From Euler flows with friction to gradient flows

Athanasios Tzavaras

Computer, Electrical and Mathematical Science & Engineering

King Abdullah University of Science and Technology KAUST

Collaborators
Jan Giesselmann (TU Darmstadt)
Corrado Lattanzio (L'Aquilla)

Xiaokai Huo (KAUST)
Ansgar Juengel (U. Vienna)

Motivation

In dynamics of polymers there are two widespread theories:

- Smoluchowski theory of diffusion (developed around 1905) that describes motion of particles in a friction dominated regime
- The Kramers and Kirkwood theories in chemical physics (developed between 1940-1950) that is based on models of Hamiltonian dynamics of many particle systems
- The passage from the latter to the former is called Kramers to Smoluchowski approximation.
- High friction or small mass approximation

Paradigm

System of particles $x(t) = \big(x_1(t), x_2(t), \dots, x_N(t)\big)$ $\quad v(t) = \frac{dx}{dt}$

$$
(K) \quad \begin{cases} \dfrac{dx}{dt} = v \\ \varepsilon \dfrac{dv}{dt} = -\dfrac{\delta V}{\delta x}(x) - \dfrac{dx}{dt} \end{cases}
\qquad
\begin{aligned} V(x) &:= V\Big((x_1, \dots, x_N)(t)\Big) \\ &= \sum_{i \neq j} K_{ij} W\big(|x_i - x_j|\big) \end{aligned}
$$

As $\varepsilon \to 0$

$$
(S) \quad \frac{dx}{dt} = -\frac{\delta V}{\delta x}(x)
$$

Energy for (K) $\quad \dfrac{d}{dt}\left(\frac{1}{2}\varepsilon |v|^2 + V(x)\right) + \left|\dfrac{dx}{dt}\right|^2 = 0$

As $\varepsilon \to 0$

Energy dissipation for (S) $\quad \dfrac{d}{dt} V(x) + \left|\dfrac{\delta V}{\delta x}\right|^2 = 0$

A characterization of gradient flows

Suppose $x(t)$ is the position of a particle system and consider

$$
\begin{aligned}
\frac{d}{dt} V(x) &= \frac{\delta V}{\delta x} \cdot \frac{dx}{dt} \\
&= \tfrac{1}{2}\Big| \frac{\delta V}{\delta x} + \frac{dx}{dt} \Big|^2 - \tfrac{1}{2}\Big| \frac{\delta V}{\delta x} \Big|^2 - \tfrac{1}{2}\Big| \frac{dx}{dt} \Big|^2 \\
&\geq -\tfrac{1}{2}\Big| \frac{\delta V}{\delta x} \Big|^2 - \tfrac{1}{2}\Big| \frac{dx}{dt} \Big|^2
\end{aligned}
$$

with equality $(=)$ if and only if $x(t)$ obeys the gradient flow

$$
\frac{dx}{dt} = -\frac{\delta V}{\delta x}(x)
$$

in which case we have the potential energy dissipation

$$
\frac{d}{dt} V(x) + \Big| \frac{\delta V}{\delta x} \Big|^2 = 0
$$

A characterization of Hamiltonian flows with friction

Suppose $(x(t), v(t))$ is the position and momenta of a particle system, $v = \frac{dx}{dt}$ and consider

$$\begin{aligned}
\frac{d}{dt}\left(\tfrac{1}{2}\varepsilon|v|^2 + V(x)\right) &= \left(\varepsilon\frac{dv}{dt} + \frac{\delta V}{\delta x}\right)\cdot v \\
&= \tfrac{1}{2}\left|\varepsilon\frac{dv}{dt} + \frac{\delta V}{\delta x} + v\right|^2 - \tfrac{1}{2}\left|\varepsilon\frac{dv}{dt} + \frac{\delta V}{\delta x}\right|^2 - \tfrac{1}{2}|v|^2 \\
&\geq -\tfrac{1}{2}\left|\varepsilon\frac{dv}{dt} + \frac{\delta V}{\delta x}\right|^2 - \tfrac{1}{2}|v|^2
\end{aligned}$$

with equality $(=)$ if and only if (K) holds

$$\varepsilon\frac{dv}{dt} = -\frac{\delta V}{\delta x}(x) - v$$

in which case we have maximal energy rate dissipation

Objective

- Systems driven by an energy $\mathcal{E}(\rho)$
  - $\mathcal{E}[\rho]$ is an energy functional, e.g.

