = \langle h, \psi_0\rangle, \\ (C_2 - \frac{C_3}{C_1})\partial_t\phi + \langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}f, \psi_1\rangle + \nabla_x \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}f, \psi_1 \right\rangle \\ \qquad = \langle h - Lf, \psi_1\rangle, \\ \langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}f, \psi_2\rangle + \frac{C_2}{3}\nabla_x a + \nabla_x \cdot \left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}f, \psi_2 \right\rangle = \langle h - Lf, \psi_2\rangle. \end{cases} \tag{4}$$

Here,

$$\int_{\mathbb{R}^3} J dp = C_1, \ \int_{\mathbb{R}^3} \frac{|p|^2 J}{p_0} dp = C_2, \ \int_{\mathbb{R}^3} |p|^2 J dp = C_3.$$

Fourier transform in x gives

$$
\begin{cases}
\partial_t \hat{f} + \frac{i\xi\cdot p}{p_0}\hat{f} - \frac{\partial_t\hat{\phi}}{p_0}J^{1/2} + L\hat{f} = 0, \\
(\partial_t^2 + \xi^2 + 2 - C_4)\hat{\phi} = -\int_{\mathbb{R}^3}\frac{J^{1/2}\hat{f}}{p_0}dp, \\
C_1\partial_t\hat{a} - \partial_t\hat{\phi} + i\xi\cdot\left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_0\right\rangle = 0, \\
(C_2 - \frac{C_3}{C_1})\partial_t\hat{\phi} + \left\langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_1\right\rangle + i\xi\cdot\left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_1\right\rangle = -\langle L\hat{f}, \psi_1\rangle, \\
\left\langle \partial_t\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_2\right\rangle + \frac{\mathscr{C}_2}{3}i\xi\hat{a} + i\xi\cdot\left\langle \frac{p}{p_0}\{\mathbf{I}-\mathbf{P}_0\}\hat{f}, \psi_2\right\rangle = -\langle L\hat{f}, \psi_2\rangle.
\end{cases}
\tag{5}
$$

The first two requations give

$$\partial_t \left[|\hat{f}|^2 + |\partial_t \hat{\phi}|^2 + |\xi|^2 |\hat{\phi}|^2 + (2 - \mathscr{C}_4)|\hat{\phi}|^2 \right] + \lambda |\{\mathbf{I} - \mathbf{P}_0\}\hat{f}|_D^2 \le 0. \quad (6)$$

This combining with the estimation on

$$\partial_t \left(\left\langle \{\mathbf{I} - \mathbf{P}_0\}\hat{f}, \psi_2 \right\rangle | i\xi \hat{a} \right) \Rightarrow |\xi|^2 |\hat{a}|^2$$

$$\partial_t \left(\xi \left\langle \{\mathbf{I} - \mathbf{P}_0\}\hat{f}, \psi_1 \right\rangle | \xi \partial_t \hat{\phi} \right) \Rightarrow |\xi|^2 |\partial_t \hat{\phi}|^2$$

and

$$\partial_t \left(\xi \partial_t \hat{\phi} | \xi \hat{\phi} \right) \Rightarrow |\xi|^2 (|\xi|^2 + 1) |\hat{\phi}|^2$$

give the desired decay estimate.

Comparison with other kinetic models

(Standard decay structure)

Heat kernel type in low frequency regime and exponential decay in high frequency regime.

- *Boltzmann equation, Vlasov-Poisson-Boltzmann system, ...*
- *Hyperbolic-parabolic system, cf. Kawashima and many others*

(Regularity loss type)

- *Vlasov-Maxwell-Boltzmann system, ...*
- *Timoshenko system, Euler-Maxwell system, cf. Kawashima and many others*

Decay estimate on the linearized system

Let $U = U_1 = [f, \phi, \nabla_x \phi, \partial_t \phi]$, then formally

$$U(t) = S(t)U_0 + \int_0^t S(t-s)[h, g, 0, 0]\, ds.$$

Hence

$$\|\nabla_x^m S(t)U_0\| \lesssim (1+t)^{-\frac{3}{4}-\frac{m}{2}} \left(\|f_0\|_{Z_1} + \|[\phi_0, \phi_1, \nabla_x \phi_0]\|_{L_x^1} \right) + (1+t)^{-\frac{j}{2}} \|\nabla_x^{m+j}[f_0, \phi_0, \phi_1, \nabla_x \phi_0]\|.$$

Energy estimates

(Estimate on the energy functional)

$$\frac{d}{dt}\mathscr{E}_{N,l}(t) + \lambda \mathscr{D}_{N,l}(t) \leq 0,$$

and

$$\frac{d}{dt}\mathscr{E}^h_{N,l}(t) + \lambda \mathscr{D}_{N,l}(t) \leq 0,$$

where $N \geq 4$ *and* $l \geq 0$.