$$\mathcal{E}(\rho) = \int h(\rho) + \kappa(\rho)\,|\nabla(\rho)|^2 dx$$

$$\begin{aligned} \partial_t \rho + \operatorname{div}(\rho u) &= 0 \\ \rho \frac{Du}{Dt} &= -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \end{aligned}$$

- High friction limit from Euler flows to gradient flows
  - Euler flow with high-friction

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = -\frac{1}{\varepsilon^2}\rho\big(u + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}\big)$$

energy dissipation identity

$$\partial_t \Big(\mathcal{E}[\rho] + \int \frac{\varepsilon^2}{2} \rho |u|^2 dx \Big) + \int \rho |u|^2 dx = 0$$

  - Limit Diffusion theory

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

$\mathcal{E}[\rho]$ is a convex functional

$$\partial_t \mathcal{E}[\rho] + \int \rho \Big| \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \Big|^2 dx = 0 \qquad \text{energy dissipation}$$

Euler flows generated by an energy functional

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\rho \frac{Du}{Dt} = \rho\big(\partial_t u + (u \cdot \nabla) u\big) = -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

where $\mathcal{E}[\rho]$ is a functional

Hamiltonian

$$\mathcal{H}(\rho, u) = \mathcal{E}(\rho) + \int \tfrac{1}{2} \rho |u|^2 dx$$

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ u \end{pmatrix} = \begin{pmatrix} 0 & -\operatorname{div} \\ -\nabla & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta \rho} \\ \frac{\delta \mathcal{H}}{\delta u} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta u} \times \operatorname{curl}_x \left(\frac{1}{\rho} \frac{\delta \mathcal{H}}{\delta u}\right) \end{pmatrix} .$$

$$\frac{d}{dt} \mathcal{H}(\rho, u) = 0$$

ex: the quantum hydrodynamics system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = -\nabla p(\rho) + 2\rho\nabla\left(\frac{\triangle\sqrt{\rho}}{\sqrt{\rho}}\right) + \rho\nabla c$$

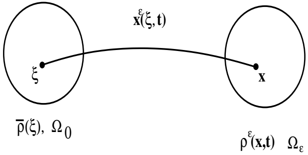
$$-\triangle c = \rho - \bar{\rho}$$

generated by the energy

$$\mathcal{E}(\rho) = \int h(\rho) + \tfrac{1}{2}\frac{1}{\rho}|\nabla\rho|^2 + \tfrac{1}{2}\rho\, c$$

with $\rho h''(\rho) = p'(\rho)$

why this structure ?



Family of maps

$$x^\varepsilon(\xi, t) \longrightarrow \begin{cases} u^\varepsilon(x,t) \\ \rho^\varepsilon(x,t) \end{cases}$$

$$\rho^\varepsilon = x^\varepsilon_{\#} \bar\rho \,, \qquad \partial_t \rho^\varepsilon + \text{div}_x(\rho^\varepsilon u^\varepsilon) = 0$$

Variation

$$x^\varepsilon(\xi, t) = x(\xi, t) + \varepsilon \delta x(\xi, t)$$

Find extrema of the action $\mathcal{L}$ over x^ε such that $\rho^\varepsilon(\cdot, t_1) = \rho_1, \ \rho^\varepsilon(\cdot, t_2) = \rho_2$

$$\mathcal{L}[x^\varepsilon] = \int_{t_1}^{t_2} \int_{\Omega_\varepsilon = x^\varepsilon(\Omega_0)} \tfrac{1}{2} \rho^\varepsilon |u^\varepsilon|^2 dx dt - \int_{t_1}^{t_2} \mathcal{E}\big[\rho^\varepsilon(\cdot, t)\big] dt$$

It turns out

$$
\begin{aligned}
x^{\varepsilon}(\xi, t) &= x(\xi, t) + \varepsilon \delta x(\xi, t) \\
\delta x(\xi, t) &= \delta \phi(x(\xi, t), t) \\
\frac{d\rho^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0} &= -\mathrm{div}_x(\rho \delta \varphi)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Big(\int_{t_1}^{t_2} \mathcal{E}\big[\rho^{\varepsilon}(\cdot, t)\big]\, dt \Big) &= \int_{t_1}^{t_2} \big\langle \frac{\delta \mathcal{E}}{\delta \rho}, \frac{d\rho^{\varepsilon}}{d\varepsilon}\Big|_{\varepsilon=0} \big\rangle \, d\tau \\
&= \int_{t_1}^{t_2} \big\langle \rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}, \delta \varphi \big\rangle \, d\tau
\end{aligned}
$$

Obtain the equations:

$$
\begin{aligned}
\partial_t \rho + \mathrm{div}\,(\rho u) &= 0 \\
\rho \frac{Du}{Dt} &= -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho}
\end{aligned}
$$

Relative Energy

Idea is to compute

$$\mathcal{K}(\rho, m \mid \bar\rho, \bar m) := \int \tfrac{1}{2}\rho\Big|\frac{m}{\rho} - \frac{\bar m}{\bar\rho}\Big|^2 dx$$

$$\mathcal{E}(\rho|\bar\rho) := \mathcal{E}(\rho) - \mathcal{E}(\bar\rho) - \left\langle \frac{\delta\mathcal{E}}{\delta\rho}(\bar\rho), \rho - \bar\rho \right\rangle$$

$\mathcal{K}(\rho, m)$ convex in (ρ, m), and require that $\mathcal{E}(\rho)$ is convex in ρ

Relative entropy calculation

$$\frac{d}{dt}\left(\int \tfrac{1}{2}\rho|u - \bar u|^2\, dx + \mathcal{E}(\rho|\bar\rho)\right) = \int -\rho\nabla_x \bar u : (u - \bar u)\otimes(u - \bar u) + \int \nabla\bar u : S(\rho|\bar\rho)\, dx$$

where

$$\mathcal{E}(\rho|\bar\rho) := \mathcal{E}(\rho) - \mathcal{E}(\bar\rho) - \left\langle \frac{\delta\mathcal{E}}{\delta\rho}(\bar\rho), \rho - \bar\rho \right\rangle$$

$$S(\rho|\bar\rho) := S(\rho) - S(\bar\rho) - \left\langle \frac{\delta S}{\delta\rho}(\bar\rho), \rho - \bar\rho \right\rangle$$

Hypothesis : $\mathcal{E}(\rho)$ satisfies for some functional $S[\rho]$

$$(*) \qquad -\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} = \nabla_x \cdot S[\rho]$$

Formula (*)

- gives meaning to weak solutions
- serves as the basis for the relative energy calculation

Note that the reative entropy calculation is based on the weak for of (*) and its variational derivative

$$\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho) \, , \, \partial_{x_j}(\rho \varphi_j) \rangle = - \int S_{ij}[\rho] \frac{\partial \varphi_j}{\partial x_i}(x) \, dx$$

$$\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho) \, , \, \partial_{x_j}(\psi \varphi_j) \rangle + \langle \langle \frac{\delta^2 \mathcal{E}}{\delta \rho^2}(\rho) \, : \, (\psi, \partial_{x_j}(\rho \varphi_j)) \rangle \rangle = - \int \langle \frac{\delta S_{ij}}{\delta \rho}(\rho), \psi \frac{\partial \varphi_j}{\partial x_i} \rangle \, dx$$

Validation of hypothesis for the Euler-Korteweg system

$$\mathcal{E}[\rho] = \int F(\rho, \nabla\rho)\, dx = \int h(\rho) + \tfrac{1}{2}\kappa(\rho)|\nabla\rho|^2\, dx$$