These estimates together with the decay estimate on the linearized system give the proof of the theorems

The local existence will be established based on the following iteration regime

$$\begin{aligned}
\partial_t f^{n+1} &+ \nabla_p\left(\sqrt{e^{2\phi^n}+p^2}\right)\cdot\nabla_x f^{n+1} - \nabla_x\left(\sqrt{e^{2\phi^n}+p^2}\right)\cdot\nabla_p f^{n+1} \\
&- \frac{e^{2\phi^n}\partial_t\phi^n}{\sqrt{e^{2\phi^n}+p^2}} J_{\phi^n} J^{-1/2} + \frac{e^{2\phi^{n+1}}\partial_t\phi^{n+1}}{\sqrt{e^{2\phi^{n+1}}+p^2}} J^{1/2} - \frac{e^{2\phi^{n+1}}\partial_t\phi^{n+1}}{\sqrt{e^{2\phi^{n+1}}+p^2}} J^{1/2} \\
&+ \frac{1}{2}\nabla_x\left(\sqrt{e^{2\phi^n}+p^2}\right)\cdot\frac{p}{\sqrt{1+p^2}} f^{n+1} + e^{2\phi^n} L f^{n+1} \\
&= e^{2\phi^n} L f^{n+1} - e^{2\phi^n} L_{\phi^n} f^{n+1},
\end{aligned}$$

$$(\partial_t^2-\Delta)\phi^n = -e^{2\phi^n}\int_{\mathbb{R}^3}\frac{J_{\phi^n}}{\sqrt{e^{2\phi^n}+p^2}}dp - e^{2\phi^n}\int_{\mathbb{R}^3}\frac{J^{1/2}f^n}{\sqrt{e^{2\phi^n}+p^2}}dp + 1,$$

starting from

$$f^0(t,x,p) = f_0(x,p),\ \phi^0(t,x) = \phi_0(x).$$

Remarks

- If the background density is nontrivial, for instance a function depending on space variable only but small in some sense, we believe the approaches presented here are also available.
- The vanishing diffusivity phenomena when $\beta = \rho(x) = 0$.

(Vlasov-Poisson-Boltzmann system)

$$\partial_t F_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x F_\varepsilon + \frac{1}{\varepsilon} \nabla_x \Phi_\varepsilon \cdot \nabla_v F_\varepsilon = \frac{1}{\varepsilon^2} Q(F_\varepsilon, F_\varepsilon),$$
$$\Delta_x \Phi_\varepsilon = \int_{\mathbb{R}^3} F_\varepsilon dv - 1,$$

with hard potential

$$Q(f,g) = \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|v - v_*|, \omega)(f'_* g' + f' g'_* - f_* g - f g_*) dv_* d\omega,$$

where

$$f'_* = f(t,x,v'_*), \quad f' = f(t,x,v'), \quad f_* = f(t,x,v_*), \quad f = f(t,x,v),$$
$$v' = v - [(v - v_*) \cdot \omega]\omega, \quad v'_* = v_* + [(v - v_*) \cdot \omega]\omega, \quad \omega \in \mathbb{S}^2.$$

(Works on VPB)

- *Renormaized solution: Mischler, '00;*
- *Perturbative solutions: Guo('01), Y.-Zhao('06), Y.-Yu-Zhao('06), Duan-Y.('10), Duan-Y.-Zhao('13), Duan-Strain('11), ...;*
- *Spectrum structure and Green's function, Li-Y.-Zhong('16,'19);*
- *Compressible fluid limit: Euler-Poisson(Guo-Jang '10); the bipolar VPB system towards a solution to the incompressible Vlasov-Navier-Stokes-Fourier system(Wang '11);*
- . . .

(Incompressible Navier-Stokes limit from the Boltzmann equation)

- *Bardos-Golse-Levermore, ('91, 93);*
- *Bardos-Ukai ('91);*
- $\cdots$;
- *Golse & Saint-Raymond, Navier-Stokes limit ('04);*
- *Guo, diffusive limit beyond Navier-Stokes ('06);*
- *many others,* $\cdots$

Diffusive limit of VPB

Set

$$F_\varepsilon = M + \varepsilon\sqrt{M} f_\varepsilon, \quad \Phi_\varepsilon = \varepsilon\phi_\varepsilon,$$

with

$$M = M(v) = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}, \quad v \in \mathbb{R}^3.$$

(VPB system for perturbation)

$$\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon - \frac{1}{\varepsilon} v\sqrt{M} \cdot \nabla_x \phi_\varepsilon - \frac{1}{\varepsilon^2} L f_\varepsilon = G_1(f_\varepsilon) + \frac{1}{\varepsilon} G_2(f_\varepsilon),$$
$$\Delta_x \phi_\varepsilon = \int_{\mathbb{R}^3} f_\varepsilon \sqrt{M} dv,$$

with the initial condition

$$f_\varepsilon(0,x,v) = f_0(x,v).$$

Here,

$$Lf_\varepsilon = \frac{1}{\sqrt{M}}[Q(M,\sqrt{M}f_\varepsilon) + Q(\sqrt{M}f_\varepsilon, M)],$$
$$G_1(f_\varepsilon) = \frac{1}{2}(v\cdot\nabla_x\phi_\varepsilon)f_\varepsilon - \nabla_x\phi_\varepsilon\cdot\nabla_v f_\varepsilon,$$
$$G_2(f_\varepsilon) = \Gamma(f_\varepsilon, f_\varepsilon) = \frac{1}{\sqrt{M}}Q(\sqrt{M}f_\varepsilon, \sqrt{M}f_\varepsilon).$$