The functional is invariant under translations $\rho(\cdot) \longrightarrow \rho(\cdot + \vec{h})$
NOETHER'S THEOREM implies

$$\begin{aligned} -\rho\nabla_x \frac{\delta\mathcal{E}}{\delta\rho} &= \nabla_x \cdot S[\rho] \\ &= \nabla_x \cdot \underbrace{\Big[\big(-\rho F_\rho + F + \rho\nabla_x \cdot F_q\big) I - F_q(\rho\,, \nabla_x\rho) \otimes \nabla_x\rho\Big]}_{\mathcal{S}(\rho)} \end{aligned}$$

Application to the quantum hydrodynamics system

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$
$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = 2\rho \nabla \left(\frac{\triangle \sqrt{\rho}}{\sqrt{\rho}} \right)$$

generated by the energy $\mathcal{E}(\rho) = \int h(\rho) + \frac{1}{2}\frac{1}{\rho}|\nabla \rho|^2$

Thm If (ρ, u) is a weak conservative solution and $(\bar{\rho}, \bar{u})$ smooth then

$$\Psi(t) = \int \tfrac{1}{2}\rho|u - \bar{u}|^2 + h(\rho|\bar{\rho}) + \tfrac{1}{2}\rho \left| \frac{\nabla \rho}{\rho} - \frac{\nabla \bar{\rho}}{\rho} \right|^2 dx$$

satisfies the stability estimate

$$\Psi(t) \leq \Psi(0) + O(|\nabla \bar{u}|) \int_0^T \Psi(\tau) d\tau$$

Gieselmann - Lattanzio - T 16 Bresch-Gisclon-Violet 18

Diffusion Theories

$$\partial_t \rho = \operatorname{div} \big(\rho \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \big)$$

where $\mathcal{E}[\rho]$ is a convex functional of ρ; and energy dissipation

$$\partial_t \mathcal{E}[\rho] + \int \rho \Big| \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \Big|^2 dx = 0$$

- Examples: porous media, generalized Keller-Segel models, Cahn-Hilliard equation

  Otto, Carillo-Toscani, Villani, Westdickenberg ...

- Diffusion theories arise as models in a friction dominated regime eg. porous media, Smoluchowski approximation in polymeric flows, drift-diffusion for semiconductors

(i) $\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$

Energy dissipation

$$\partial_t \mathcal{E}[\rho] + \int \rho \Big| \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \Big|^2 dx = 0 \qquad (*)$$

(ii) Minimization via Wasserstein distance, Jordan-Kinderlehrer-Otto scheme

$$\rho^k \text{ is the minimizer of the problem} \quad \min \left\{ \frac{1}{2h} d_W(\rho, \rho^{k-1})^2 + \mathcal{E}[\rho] \right\}$$

(iii) Let (ρ, u) satisfy $\partial_t \rho + \mathrm{div}\,(\rho u) = 0$. Then

$$\partial_t \mathcal{E}[\rho] = \langle \rho u, \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \rangle \geq \int -\rho |u|^2 - \rho |\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}|^2 dx$$

with equality iff

$$u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \qquad \text{and then } (*) \text{ holds}$$

Diffusion theory

$$\partial_t \rho + \operatorname{div}(\rho u) = 0 \qquad u = -\nabla_x \frac{\delta \mathcal{E}}{\delta \rho}$$

where $\mathcal{E}[\rho]$ is a convex functional

$$\partial_t \mathcal{E}[\rho] + \int \rho \Big| \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \Big|^2 dx = 0 \qquad \text{energy dissipation}$$

Relaxation to the diffusion theory

$$\partial_t \rho + \operatorname{div}(\rho u) = 0$$

$$\partial_t (\rho u) + \operatorname{div}(\rho u \otimes u) = -\frac{1}{\varepsilon^2} \rho \Big(u + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \Big)$$

has the energy dissipation structure

$$\partial_t \Big(\mathcal{E}[\rho] + \int \frac{\varepsilon^2}{2} \rho |u|^2 dx \Big) + \int \rho |u|^2 dx = 0$$

maximal energy dissipation

Suppose (ρ, u) satisfies $\partial_t \rho + \mathrm{div}\,(\rho u) = 0$

$$
\begin{aligned}
\frac{d}{dt}\Big(\int_\Omega \varepsilon \tfrac{1}{2}\rho |u|^2 \, dx + \mathcal{E}[\rho] \Big) &= \langle \rho u, \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \rangle \\
&= \int_\Omega \tfrac{1}{2}\rho \left| \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} + u \right|^2 - \int_\Omega \tfrac{1}{2}\rho \left| \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \right|^2 - \int_\Omega \tfrac{1}{2}\rho |u|^2 dx \\
&\geq - \int_\Omega \tfrac{1}{2}\rho \left| \varepsilon \frac{Du}{Dt} + \nabla_x \frac{\delta \mathcal{E}}{\delta \rho} \right|^2 - \int_\Omega \tfrac{1}{2}\rho |u|^2 dx
\end{aligned}
$$

with equality if and only if

$$
\varepsilon \rho \frac{Du}{Dt} = -\rho \nabla \frac{\delta \mathcal{E}}{\delta \rho} - \rho u
$$

Relative entropy for the relaxation system and the limiting diffusion theory

Relaxation system:

$$\frac{d}{dt}\left(\mathcal{E}(\rho\,|\bar\rho) + \int \frac{\varepsilon^2}{2}|u-\bar u|^2 dx\right) + \int \rho|u-\bar u|^2 dx$$
$$= -\int \left(\varepsilon^2 \rho \nabla_x \bar u : (u-\bar u)\otimes(u-\bar u) + \nabla \bar u : S(\rho\,|\bar\rho)\right) dx$$

Limiting diffusion theory ($\varepsilon = 0$):

$$\frac{d}{dt}\mathcal{E}(\rho\,|\bar\rho) + \int \rho \left|\nabla_x \frac{\delta\mathcal{E}(\rho)}{\delta\rho} - \nabla_x \frac{\delta\mathcal{E}(\bar\rho)}{\delta\rho}\right|^2 dx = -\int \nabla \bar u : S(\rho\,|\bar\rho) dx$$

Lattanzio - AT 17

Ex Cahn-Hilliard equation

$\mathcal{E}(\rho)$ entropy functional

$$\mathcal{E}[\rho] = \int F(\rho, \nabla\rho)\, dx,$$
$$\left\langle \frac{\delta \mathcal{E}}{\delta \rho}(\rho), \varphi \right\rangle = \int F_\rho \varphi + F_q \cdot \nabla \varphi$$

Generated diffusion equation

$$\partial_t \rho - \nabla_x \cdot \rho \nabla_x \left(F_\rho - \nabla_x \cdot F_q \right) = 0$$

special case $F = h(\rho) + \frac{C_\kappa}{2} |\nabla\rho|^2$ yields the Cahn-Hilliard equation

$$\partial_t \rho = \triangle p(\rho) - C_\kappa \nabla_x \cdot \rho \nabla (\triangle \rho)$$

Otto-Slepcev

from Euler-Korteweg to Cahn-Hilliard

Let (ρ, u) be a dissipative solution of the Euler-Korteweg system (with $\kappa = 1$) with friction and $\bar{\rho}$ a smooth solution of the Cahn-Hilliard equation

$$\rho_t = \operatorname{div}\left(\nabla p(\rho) - C_\kappa \rho \nabla \triangle \rho\right)$$

Thm If $p'(\rho) > 0$, $h(\rho) = \frac{1}{\gamma-1}\rho^\gamma + o(\rho^\gamma)$ and $|p''(\rho)| \leq A\frac{p'(\rho)}{\rho}$, and

$$\gamma \geq 2 \qquad \text{or} \qquad \left\{\ 1 < \gamma < 2 \quad \text{and} \quad \int \rho_0 = \int \bar{\rho}_0\ \right\}$$

then

$$\Psi(t) = \int_{\mathbb{T}^d} \tfrac{1}{2}\rho|u - \bar{u}|^2 + h(\rho|\bar{\rho}) + \frac{1}{2}|\nabla(\rho - \bar{\rho})|^2 dx \to 0$$

as $\varepsilon \to 0$ where $\frac{1}{\varepsilon}$ is the friction coefficient.