And

$$Lf(v) = (Kf)(v) - \nu(v)f(v),$$

with

$$\nu_0(1+|v|)^\gamma \le \nu(v) \le \nu_1(1+|v|)^\gamma.$$

The nullspace of the operator L is spanned by

$$\chi_0 = \sqrt{M}, \quad \chi_j = v_j\sqrt{M}\ (j = 1,2,3), \quad \chi_4 = \frac{(|v|^2 - 3)\sqrt{M}}{\sqrt{6}}.$$

Denote

$$f = P_0 f + P_1 f,$$
$$P_0 f = \sum_{k=0}^{4}(f,\chi_k)\chi_k,\ P_1 f = f - P_0 f,$$

$$(Lf,f) \leq -\mu\|P_1 f\|^2, \quad f \in D(L),$$

where $D(L)$ is the domains of L given by

$$D(L) = \left\{f \in L^2(\mathbb{R}^3)\,|\,\nu(v)f \in L^2(\mathbb{R}^3)\right\}.$$

(Incompressible Navier-Stokes-Poisson-Fourier (NSPF) system)

$$\nabla_x \cdot m = 0, \quad n + \sqrt{\tfrac{2}{3}}q - \phi = 0,$$
$$\partial_t m - \kappa_0 \Delta_x m + \nabla_x p = n \nabla_x \phi - \nabla_x \cdot (m \otimes m),$$
$$\partial_t \Big(q - \sqrt{\tfrac{2}{3}}n\Big) - \kappa_1 \Delta_x q = \sqrt{\tfrac{2}{3}} m \cdot \nabla_x \phi - \tfrac{5}{3} \nabla_x \cdot (qm),$$
$$\Delta_x \phi = n,$$

where p is the pressure, and the initial data given by

$$m(0) = (f_0, v\chi_0), \quad q(0) - \sqrt{\tfrac{2}{3}}n(0) = \Big(f_0, \chi_4 - \sqrt{\tfrac{2}{3}}\chi_0\Big),$$
$$\nabla_x \cdot m(0) = 0, \quad n(0) - \Delta_x^{-1} n(0) + \sqrt{\tfrac{2}{3}} q(0) = 0.$$

$$\kappa_0 = -(L^{-1}P_1(v_1\chi_2), v_1\chi_2), \quad \kappa_1 = -(L^{-1}P_1(v_1\chi_4), v_1\chi_4).$$

(Theorem Li-Y.-Zhong, 2019))

Let $(f_\varepsilon, \phi_\varepsilon) = (f_\varepsilon(t,x,v), \phi_\varepsilon(t,x))$ *be the global solution to the VPB system and* $(n,m,q,\phi) = (n,m,q,\phi)(t,x)$ *the global solution to the NSPF system. There exists a small constant* $\delta_0 > 0$ *such that if* $\|f_0\|_{H^6_1} + \|f_0\|_{L^2_v(L^1_x)} + \|\nabla_x \Delta_x^{-1}(f_0, \chi_0)\|_{L^p_x} \le \delta_0$ *with* $p \in (1,2)$*, then*

$$\|f_\varepsilon(t) - u(t)\|_{L^\infty_x(L^2_v)} + \|\nabla_x \phi_\varepsilon(t) - \nabla_x \phi(t)\|_{L^\infty_x} \le C\delta_0(\varepsilon^a (1+t)^{-\frac{1}{2}} + (1+\varepsilon^{-1}t)^{-b}),$$

where

$$u(t,x,v) = n(t,x)\chi_0 + m(t,x)\cdot v\chi_0 + q(t,x)\chi_4,$$

with $b = \min\{1, p'\}$, $p' = 3/p - 3/2 \in (0, 3/2)$, *and* $a = b$ *when* $b < 1$; $a = 1 + 2\log_\varepsilon |\ln \varepsilon|$ *when* $b = 1$.

(Theorem, continued)

Moreover, if the initial data f_0 satisfies

$$f_0(x,v) = n_0(x)\chi_0 + m_0(x)\cdot v\chi_0 + q_0(x)\chi_4,$$
$$\nabla_x \cdot m_0 = 0, \quad n_0 - \Delta_x^{-1} n_0 + \sqrt{\tfrac{2}{3}}\, q_0 = 0,$$

and $\|f_0\|_{H^6_1} + \|f_0\|_{L^2_v(L^1_x)} \le \delta_0$, then we have

$$\|f_\varepsilon(t) - u(t)\|_{L^\infty_x(L^2_v)} + \|\nabla_x\phi_\varepsilon(t) - \nabla_x\phi(t)\|_{L^\infty_x} \le C\delta_0\varepsilon(1+t)^{-\frac{3}{4}}.$$

(Idea of the proof)

- *Apply the approach by Bardos-Ukai, '91 for the classical INS from the Boltzmann equation;*
- *Sharpen the spectrum estimate including ε effect;*
- *Give preciese estimate on the initial layer.*

THANK YOU!