Multi-component mixtures

(ρ_i, v_i, e_i) model partial balance equations

$$
\begin{aligned}
\partial_t \rho_i + \operatorname{div}(\rho_i v_i) &= 0 \\
\partial_t (\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) &= \operatorname{div} S_i + f_i \\
\partial_t (\rho_i e_i + \tfrac{1}{2}\rho_i |v_i|^2) + \operatorname{div}(\rho_i v_i e_i + \tfrac{1}{2}\rho_i v_i |v_i|^2) &= \operatorname{div}(v_i \cdot S_i + q_i) + h_i
\end{aligned}
$$

where $\sum f_i = 0$, $\sum h_i = 0$

f_i partial forces, h_i partial energies

Class I models: based on $\rho_1, \ldots, \rho_n$ and v barycentric velocity
Class II models: based on $\rho_1, \ldots, \rho_n$ and $v_1, \ldots v_n$

Thermodynamical properties are well understood for reacting mixtures

Bothe - Dreyer 15

Eckart, I. Mueller, Ruggeri, ...

High-friction limits for multi-component flows

$(\rho_1, ..., \rho_n, v_1, ..., v_n)$ the densities/velocities of a multi-component fluid

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v_i) = 0 \\ \partial_t(\rho_i v_i) + \operatorname{div}(\rho_i v_i \otimes v_i) = -\rho_i \nabla \dfrac{\delta \mathcal{E}}{\delta \rho_i} - \frac{1}{\varepsilon} \displaystyle\sum_j b_{ij} \rho_i \rho_j (v_i - v_j) \end{cases}$$

$$\mathcal{E}(\rho_1, ..., \rho_n) = \int \sum_i h(\rho_i) + \sum_i \tfrac{1}{2} \kappa(\rho_i) |\nabla \rho_i|^2$$

$$\frac{d}{dt} \left(\mathcal{E}(\rho) + \int \sum_i \tfrac{1}{2} \rho_i |v_i|^2 \right) + \tfrac{1}{2\varepsilon} \int \sum_{i,j} b_{i,j} \rho_i \rho_j |v_1 - v_j|^2 = 0$$

We are interested in the high friction limit $\varepsilon \to 0$

work by X. Huo, A. Juengel, A.T. 2019

Moments

$$\rho = \sum \rho_i \quad \rho v = \sum_i \rho_i v_i \quad u_i := v_i - v$$

ρ total density (conserved); v barycentric velocity; u_i relative velocities

Solvability of linear system

$$\begin{cases} -\sum_j b_{ij}\rho_i\rho_j(u_i - u_j) = d_i & \text{where } \sum_i d_i = 0 \\ \sum_i \rho_i u_i = 0 \end{cases}$$

Hypothesis $\quad b_{ij} = b_{ji},\ b_{ij} \geq 0, \quad \mathcal{N}(\mathcal{B}) = \text{span}\{(1, 1, ..., 1)\}$ (H)

$$\text{Solution} \begin{cases} \rho_i u_i = -\sum_{j,k=1}^{n-1} (Q^{-1})_{ij}(\tau^{-1})_{jk} d_k \\ \rho_n u_n = -\sum_{i=1}^{n-1} \rho_i u_i \end{cases}$$

where $\tau^{(n-1)\times(n-1)}$ invertible and $Q^{(n-1)\times(n-1)} > 0$ Wen-An Yong

Chapman-Enskog expansion

Set $\rho = \sum_i \rho_i$ $\qquad v = \frac{1}{\rho}\sum_i \rho_i v_i$

v is the mean velocity, $\quad u_i := v_i - v \quad$ the relative velocities

A Chapman-Enskog expansion calculates the (formal) effective equation which up to order $O(\varepsilon^2)$ is the system for $(\rho_1, ..., \rho_n)$, v, $(u_1, ..., u_n)$

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v) = -div(\rho_i u_i) \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = -\sum_i \rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} \end{cases}$$

$$-\sum_j b_{ij}\rho_i\rho_j(u_i - u_j) = \varepsilon \left(\rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} - \frac{\rho_i}{\rho} \sum_j \rho_j \nabla \frac{\delta \mathcal{E}}{\delta \rho_j} \right)$$

$$\sum_i \rho_i u_i = 0$$

It can be expressed in the form

$$\begin{cases} \partial_t \rho_i + \operatorname{div}(\rho_i v) = \varepsilon \operatorname{div} \sum_{j=1}^{n} D_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_j} \\ \partial_t(\rho v) + \operatorname{div}(\rho v \otimes v) = - \sum_{i=1}^{n} \rho_i \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} \end{cases}$$

where

$$D^{n \times n} = G Q^{-1}(\rho) \tau^{-1} Q^{-1}(\rho) G^T \qquad D \geq 0$$

Note

$$\partial_t \sum \rho_i + \operatorname{div}\left(\sum \rho_i v\right) = 0$$

$$\partial_t \Big(\mathcal{E}[\rho] + \int \tfrac{1}{2} \big(\sum_i \rho_i\big) |v|^2 \, dx \Big) + \varepsilon \int \sum_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} D_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_j} = 0$$

Consider the case $v = 0$. Then $\partial_t \rho = 0$ and $\rho = \sum_i \rho_i = const$

Set $\tilde{\rho} = (\rho_1, ..., \rho_{n-1})$, $\rho_n = \rho - \sum_{i=1}^{n-1} \rho_i$

$$\tilde{\mathcal{E}}(\rho_1, ..., \rho_{n-1}) = \mathcal{E}(\rho_1, ..., \rho_{n-1}, \rho - \sum_{i=1}^{n-1} \rho_i)$$

Lemma We can write

$$\partial_t \tilde{\rho}_i = \text{div} \left(\sum_{j=1}^{n-1} \tilde{D}_{ij} \nabla \frac{\delta \tilde{\mathcal{E}}}{\delta \rho_j} \right) \qquad i = 1, ..., n-1$$

where $\tilde{D}^{(n-1)\times(n-1)} > 0$. The system is a gradient flow, i.e.

$$\frac{d\tilde{\mathcal{E}}}{dt} = -\varepsilon \int \int \sum_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_i} \cdot \tilde{D}_{ij} \nabla \frac{\delta \mathcal{E}}{\delta \rho_j}$$

and it is parabolic, i.e. e.v of $\tilde{D}\mathcal{E}''$ are real, positive.

$$(H) \quad \text{Hypothesis} \qquad -\rho_i \nabla \frac{\delta \tilde{\mathcal{E}}}{\delta \rho_i} = \operatorname{div} S_i(\rho) \quad i = 1, ..., n$$

Thm Under hypothesis (H)
if (ρ_i, v_i) is solution of the relaxation system
and $(\hat{\rho}_i, \hat{v})$ solution of the Chapman-Enskog expansion, $\hat{v}_i = \hat{v} + \hat{u}_i$
then the relative entropy

$$\chi(t) = \int \sum_i \tfrac{1}{2}\rho_i |v_i - \hat{v}_i|^2 + |\rho_i - \hat{\rho}_i|^2 + \tfrac{1}{2}\kappa(\rho_i)|k_i(\rho_i)\nabla\rho_i - k_i(\hat{\rho}_i)\nabla\hat{\rho}_i|^2$$

satisfies

$$\chi(t) \leq C(\chi(0) + \varepsilon^2)$$

Huo - Juengel - T. 2019

Remarks

- Hypothesis (H) excludes p-n models for semi-conductors
- Convergence to zero-relaxation limit for gas dynamic and Euler-Korteweg like models

Thank You